Exercise 2.12 (Order of matrix) What are the orders of the matrices that guarantee that $\boldsymbol{ABC}$ is defined?

Solution

Let $\boldsymbol{A}$ $(m \times n)$, $\boldsymbol{B}$ $(p \times q)$, and $\boldsymbol{C}$ $(r \times s)$. Then $\boldsymbol{ABC}$ is defined when $n = p$ and $q = r$, but also when $m = n = 1$ and $q = r$, when $p = q = 1$ and $n = r$, or when $r = s = 1$ and $n = p$. It is also defined when any two of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are scalars.

*__Exercise 2.13 (Generalization of $x^2 = 0 \iff x = 0$)__ For real matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, show that:

(a) $\boldsymbol{A}'\boldsymbol{A} = \mathbf{O}$ if and only if $\boldsymbol{A} = \mathbf{O}$;

(b) $\boldsymbol{AB} = \mathbf{O}$ if and only if $\boldsymbol{A}'\boldsymbol{AB} = \mathbf{O}$;

(c) $\boldsymbol{AB} = \boldsymbol{AC}$ if and only if $\boldsymbol{A}'\boldsymbol{AB} = \boldsymbol{A}'\boldsymbol{AC}$.

(d) Why do we require the matrices to be real?

Solution

(a) Clearly, $\boldsymbol{A} = \mathbf{O}$ implies $\boldsymbol{A}'\boldsymbol{A} = \mathbf{O}$. Conversely, assume $\boldsymbol{A}'\boldsymbol{A} = \mathbf{O}$. Then, for all j, the j-th diagonal element of $\boldsymbol{A}'\boldsymbol{A}$ is zero, that is, $\sum_i a_{ij}^2 = 0$. This implies that $a_{ij} = 0$ for all i and j, and hence that $\boldsymbol{A} = \mathbf{O}$. Contrast this result with Exercise 2.8(b).

(b) Clearly, $\boldsymbol{AB} = \mathbf{O}$ implies $\boldsymbol{A}'\boldsymbol{AB} = \mathbf{O}$. Conversely, if $\boldsymbol{A}'\boldsymbol{AB} = \mathbf{O}$, then

$$(\boldsymbol{AB})'(\boldsymbol{AB}) = \boldsymbol{B}'\boldsymbol{A}'\boldsymbol{AB} = \mathbf{O}$$

and hence $\boldsymbol{AB} = \mathbf{O}$, by (a).

(c) This follows by replacing $\boldsymbol{B}$ by $\boldsymbol{B} - \boldsymbol{C}$ in (b).

(d) Consider $\boldsymbol{a} = (1 + \mathrm{i}, 1 - \mathrm{i})'$. Then $\boldsymbol{a}'\boldsymbol{a} = (1 + \mathrm{i})^2 + (1 - \mathrm{i})^2 = 0$, even though $\boldsymbol{a} \neq \mathbf{0}$. Hence, the above statements are, in general, not true for complex matrices. However, they are true if we replace $'$ by *.

Exercise 2.14 (Multiplication, 3)

(a) Show that $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$ for conformable $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$.

(b) Show that $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$ for conformable $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$.

Solution

(a) Let $\boldsymbol{D} := \boldsymbol{AB}$ and $\boldsymbol{E} := \boldsymbol{BC}$. Then,

$$\begin{aligned}(\boldsymbol{DC})_{ik} &= \sum_j d_{ij}c_{jk} = \sum_j \left(\sum_h a_{ih}b_{hj}\right) c_{jk} \\ &= \sum_h a_{ih} \left(\sum_j b_{hj}c_{jk}\right) = \sum_h a_{ih}e_{hk} = (\boldsymbol{AE})_{ik}.\end{aligned}$$

Hence, $DC = AE$.
(b) Let $D := B + C$. Then,

$$(AD)_{ij} = \sum_h a_{ih}d_{hj} = \sum_h a_{ih}(b_{hj} + c_{hj})$$
$$= \sum_h a_{ih}b_{hj} + \sum_h a_{ih}c_{hj} = (AB)_{ij} + (AC)_{ij}.$$

Exercise 2.15 (Transpose and products)
(a) Show that $(AB)' = B'A'$.
(b) Show that $(ABC)' = C'B'A'$.
(c) Under what condition is $(AB)' = A'B'$?

Solution
(a) We have

$$(B'A')_{ij} = \sum_h (B')_{ih}(A')_{hj} = \sum_h (B)_{hi}(A)_{jh}$$
$$= \sum_h (A)_{jh}(B)_{hi} = (AB)_{ji}.$$

(b) Let $D := BC$. Then, using (a),

$$(ABC)' = (AD)' = D'A' = (BC)'A' = C'B'A'.$$

(c) This occurs if and only if $AB = BA$, that is, if and only if A and B commute.

Exercise 2.16 (Partitioned matrix) Let A and B be 3×5 matrices, partitioned as

$$A = \left(\begin{array}{ccc|cc} 1 & 3 & -2 & 1 & 2 \\ 6 & 8 & 0 & -1 & 6 \\ \hline 0 & 0 & 1 & 4 & 1 \end{array}\right), \quad B = \left(\begin{array}{ccc|cc} 1 & -3 & -2 & 4 & 1 \\ 6 & 2 & 6 & 2 & 0 \\ \hline 1 & 0 & 2 & 0 & 1 \end{array}\right),$$

and let C be a 5×4 matrix, partitioned as

$$C = \left(\begin{array}{cc|cc} 1 & 0 & 5 & 1 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 3 & 1 \\ \hline 3 & 5 & 0 & 2 \\ 2 & -1 & 3 & 1 \end{array}\right).$$

Denoting the submatrices by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

show that

$$A + B = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix},$$

$$AC = \begin{pmatrix} A_{11}C_{11} + A_{12}C_{21} & A_{11}C_{12} + A_{12}C_{22} \\ A_{21}C_{11} + A_{22}C_{21} & A_{21}C_{12} + A_{22}C_{22} \end{pmatrix},$$

and

$$A' = \begin{pmatrix} A'_{11} & A'_{21} \\ A'_{12} & A'_{22} \end{pmatrix}.$$

Solution

$$A + B = \left(\begin{array}{ccc|cc} (1+1) & (3-3) & (-2-2) & (1+4) & (2+1) \\ (6+6) & (8+2) & (0+6) & (-1+2) & (6+0) \\ \hline (0+1) & (0+0) & (1+2) & (4+0) & (1+1) \end{array}\right)$$

$$= \left(\begin{array}{ccc|cc} 2 & 0 & -4 & 5 & 3 \\ 12 & 10 & 6 & 1 & 6 \\ \hline 1 & 0 & 3 & 4 & 2 \end{array}\right) = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix},$$

$$AC = \left(\begin{array}{ccc|cc} 1 & 3 & -2 & 1 & 2 \\ 6 & 8 & 0 & -1 & 6 \\ \hline 0 & 0 & 1 & 4 & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & 0 & 5 & 1 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 3 & 1 \\ \hline 3 & 5 & 0 & 2 \\ 2 & -1 & 3 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 10 & 9 & 5 & 3 \\ 15 & 5 & 48 & 10 \\ \hline 13 & 19 & 6 & 10 \end{array}\right) = \begin{pmatrix} A_{11}C_{11} + A_{12}C_{21} & A_{11}C_{12} + A_{12}C_{22} \\ A_{21}C_{11} + A_{22}C_{21} & A_{21}C_{12} + A_{22}C_{22} \end{pmatrix},$$

and

$$A' = \left(\begin{array}{cc|c} 1 & 6 & 0 \\ 3 & 8 & 0 \\ -2 & 0 & 1 \\ \hline 1 & -1 & 4 \\ 2 & 6 & 1 \end{array}\right) = \begin{pmatrix} A'_{11} & A'_{21} \\ A'_{12} & A'_{22} \end{pmatrix}.$$

Exercise 2.17 (Sum of outer products) Let $A := (a_1, a_2, \dots, a_n)$ be an $m \times n$ matrix.
(a) Show that $AA' = \sum_i a_i a_i'$.
(b) Show that $A'A = (a_i' a_j)$.

Solution
We write

$$AA' = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)\begin{pmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_n' \end{pmatrix}, \quad A'A = \begin{pmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_n' \end{pmatrix}(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n),$$

and the results follow.

***Exercise 2.18 (Identity matrix)**
(a) Show that $\boldsymbol{I}\boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x}$, and that this relation uniquely determines $\boldsymbol{I}$.
(b) Show that $\boldsymbol{I}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{I} = \boldsymbol{A}$ for any matrix $\boldsymbol{A}$, and specify the orders of the identity matrices.

Solution
(a) If $\boldsymbol{A} = \boldsymbol{I}$, then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ holds for all $\boldsymbol{x}$. Conversely, if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ holds for all $\boldsymbol{x}$, then it holds in particular for the unit vectors $\boldsymbol{x} = \boldsymbol{e}_j$. This gives $\boldsymbol{A}\boldsymbol{e}_j = \boldsymbol{e}_j$, so that $a_{ij} = \boldsymbol{e}_i'\boldsymbol{A}\boldsymbol{e}_j = \boldsymbol{e}_i'\boldsymbol{e}_j$, which is zero when $i \neq j$ and one when $i = j$. Hence, $\boldsymbol{A} = \boldsymbol{I}$.
(b) Let $\boldsymbol{A}$ be an $m \times n$ matrix, and let $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ denote its columns. Then,

$$\boldsymbol{I}_m\boldsymbol{A} = (\boldsymbol{I}_m\boldsymbol{a}_1, \boldsymbol{I}_m\boldsymbol{a}_2, \ldots, \boldsymbol{I}_m\boldsymbol{a}_n) = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n) = \boldsymbol{A},$$

using (a). Since $\boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}$ for every $\boldsymbol{A}$, it follows that $\boldsymbol{I}_n\boldsymbol{A}' = \boldsymbol{A}'$ for every $\boldsymbol{A}$, and hence that $\boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{A}$.

Exercise 2.19 (Diagonal matrix, permutation)
(a) Is the 3×3 matrix

$$\boldsymbol{A} := \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}$$

a diagonal matrix?
(b) With $\boldsymbol{A}$ defined in (a), show that $\boldsymbol{A}\boldsymbol{A}'$ and $\boldsymbol{A}'\boldsymbol{A}$ are diagonal matrices.

Solution
(a) Although one might argue that a square matrix has two diagonals, only the diagonal $(a_{11}, a_{22}, \ldots, a_{nn})$ is called *the* diagonal. So, the matrix $\boldsymbol{A}$ is *not* a diagonal matrix, unless $a = c = 0$.
(b) We have

$$\boldsymbol{A}\boldsymbol{A}' = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix} = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix},$$

and, similarly,

$$A'A = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}.$$

Exercise 2.20 (Diagonal matrices, commutation) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be diagonal matrices. Show that $\boldsymbol{AB}$ is also diagonal and that $\boldsymbol{AB} = \boldsymbol{BA}$.

Solution

Let $\boldsymbol{A} := \text{diag}(a_1, a_2, \dots, a_n)$ and $\boldsymbol{B} := \text{diag}(b_1, b_2, \dots, b_n)$. Then,

$$\boldsymbol{AB} = \text{diag}(a_1 b_1, \dots, a_n b_n) = \text{diag}(b_1 a_1, \dots, b_n a_n) = \boldsymbol{BA}.$$

A diagonal matrix is the simplest generalization of a scalar, and essentially all properties of scalars also hold for diagonal matrices.

Exercise 2.21 (Triangular matrix)

(a) Consider the lower triangular matrices

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

Show that $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are lower triangular, but that $\boldsymbol{AB} \neq \boldsymbol{BA}$.

(b) Show that the product of two lower triangular matrices is always lower triangular.

Solution

(a) We have

$$\boldsymbol{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{BA} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}.$$

(b) Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be lower triangular $n \times n$ matrices. Consider the ij-th element of $\boldsymbol{AB}$. We will show that $(\boldsymbol{AB})_{ij} = 0$ for $i < j$. Now,

$$(\boldsymbol{AB})_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{k=1}^{i} a_{ik} b_{kj} + \sum_{k=i+1}^{n} a_{ik} b_{kj}.$$

In the first sum, $b_{kj} = 0$ for all $k \leq i < j$; in the second sum, $a_{ik} = 0$ for all $k > i$. Hence, $(\boldsymbol{AB})_{ij} = 0$ for $i < j$, that is, $\boldsymbol{AB}$ is lower triangular.

Exercise 2.22 (Symmetry) Let $\boldsymbol{A}$ be a square real matrix.

(a) Show that $\boldsymbol{A} + \boldsymbol{A}'$ is symmetric, even if $\boldsymbol{A}$ is not symmetric.

(b) Show that $\boldsymbol{AB}$ is not necessarily symmetric if $\boldsymbol{A}$ and $\boldsymbol{B}$ are.

(c) Show that $\boldsymbol{A}'\boldsymbol{BA}$ is symmetric if $\boldsymbol{B}$ is symmetric, but that the converse need not be true.

Solution
(a) Since $(A + B)' = A' + B'$, we have $(A + A')' = A' + (A')' = A' + A = A + A'$. Hence, $A + A'$ is symmetric.
(b) For example,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(c) We have $(A'BA)' = A'B'(A')' = A'BA$. To prove that the converse is not necessarily true, let e_i and e_j be unit vectors and define $A := e_i e_j'$. Then, for any matrix B, $A'BA = e_j e_i' B e_i e_j' = b_{ii} e_j e_j'$, which is symmetric.

Exercise 2.23 (Skew-symmetry) Let A be a square real matrix.
(a) Show that $A - A'$ is skew-symmetric.
(b) Hence, show that A can be decomposed as the sum of a symmetric and a skew-symmetric matrix.
(c) If A is skew-symmetric, show that its diagonal elements are all zero.

Solution
(a) We have $(A - A')' = A' - A = -(A - A')$.
(b) We write

$$A = \frac{A + A'}{2} + \frac{A - A'}{2}.$$

The first matrix on the right-hand side is symmetric; the second is skew-symmetric.
(c) Since the diagonal elements of A' are the diagonal elements of A, the defining equation $A' = -A$ implies that $a_{ii} = -a_{ii}$ for all i. Hence, $a_{ii} = 0$ for all i.

Exercise 2.24 (Trace as linear operator) The trace of a square matrix A is the sum of its diagonal elements, and is written as $\operatorname{tr}(A)$ or $\operatorname{tr} A$. Let A and B be square matrices of the same order, and let λ and μ be scalars. Show that:
(a) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$;
(b) $\operatorname{tr}(\lambda A + \mu B) = \lambda \operatorname{tr}(A) + \mu \operatorname{tr}(B)$;
(c) $\operatorname{tr}(A') = \operatorname{tr}(A)$;
(d) $\operatorname{tr}(AA') = \operatorname{tr}(A'A) = \sum_{ij} a_{ij}^2$;
(e) $\operatorname{tr}(aa') = a'a = \sum_i a_i^2$ for any vector a.

Solution
(a)–(b) This follows by direct verification or by noting that the trace is a linear operator.
(c) In the trace operation only *diagonal* elements are involved; what happens outside the diagonal is irrelevant.
(d) We have

$$\operatorname{tr} AA' = \sum_i (AA')_{ii} = \sum_i \sum_j a_{ij}^2 = \sum_j \sum_i a_{ij}^2 = \sum_j (A'A)_{jj} = \operatorname{tr} A'A.$$

(e) This follows from (d) because $\operatorname{tr} a'a = a'a$, since $a'a$ is a scalar.

Exercise 2.25 (Trace of $A'A$) For any real matrix $\boldsymbol{A}$, show that $\operatorname{tr} \boldsymbol{A}'\boldsymbol{A} \geq 0$, with $\operatorname{tr} \boldsymbol{A}'\boldsymbol{A} = 0$ if and only if $\boldsymbol{A} = \mathbf{O}$.

Solution
Since $\operatorname{tr} \boldsymbol{A}'\boldsymbol{A} = \sum_{ij} a_{ij}^2$ and $\boldsymbol{A}$ is real, the result follows.

Exercise 2.26 (Trace, cyclical property)
(a) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ matrices. Prove that

$$\operatorname{tr}(\boldsymbol{A}'\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{B}\boldsymbol{A}') = \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}') = \operatorname{tr}(\boldsymbol{B}'\boldsymbol{A}).$$

(b) Show that $\operatorname{tr}(\boldsymbol{A}\boldsymbol{a}\boldsymbol{a}') = \boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}$ for any square $\boldsymbol{A}$ and conformable $\boldsymbol{a}$.
(c) Show that $\operatorname{tr}(\boldsymbol{ABC}) = \operatorname{tr}(\boldsymbol{CAB}) = \operatorname{tr}(\boldsymbol{BCA})$ and specify the restrictions on the orders of $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$.
(d) Is it also true that $\operatorname{tr}(\boldsymbol{ABC}) = \operatorname{tr}(\boldsymbol{ACB})$?

Solution
(a) In view of Exercise 2.24(c) it is sufficient to prove $\operatorname{tr}(\boldsymbol{A}'\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{B}\boldsymbol{A}')$. We have

$$\operatorname{tr}(\boldsymbol{A}'\boldsymbol{B}) = \sum_j (\boldsymbol{A}'\boldsymbol{B})_{jj} = \sum_j \sum_i a_{ij} b_{ij} = \sum_i \sum_j b_{ij} a_{ij} = \sum_i (\boldsymbol{B}\boldsymbol{A}')_{ii} = \operatorname{tr}(\boldsymbol{B}\boldsymbol{A}').$$

(b) This follows from (a).
(c) Let $\boldsymbol{A}$ $(m \times n)$, $\boldsymbol{B}$ $(n \times p)$, and $\boldsymbol{C}$ $(p \times m)$, so that $\boldsymbol{ABC}$ is defined and square. Then, using (a),

$$\operatorname{tr}(\boldsymbol{ABC}) = \operatorname{tr}((\boldsymbol{AB})\boldsymbol{C}) = \operatorname{tr}(\boldsymbol{C}(\boldsymbol{AB})) = \operatorname{tr}(\boldsymbol{CAB}),$$

and similarly for the second equality.
(d) No, this is not true. The expression $\boldsymbol{ACB}$ is not even defined in general.

Exercise 2.27 (Trace and sum vector) Show that

$$\boldsymbol{\imath}'\boldsymbol{A}\boldsymbol{\imath} = \boldsymbol{\imath}'(\operatorname{dg} \boldsymbol{A})\boldsymbol{\imath} + \operatorname{tr}\left((\boldsymbol{\imath}\boldsymbol{\imath}' - \boldsymbol{I}_n)\boldsymbol{A}\right)$$

for any $n \times n$ matrix $\boldsymbol{A}$.

Solution
We write

$$\operatorname{tr}\left((\boldsymbol{\imath}\boldsymbol{\imath}' - \boldsymbol{I}_n)\boldsymbol{A}\right) = \operatorname{tr}(\boldsymbol{\imath}\boldsymbol{\imath}'\boldsymbol{A}) - \operatorname{tr}(\boldsymbol{I}_n\boldsymbol{A}) = \operatorname{tr}(\boldsymbol{\imath}'\boldsymbol{A}\boldsymbol{\imath}) - \operatorname{tr}(\boldsymbol{A}) = \boldsymbol{\imath}'\boldsymbol{A}\boldsymbol{\imath} - \boldsymbol{\imath}'(\operatorname{dg} \boldsymbol{A})\boldsymbol{\imath}.$$

Exercise 2.28 (Orthogonal matrix, representation) A real square matrix $\boldsymbol{A}$ is orthogonal if $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}' = \boldsymbol{I}$.

(a) Show that every orthogonal 2×2 matrix takes one of the two forms

$$A_1 := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad A_2 := \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix},$$

and describe its effect on a 2×1 vector $\boldsymbol{x}$.
(b) Show that, if a matrix $\boldsymbol{A}$ is orthogonal, its rows form an orthonormal set.
(c) Show that, if a matrix $\boldsymbol{A}$ is orthogonal, its columns also form an orthonormal set.

Solution
(a) This is essentially a generalization of the fact that any normalized real 2×1 vector $\boldsymbol{x}$ has a representation $\boldsymbol{x} = (\cos\theta, \sin\theta)'$. Let

$$\boldsymbol{A} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The equations $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}' = \boldsymbol{I}$ yield

$$a^2 + b^2 = 1, \quad a^2 + c^2 = 1, \quad b^2 + d^2 = 1, \quad c^2 + d^2 = 1,$$

and

$$ab + cd = 0, \quad ac + bd = 0,$$

implying

$$a^2 = d^2, \quad b^2 = c^2, \quad a^2 + b^2 = 1, \quad ab + cd = 0.$$

This gives

$$a = \cos\theta, \quad b = -\sin\theta, \quad c = \pm\sin\theta, \quad d = \pm\cos\theta.$$

The matrix $\boldsymbol{A}_1$ rotates any vector $\boldsymbol{x} := (x, y)'$ by an angle θ in the positive (counterclockwise) direction. For example, when $\theta = \pi/2$,

$$\boldsymbol{A}_1\boldsymbol{x} = \begin{pmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

The matrix $\boldsymbol{A}_2$ satisfies

$$\boldsymbol{A}_2\boldsymbol{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{A}_1\boldsymbol{x},$$

so that $\boldsymbol{x}$ is rotated counterclockwise by an angle θ, and then reflected across the x-axis.
(b) Let $\boldsymbol{a}'_{1\bullet}, \dots, \boldsymbol{a}'_{n\bullet}$ denote the rows of $\boldsymbol{A}$. From $\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{I}_n$ it follows that $\boldsymbol{a}'_{i\bullet}\boldsymbol{a}_{i\bullet} = 1$ and $\boldsymbol{a}'_{i\bullet}\boldsymbol{a}_{j\bullet} = 0$ $(i \neq j)$. Hence, the rows form an orthonormal set.
(c) Let $\boldsymbol{a}_{\bullet 1}, \dots, \boldsymbol{a}_{\bullet n}$ denote the columns of $\boldsymbol{A}$. Then, from $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{I}_n$ it follows that $\boldsymbol{a}'_{\bullet i}\boldsymbol{a}_{\bullet i} = 1$ and $\boldsymbol{a}'_{\bullet i}\boldsymbol{a}_{\bullet j} = 0$ $(i \neq j)$. Hence, the columns also form an orthonormal set.

Exercise 2.29 (Permutation matrix) A square matrix $\boldsymbol{A}$ is called a permutation matrix if each row and each column of $\boldsymbol{A}$ contains a single element 1, and the remaining elements are zero.

(a) Show that there exist 2 permutation matrices of order 2.
(b) Show that there exist 6 permutation matrices of order 3, and determine which of these transforms the matrix $\boldsymbol{A}$ of Exercise 2.19(a) into $\operatorname{diag}(a, b, c)$.
(c) Show that there exist $n!$ permutation matrices of order n.
(d) Show that every permutation matrix is orthogonal.

Solution
(a) The permutation matrices of order 2 are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The latter matrix permutes (or swaps) the axes by premultiplication, since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}.$$

(b) The permutation matrices of order 3 are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

To write the matrix $\boldsymbol{A}$ of Exercise 2.19 as $\operatorname{diag}(a, b, c)$, we need to swap the first and third columns. This is achieved by postmultiplying $\boldsymbol{A}$ by the last of the six displayed matrices; premultiplying would have swapped the rows instead.
(c) We proceed by induction. Suppose there are $(n-1)!$ permutation matrices of order $n-1$. For each $(n-1) \times (n-1)$ permutation matrix there are precisely n ways to form an $n \times n$ permutation matrix. Hence, there exist $n!$ permutation matrices of order n.
(d) Each row $\boldsymbol{p}'_{i\bullet}$ of the permutation matrix $\boldsymbol{P}$ contains one 1 and $(n-1)$ zeros. Hence, $\boldsymbol{p}'_{i\bullet}\boldsymbol{p}_{i\bullet} = 1$. Another row, say $\boldsymbol{p}'_{j\bullet}$, also contains only one 1, but in a different place. Hence, $\boldsymbol{p}'_{i\bullet}\boldsymbol{p}_{j\bullet} = 0$ $(i \neq j)$. Thus $\boldsymbol{P}$ is orthogonal.

Exercise 2.30 (Normal matrix) A real square matrix $\boldsymbol{A}$ is normal if $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}'$.
(a) Show that every symmetric matrix is normal.
(b) Show that every orthogonal matrix is normal.
(c) Let $\boldsymbol{A}$ be a normal 2×2 matrix. Show that $\boldsymbol{A}$ is either symmetric or has the form

$$\boldsymbol{A} = \lambda \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \quad (\lambda \neq 0).$$

Solution
(a) If $\boldsymbol{A} = \boldsymbol{A}'$ then $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}'$.
(b) If $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}' = \boldsymbol{I}$, then clearly $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}'$.
(c) Let

$$\boldsymbol{A} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The definition $A'A = AA'$ implies that

$$\begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$$

and hence that $b^2 = c^2$ and $(a - d)(b - c) = 0$. This gives either $b = c$ (symmetry) or $b = -c \neq 0$ and $a = d$.

Exercise 2.31 (Commuting matrices) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Show that the class of matrices B satisfying $AB = BA$ is given by

$$B = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 2 \\ 3 & 3 \end{pmatrix}.$$

Solution
Let

$$B := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the equation $AB = BA$ gives

$$\begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix} = \begin{pmatrix} a + 3b & 2a + 4b \\ c + 3d & 2c + 4d \end{pmatrix},$$

which leads to

$$3b - 2c = 0, \quad 2a + 3b - 2d = 0, \quad a + c - d = 0.$$

Hence,

$$c = (3/2)b \quad \text{and} \quad d = a + (3/2)b,$$

and the result follows.

Exercise 2.32 (Powers, quadratic's solution) Consider a real square matrix A of order 2.
(a) Show that $A^2 = O$ has a unique *symmetric* solution, namely $A = O$.
(b) Show that, in general, $A^2 = O$ has an infinite number of solutions, given by $A = pq'$ with $p'q = 0$.

Solution
(a) Again, let

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The equation $\boldsymbol{A}^2 = \mathbf{O}$ can then be written as

$$\begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

with general solution $a = -d, a^2 + bc = 0$. If A is symmetric, then $b = c$, and hence all elements are zero. (This also follows from Exercise 2.13.)
(b) If $\boldsymbol{A}$ is not symmetric, then the solution is given by $a = -d, a^2 + bc = 0, b \neq c$. If $a = 0$, the solutions are

$$\boldsymbol{A} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & b \end{pmatrix} \quad \text{and} \quad \boldsymbol{A} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} c & 0 \end{pmatrix}.$$

If $a \neq 0$, then all elements of $\boldsymbol{A}$ are nonzero and

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ -a^2/b & -a \end{pmatrix} = \begin{pmatrix} 1 \\ -a/b \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix}.$$

All three cases are of the form $\boldsymbol{A} = \boldsymbol{p}\boldsymbol{q}'$ with $\boldsymbol{p}'\boldsymbol{q} = 0$. Conversely, if $\boldsymbol{A} = \boldsymbol{p}\boldsymbol{q}'$ then $\boldsymbol{A}^2 = \boldsymbol{p}\boldsymbol{q}'\boldsymbol{p}\boldsymbol{q}' = \boldsymbol{p}(\boldsymbol{q}'\boldsymbol{p})\boldsymbol{q}' = \mathbf{O}$, whenever $\boldsymbol{p}'\boldsymbol{q} = 0$.

Exercise 2.33 (Powers of a symmetric matrix) Show that A^p is symmetric when A is symmetric.

Solution
We have

$$(\boldsymbol{A}^p)' = (\boldsymbol{A}\boldsymbol{A}\ldots\boldsymbol{A})' = \boldsymbol{A}'\boldsymbol{A}'\ldots\boldsymbol{A}' = \boldsymbol{A}\boldsymbol{A}\ldots\boldsymbol{A} = \boldsymbol{A}^p.$$

Exercise 2.34 (Powers of the triangle) Consider an $n \times n$ triangular matrix $\boldsymbol{A}$. Show that the powers of $\boldsymbol{A}$ are also triangular and that the diagonal elements of $\boldsymbol{A}^p$ are given by a_{ii}^p for $i = 1, \ldots, n$.

Solution
Assume that $\boldsymbol{A}$ is lower triangular. It suffices to prove the result for $p = 2$. Exercise 2.21(b) shows that the product of two lower triangular matrices is again lower triangular. Let $\boldsymbol{B} := \boldsymbol{A}^2$. Then its i-th diagonal element is given by $b_{ii} = \sum_k a_{ik}a_{ki} = a_{ii}^2$, since either $a_{ki} = 0$ or $a_{ik} = 0$ when $k \neq i$.

Exercise 2.35 (Fibonacci sequence) Consider the 2×2 matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that

$$\boldsymbol{A}^n = \begin{pmatrix} x_n & x_{n-1} \\ x_{n-1} & x_{n-2} \end{pmatrix}$$

with $x_{-1} := 0$, $x_0 := 1$, and $x_n := x_{n-1} + x_{n-2}$ $(n \geq 1)$. (This is the *Fibonacci sequence*: $1, 2, 3, 5, 8, 13, \dots$.)

Solution

Since $\boldsymbol{A}$ is symmetric, we know from Exercise 2.33 that $\boldsymbol{A}^n$ is symmetric. Let

$$\boldsymbol{A}^n := \begin{pmatrix} x_n & b_n \\ b_n & c_n \end{pmatrix}.$$

Then,

$$\boldsymbol{A}^{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n & b_n \\ b_n & c_n \end{pmatrix} = \begin{pmatrix} x_n + b_n & b_n + c_n \\ x_n & b_n \end{pmatrix} = \begin{pmatrix} x_{n+1} & b_{n+1} \\ b_{n+1} & c_{n+1} \end{pmatrix}.$$

Hence, $b_{n+1} = x_n$, $c_{n+1} = b_n = x_{n-1}$, and $x_{n+1} = x_n + b_n = x_n + x_{n-1}$. The condition $b_{n+1} = b_n + c_n$ is then automatically fulfilled. Thus,

$$\boldsymbol{A}^{n+1} = \begin{pmatrix} x_n + x_{n-1} & x_n \\ x_n & x_{n-1} \end{pmatrix}.$$

Exercise 2.36 (Difference equations) Consider the 2×2 matrices

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

(a) Show that $\boldsymbol{B} = \boldsymbol{A}^2$ and $\boldsymbol{B}^2 = -\boldsymbol{A}$.
(b) Compute $\boldsymbol{A}^2, \boldsymbol{A}^3, \dots, \boldsymbol{A}^6$.
(c) Conclude that $\boldsymbol{A}^6 = \boldsymbol{I}$ and $\boldsymbol{B}^3 = \boldsymbol{I}$.
(d) What is the relationship between the matrix $\boldsymbol{A}$ and the second-order difference equation $x_n = x_{n-1} - x_{n-2}$?

Solution

(a)–(c) We find

$$\boldsymbol{A}^2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \boldsymbol{B},$$

and further $\boldsymbol{A}^3 = -\boldsymbol{I}$, $\boldsymbol{A}^4 = -\boldsymbol{A}$, $\boldsymbol{A}^5 = -\boldsymbol{B}$, and $\boldsymbol{A}^6 = \boldsymbol{I}$. Hence, $\boldsymbol{B}^2 = \boldsymbol{A}^4 = -\boldsymbol{A}$ and $\boldsymbol{B}^3 = \boldsymbol{A}^6 = \boldsymbol{I}$.
(d) Let $\boldsymbol{z}_n := (x_n, x_{n-1})'$ for $n = 0, 1, \dots$. Then,

$$\boldsymbol{z}_n = \boldsymbol{A}\boldsymbol{z}_{n-1} \iff \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} = \begin{pmatrix} x_{n-1} - x_{n-2} \\ x_{n-1} \end{pmatrix},$$

so that the first-order vector equation $\boldsymbol{z}_n = \boldsymbol{A}\boldsymbol{z}_{n-1}$ is equivalent to the second-order difference equation $x_n = x_{n-1} - x_{n-2}$. Hence, the solution $\boldsymbol{z}_n = \boldsymbol{A}^n \boldsymbol{z}_0$ of the vector equation also solves the difference equation.

Exercise 2.37 (Idempotent) A square matrix $\boldsymbol{A}$ is idempotent if $\boldsymbol{A}^2 = \boldsymbol{A}$.
(a) Show that the only idempotent *symmetric* 2×2 matrices are

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{A} = \boldsymbol{a}\boldsymbol{a}' \quad (\boldsymbol{a}'\boldsymbol{a} = 1).$$

(b) Recall that $\boldsymbol{\imath} := (1, 1, \dots, 1)'$, denotes the $n \times 1$ vector of ones. Show that the matrix $\boldsymbol{I}_n - (1/n)\boldsymbol{\imath}\boldsymbol{\imath}'$ is idempotent and symmetric. What is the intuition behind this fact?
(c) Give an example of an $n \times n$ idempotent matrix that is *not* symmetric.

Solution
(a) The 2×2 matrix $\boldsymbol{A}$ is symmetric idempotent if and only if

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} a & b \\ b & d \end{pmatrix},$$

that is, if and only if,

$$a^2 + b^2 = a, \quad b(a + d) = b, \quad d^2 + b^2 = d.$$

We distinguish between $a + d \neq 1$ and $a + d = 1$. If $a + d \neq 1$, then $b = 0$ and $a = d = 1$ or $a = d = 0$. If $a + d = 1$, then $b^2 = a(1 - a)$, so that $0 \leq a \leq 1$, $0 \leq d \leq 1$, and $b = \pm\sqrt{a(1 - a)}$. Then,

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} a & \pm\sqrt{a(1-a)} \\ \pm\sqrt{a(1-a)} & 1 - a \end{pmatrix} = \begin{pmatrix} \sqrt{a} \\ \pm\sqrt{1-a} \end{pmatrix} \begin{pmatrix} \sqrt{a} \\ \pm\sqrt{1-a} \end{pmatrix}',$$

which is of the form $\boldsymbol{a}\boldsymbol{a}'$ $(\boldsymbol{a}'\boldsymbol{a} = 1)$. Conversely, $(\boldsymbol{a}\boldsymbol{a}')(\boldsymbol{a}\boldsymbol{a}') = \boldsymbol{a}(\boldsymbol{a}'\boldsymbol{a})\boldsymbol{a}' = \boldsymbol{a}\boldsymbol{a}'$ if $\boldsymbol{a}'\boldsymbol{a} = 1$.
(b) Let $\boldsymbol{M} := \boldsymbol{I}_n - (1/n)\boldsymbol{\imath}\boldsymbol{\imath}'$. Then,

$$\begin{aligned} \boldsymbol{M}^2 &= (\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{\imath}\boldsymbol{\imath}')(\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{\imath}\boldsymbol{\imath}') = \boldsymbol{I}_n - \frac{1}{n}\boldsymbol{\imath}\boldsymbol{\imath}' - \frac{1}{n}\boldsymbol{\imath}\boldsymbol{\imath}' + \frac{1}{n^2}\boldsymbol{\imath}\boldsymbol{\imath}'\boldsymbol{\imath}\boldsymbol{\imath}' \\ &= \boldsymbol{I}_n - \frac{2}{n}\boldsymbol{\imath}\boldsymbol{\imath}' + \frac{1}{n^2}\boldsymbol{\imath}(\boldsymbol{\imath}'\boldsymbol{\imath})\boldsymbol{\imath}' = \boldsymbol{I}_n - \frac{2}{n}\boldsymbol{\imath}\boldsymbol{\imath}' + \frac{1}{n}\boldsymbol{\imath}\boldsymbol{\imath}' = \boldsymbol{M}. \end{aligned}$$

To understand the intuition, consider the vector equation $\boldsymbol{y} = \boldsymbol{M}\boldsymbol{x}$. We have

$$\boldsymbol{y} = \boldsymbol{M}\boldsymbol{x} = (\boldsymbol{I}_n - \frac{1}{n}\boldsymbol{\imath}\boldsymbol{\imath}')\boldsymbol{x} = \boldsymbol{x} - \frac{1}{n}\boldsymbol{\imath}(\boldsymbol{\imath}'\boldsymbol{x}) = \boldsymbol{x} - \overline{x}\boldsymbol{\imath},$$

where $\overline{x} := (1/n)\boldsymbol{\imath}'\boldsymbol{x}$ (the average). Hence, $y_i = x_i - \overline{x}$, and the transformation $\boldsymbol{M}$ thus puts $\boldsymbol{x}$ in deviations from its mean. Now consider $\boldsymbol{z} = \boldsymbol{M}\boldsymbol{y}$ and note that $\overline{y} = 0$. Hence, $\boldsymbol{z} = \boldsymbol{y}$, that is, $\boldsymbol{M}^2\boldsymbol{x} = \boldsymbol{M}\boldsymbol{x}$ for every $\boldsymbol{x}$. This gives $\boldsymbol{M}^2 = \boldsymbol{M}$. Associated with an idempotent matrix is an idempotent *operation* (in this case: "put the elements of a vector in deviation form"). Once the operation has been performed, repeating it has no further effect.
(c) In econometrics most idempotent matrices will be symmetric. But the matrix $\boldsymbol{A} = \boldsymbol{a}\boldsymbol{b}'$ with $\boldsymbol{b}'\boldsymbol{a} = 1$ is idempotent but not symmetric (unless $\boldsymbol{a} = \boldsymbol{b}$ or one of the vectors is the null vector).

Exercise 2.38 (Inner product, matrix) For two real matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same order, the inner product is defined as $\langle \boldsymbol{A}, \boldsymbol{B}\rangle := \sum_i \sum_j a_{ij} b_{ij} = \operatorname{tr} \boldsymbol{A}'\boldsymbol{B}$. Prove that:
(a) $\langle \boldsymbol{A}, \boldsymbol{B}\rangle = \langle \boldsymbol{B}, \boldsymbol{A}\rangle$;
(b) $\langle \boldsymbol{A}, \boldsymbol{B}+\boldsymbol{C}\rangle = \langle \boldsymbol{A}, \boldsymbol{B}\rangle + \langle \boldsymbol{A}, \boldsymbol{C}\rangle$;
(c) $\langle \lambda\boldsymbol{A}, \boldsymbol{B}\rangle = \lambda\langle \boldsymbol{A}, \boldsymbol{B}\rangle$;
(d) $\langle \boldsymbol{A}, \boldsymbol{A}\rangle \geq 0$, with $\langle \boldsymbol{A}, \boldsymbol{A}\rangle = 0 \iff \boldsymbol{A} = \mathbf{O}$.

Solution
We need to show that $\operatorname{tr} \boldsymbol{A}'\boldsymbol{B} = \operatorname{tr} \boldsymbol{B}'\boldsymbol{A}$, $\operatorname{tr} \boldsymbol{A}'(\boldsymbol{B}+\boldsymbol{C}) = \operatorname{tr} \boldsymbol{A}'\boldsymbol{B} + \operatorname{tr} \boldsymbol{A}'\boldsymbol{C}$, $\operatorname{tr}(\lambda\boldsymbol{A})'\boldsymbol{B} = \lambda \operatorname{tr} \boldsymbol{A}'\boldsymbol{B}$, $\operatorname{tr} \boldsymbol{A}'\boldsymbol{A} \geq 0$, and $\operatorname{tr} \boldsymbol{A}'\boldsymbol{A} = 0 \iff \boldsymbol{A} = \mathbf{O}$. All these properties have been proved before.

***Exercise 2.39 (Norm, matrix)** For a real matrix $\boldsymbol{A}$, we define the norm as

$$\|\boldsymbol{A}\| := \langle \boldsymbol{A}, \boldsymbol{A}\rangle^{1/2} = \sqrt{\sum_i \sum_j a_{ij}^2} = \sqrt{\operatorname{tr} \boldsymbol{A}'\boldsymbol{A}}.$$

Show that:
(a) $\|\lambda\boldsymbol{A}\| = |\lambda| \cdot \|\boldsymbol{A}\|$;
(b) $\|\boldsymbol{A}\| \geq 0$, with $\|\boldsymbol{A}\| = 0$ if and only if $\boldsymbol{A} = \mathbf{O}$;
(c) $\|\boldsymbol{A}+\boldsymbol{B}\| \leq \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$ *(triangle inequality)*.

Solution
(a) We have

$$\|\lambda\boldsymbol{A}\| = \sqrt{\operatorname{tr}(\lambda\boldsymbol{A})'(\lambda\boldsymbol{A})} = \sqrt{\lambda^2 \operatorname{tr} \boldsymbol{A}'\boldsymbol{A}} = |\lambda|\sqrt{\operatorname{tr} \boldsymbol{A}'\boldsymbol{A}} = |\lambda| \cdot \|\boldsymbol{A}\|.$$

(b) Further, $\|\boldsymbol{A}\| = \sqrt{\operatorname{tr} \boldsymbol{A}'\boldsymbol{A}} \geq 0$, with $\|\boldsymbol{A}\| = 0$ if and only if $\boldsymbol{A} = \mathbf{O}$, according to Exercise 2.25.
(c) Finally, let $\boldsymbol{A} := (a_{ij})$ and $\boldsymbol{B} := (b_{ij})$ be $m \times n$ matrices, and define $mn \times 1$ vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a}$ contains the elements of $\boldsymbol{A}$ in a specific order and $\boldsymbol{b}$ contains the elements of $\boldsymbol{B}$ *in the same order*. For example,

$$\boldsymbol{a} := (a_{11}, a_{21}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{mn})',$$

which we shall later write as vec $\boldsymbol{A}$; see Chapter 10. Then,

$$\operatorname{tr} \boldsymbol{A}'\boldsymbol{B} = \sum_{ij} a_{ij} b_{ij} = \boldsymbol{a}'\boldsymbol{b}$$

and similarly, $\operatorname{tr} \boldsymbol{A}'\boldsymbol{A} = \boldsymbol{a}'\boldsymbol{a}$ and $\operatorname{tr} \boldsymbol{B}'\boldsymbol{B} = \boldsymbol{b}'\boldsymbol{b}$. Hence,

$$\begin{aligned}\|\boldsymbol{A}+\boldsymbol{B}\| &= \sqrt{\operatorname{tr}(\boldsymbol{A}+\boldsymbol{B})'(\boldsymbol{A}+\boldsymbol{B})} = \sqrt{(\boldsymbol{a}+\boldsymbol{b})'(\boldsymbol{a}+\boldsymbol{b})} = \|\boldsymbol{a}+\boldsymbol{b}\| \\ &\leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\| = \sqrt{\boldsymbol{a}'\boldsymbol{a}} + \sqrt{\boldsymbol{b}'\boldsymbol{b}} = \sqrt{\operatorname{tr} \boldsymbol{A}'\boldsymbol{A}} + \sqrt{\operatorname{tr} \boldsymbol{B}'\boldsymbol{B}} = \|\boldsymbol{A}\| + \|\boldsymbol{B}\|,\end{aligned}$$

using the triangle equality for vectors (Exercise 1.10(c)).