Chapter 36

EQUILIBRIUM ANALYSIS WITH NON-CONVEX TECHNOLOGIES

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Contents

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1. Introduction

Let us begin by briefly surveying the "state of the art" regarding the Arrow–Debreu model of a Walrasian economy consisting of a finite number of agents and commodities, where we assume perfect information, complete markets, no market imperfections such as externalities, public goods, or non-convexities in consumption or production, firms are price-taking profit maximizers and households are price-taking utility maximizers. In such a world, the basic properties of the classical Arrow–Debreu model consist of the existence of competitive equilibria, the first and second welfare theorems, the computation of equilibria and the local uniqueness and finiteness of equilibria.

For the purpose of this chapter it is useful to adopt Walras' original conception of a competitive equilibrium as a solution to a (non-linear) system of equations. There are two general methods for solving non-linear systems of equations. The first method consists of converting the problem into an equivalent fixed point problem and then invoking the appropriate fixed-point theorem such as Brouwer's fixed-point theorem or its generalization, the Kakutani fixed-point theorem. This was the approach used by Gale (1955), Nikaido (1956), McKenzie (1954) and Arrow and Debreu (1954) in the 1950s to establish the existence of a competitive equilibrium. The convexity assumptions in their models were crucial for the fixed-point arguments used in their proofs; in particular, the assumption that firms' production sets are convex. Convexity also appears to be crucial in the establishment of the second welfare theorem where the principal tool of analysis is the separating hyperplane theorem; see Arrow (1951) and Debreu (1951). The culmination of the research on the existence and optimality of competitive equilibria during this period is Debreu's *Theory of Value*, published in 1959.

The existence proof in *Theory of Value* is non-constructive. The first constructive proof of the existence of a competitive equilibrium was given by Scarf (1967). Shortly thereafter, Scarf (1973) published his influential monograph, *Computation of Economic Equilibria*. Scarf's constructive proof first consisted of giving an algorithm for computing an approximate fixed-point of a continuous map of the simplex into itself. Following Debreu, he defined a continuous map from the price simplex into itself, derived from the excess market demand function for the given economy; the fixed points of this map are the equilibrium prices. They are then computed using the algorithm.

Convexity plays an essential role in Scarf's analysis, both in the derivation of the market excess demand function from optimizing behavior on the part of agents and in the existence of a fixed-point which follows from Brouwer's theorem. Scarf's algorithm and its generalizations are the primary means of doing comparative statics in general equilibrium models. Computable general equilibrium models have replaced activity analysis and input–output analysis as the basic method of analysing tax policy in national economies or trade policies between nations [see Scarf and Shoven (1984) or Shoven and Whalley (1984)].

In a seminal paper, Debreu (1970) introduced the techniques of differential topology into equilibrium analysis to resolve the question of the uniqueness of equilibrium prices in an exchange economy. Using Sard's theorem and the inverse function theorem, he was able to show that almost all exchange economies – parameterized by individual endowments – have a finite number of locally unique equilibrium prices. In addition to the standard convexity assumptions on tastes, we now require that agents' characteristics are smooth [see Debreu (1972)]. Debreu's paper on the finiteness and local uniqueness of equilibrium prices inspired a number of other applications of differential topology to problems of equilibrium analysis.

First, Dierker (1972) gave a degree-theoretic existence proof, using index theory. Then in a series of papers published in the *Journal of Mathematical Economics* over a two year period from 1974 to 1976, Smale used the methods of global analysis to establish all the properties of the Arrow–Debreu model of a Walrasian economy, i.e. existence, optimality, computation, local uniqueness and finiteness of the set of equilibrium prices. He assumed that both households and firms have smooth characteristics, i.e. smooth utility and production functions. Smale's research on existence and optimality differs in a fundamental way from previous work in that he eschews fixed point methods and resurrects the discredited method of counting equations and unknowns, originally used by Walras in his "proof" of existence. This technique properly formulated within the theory of global analysis (differential topology) is a powerful tool for analysing the existence and uniqueness of solutions of non-linear systems of equations. In particular, we have in mind degree theory as our second method for proving the existence of a competitive equilibrium and for "counting" the number of equilibria.

For expositional reasons, we wish to distinguish between fixed-point arguments requiring convexity and degree-theoretic proofs which do not assume convexity.

Smale defines extended price equilibria as a solution to a system of equations consisting of the first-order conditions for profit maximization, the first-order conditions for utility maximization subject to a budget constraint and the market clearing conditions. Moreover, he assumes neither convexity of the utility or production functions and shows that for an open, dense family of economies – parameterized by endowments, utility functions and production functions – the set of extended price equilibria is locally unique [see Sections 5 and 6 in Smale (1974a)] and is finite if the set of attainable allocations is compact. Smale's results appear to have gone unnoticed by those working on the properties of general equilibrium models with non-convex production sets. Although he does not prove the existence of extended price equilibria in this paper, in an earlier paper on pure exchange – where again he does not assume convexity of utility functions – he shows that extended price equilibria exist, and presumably this proof can be extended to the case with production. It is interesting to notice that his proof for the pure exchange case rests on a degree-theoretic argument [see the appendix to Smale (1974b)]. Also by restricting utility functions to be convex, but only requiring production sets to be submanifolds, Smale gives a first-order characterization of Pareto optimal allocations in terms of marginal cost pricing [see Proposition 2, Section 4, in Smale (1976a)].

An important application of degree theory to equilibrium analysis is the use of the homotopy invariance theorem as a means of proving constructive existence theorems. Homotopy methods for solving fixed-point problems were introduced by Eaves (1972) and are the basis for the second generation of Scarf algorithms. The homotopy methods for solving non-linear systems of equations do not transform the given non-linear system of equations into an equivalent fixed-point problem, instead they construct a simple system of equations which are trivial to solve and continuously deform this simple system into the given system. By following the solution of this parameterized family of equation systems, one is led to the solution of the original system of equations. The homotopy invariance theorem states that (under certain conditions) if the simple system has an odd number of solutions then the original system, which is homotopically equivalent, also has an odd number of solutions. For a general discussion of homotopy or path following methods for solving non-linear systems of equations or equilibrium systems of equations for economic models, see Garcia and Zangwill (1981).

Having completed our survey of the basic properties of the Arrow–Debreu model of a Walrasian economy, we now turn to the central topic of this chapter: the normative implications of non-convex production sets on the firm's pricing policy. Non-convexities in production can arise from indivisibilities, fixed costs or increasing returns to scale.

The non-convex firms in the model that we shall consider can be thought of as privately owned public utilities, which are regulated. This type of market structure is common in the United States but less prevalent in Europe, for example, Electricité de France is a state-owned public utility. Since we discuss public utilities which are privately owned, any pricing rule imposed by the regulator must produce a fair rate of return on capital or normal economic profits. Consequently these firms cannot run at a loss; see Brown and Sibley (1986) for a partial equilibrium analysis of public utility pricing. The origins of the pricing rules that we shall study for regulated public utilities may be found in the marginal cost pricing controversy of the 1930s.

This controversy begins with Hotelling's classic article on optimal railroad and utility rates [see Hotelling (1938)]. He argues that marginal cost pricing together with income taxes is Pareto superior to average cost pricing. This controversy and its welfare antecedents are ably surveyed and analysed by Ruggles (1949) and (1950) in two companion pieces. As Ruggles points out in her second paper, the basis of the marginal cost pricing principle is that marginal cost pricing meets the marginal conditions for Pareto optimality; see the summary and evaluation section of her article. Ruggles then asks the following two questions: (1) Is meeting the marginal conditions a sufficient basis for recommending a pricing system? and (2) Does the marginal cost pricing system meet these conditions?

She observes that the first-order conditions are only necessary conditions for a maximum of welfare and that the marginal cost pricing principle does not meet these conditions if the means for raising the subsidies necessary to cover the losses incurred by firms, with decreasing average cost technologies, are taken into account. This criticism applies in particular to Hotelling who advocated income taxes as a superior alternative to excise taxes in raising the necessary subsidies, but, as Ruggles points out, Wald had demonstrated that income taxes are an excise tax on leisure, hence they are not lump sum taxes, as Hotelling suggested. Moreover, Wald (1945) shows in some instances, depending on the relevant elasticities, income taxes are inferior to other excise taxes for raising the necessary revenue.

Finally, Ruggles considers other pricing principles such as average cost pricing, multi-part pricing or two-part tariffs and price discrimination, all of which have been suggested as alternative pricing systems because of the need to raise revenue to cover the losses incurred by marginal cost pricing in decreasing average cost firms. In the remainder of this chapter we will discuss the existence, optimality, computation, finiteness and local uniqueness of equilibria for three representative pricing rules, for privately owned regulated public monopolies, in a general equilibrium model. We have chosen as our representative pricing rules, marginal cost pricing, average cost pricing and two-part tariffs. Not only are these pricing policies the subject of much normative discussion in the public utilities literature, but they are also the most commonly observed instances of regulatory pricing policy. Before discussing explicit models, let us survey what we know about these various pricing rules.

The major methodological innovation in the general equilibrium analysis of firms with pricing rules has been the introduction of the methods of non-smooth analysis, as an alternative to both global analysis (differential topology) and to convex analysis, for investigating the existence and optimality of equilibria. These methods were introduced by Guesnerie (1975) in a seminal paper on Pareto optimality in general equilibrium models with non-convex production sets.

Non-smooth analysis extends the local approximation of manifolds by tangent planes, and the analogous local approximation of convex sets by tangent cones to sets which are neither smooth nor convex. Here the local approximation is also by cones; in the case of Guesnerie, it is the cone of interior displacements [see Dubovickii and Miljutin (1965)]. Subsequently Cornet (1982) introduced Clarke's tangent cone as the appropriate local approximation for economic analysis. To appreciate Cornet's contribution, we must consider the polar cones of these generalized tangent cones, called normal cones. The normal cone to a point on the boundary of a production set represents the marginal rates of transformation at that point. In non-smooth analysis, the normal cone is a formal extension of the notion of normal vector to a hypersurface and the notion of cone of normals to a convex set. The Clarke normal cone has a number of desirable properties. First, its polar cone, the Clarke tangent cone, is always convex; second, if the production set is non-empty, closed and has free disposal, then the Clarke normal cone is always non-empty and closed [see Cornet (1982)]. Finally, Clarke has shown that if a price vector maximizes profits at an efficient production plan then this price vector must lie in the (Clarke) normal cone at that point. Hence in the recent economic literature the Clarke normal cone is used to describe the necessary marginal conditions for profit maximization. For a detailed discussion of non-smooth analysis, see Clarke (1983) or Rockafellar (1981).

In contrast, the cone of interior displacements need not be convex at "kinks" (Figure 36.1), hence Guesnerie was forced to exclude this family of technologies from his analysis. Moreover, when the normal cone of Dubovickii and Miljutin is convex, closed and upper hemi-continuous, as assumed in Beato

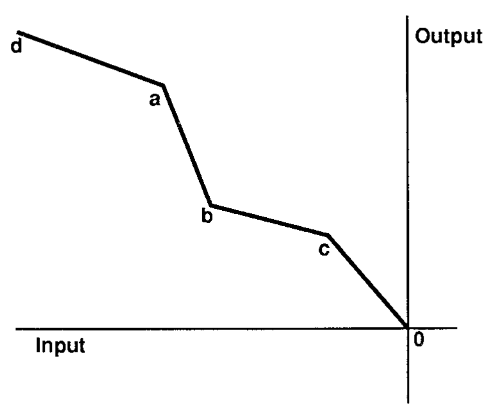

Figure 36.1. The cones of interior displacements at the production plans (a) and (c) are convex, e.g. at (*a*) it is the cone generated by the vectors *ad* and *ab* shifted to the origin. But at the "kink", production plan (*b*), it is the whole production set, which is not convex.

(1982), then it coincides with the Clarke normal cone [see Cornet (1987)]. It is the convexity of the tangent cones, which replaces the convexity of the production sets, that is crucial for equilibrium analysis. See Khan and Vohra (1987b) for additional discussion of the cone of interior displacements and the Clarke tangent cone in economic models.

Using the normal cone of Dubovickii and Miljutin and focusing only on cases where it is non-empty and convex, Guesnerie was the first to extend Smale's necessary conditions for Pareto optimal allocations from economies with smooth non-convex production sets to those with non-smooth production sets. His welfare analysis has recently been extended to models with non-convex technologies and pure public goods in a paper by Khan and Vohra (1987b); see also the extension of Guesnerie's model to infinite dimensional commodity spaces by Bonnisseau and Cornet (1988b). Both papers use the Clarke normal cone. Guesnerie also presented the first examples of general equilibrium models with non-convex production sets where all of the marginal cost pricing equilibria fail to be Pareto optimal.

Subsequently, other examples were given by Brown and Heal (1979). The intuition underlying all of these examples is clear. As illustrated by Brown and Heal, if the aggregate production set is non-convex then the community indifference curve defined by a marginal cost pricing equilibrium may "cut inside the production possibility set" (see Figure 36.2). Simply put, satisfying first-order conditions, in general, will not suffice for global optimality in the presence of non-convexities – a point already made by Ruggles in 1950. Continuing our discussion of the optimality of marginal cost pricing, Beato and Mas-Colell (1983) in an influential paper presented the first example of

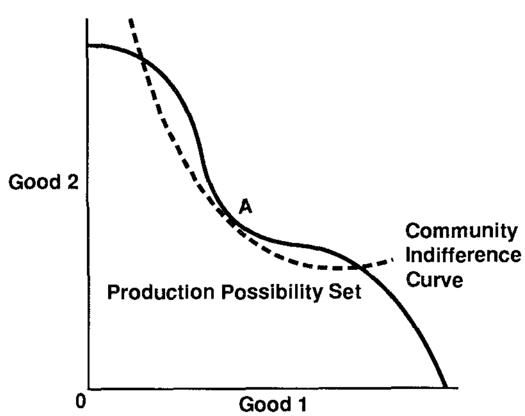

Figure 36.2. The community indifference curve which is tangent to the production possibility frontier at the production plan A "cuts inside the production possibility set."

marginal cost pricing equilibria which were inefficient, i.e. inside the social production possibility set. Dierker (1986) and Quinzii (1991) have both given sufficient conditions for a marginal cost pricing equilibrium to be Pareto optimal. Their conditions are in terms of the relative curvature of the social indifference curve, at equilibrium, with respect to the boundary of the aggregate production possibility set, i.e. the social indifference curve does not "cut inside" the aggregate production possibility set.

Returning to Guesnerie's paper, we can interpret his fixed structure of revenues condition, in a private ownership economy, as a means of imposing lump sum taxes to cover losses incurred by marginal cost pricing, without disturbing the marginal conditions for Pareto optimality. In the Arrow–Debreu model of a private ownership economy, the shareholdings of households are exogenously specified and carry limited liability. Of course, in the classical model, all firms are profit maximizers with convex technologies containing the zero vector. Hence profits are always non-negative and the assumption of limited liability is unnecessary. With non-convex production sets and firms regulated to satisfy the first-order conditions for profit maximization – the modern formulation of the marginal cost pricing principle – the assumption of unlimited liability is of some import. By dropping the assumption of limited liability, fixing the income distribution exogenously by giving each agent a fixed proportion of net social wealth (the fixed structure of revenues condition), and assuming positivity of net social wealth, we can cover all losses and maintain the first-order conditions for optimality.

Brown and Heal (1983), by assuming both a fixed structure of revenues and homothetic preferences, prove the existence of at least one Pareto optimal marginal cost pricing equilibrium. Their result is an immediate consequence of Eisenberg's aggregation theorem. The existence of at least one Pareto optimal marginal cost pricing equilibrium for a much larger class of economies follows from Jerison (1984), where he gives necessary and sufficient conditions for aggregation of preferences, if the income distribution is independent of prices, e.g. a fixed structure of revenues.

We now give a formal definition of a marginal cost pricing equilibrium. A marginal cost pricing equilibrium (MCP equilibrium) is a family of consumption plans, production plans, lump sum taxes and prices such that households are maximizing utility subject to their budget constraints and firms' production plans satisfy the first-order conditions for profit maximization, i.e. at the given production plans the market prices lie in the Clarke normal cones; lump sum taxes cover the losses of all firms with non-convex production sets; and all markets clear. It is important to point out that if all firms have convex technologies which include the zero vector, then the notion of a MCP equilibrium reduces to the notion of a Walrasian equilibrium in the classical Arrow–Debreu model. The first existence proof of a MCP equilibrium for a private ownership economy with a single firm with a non-smooth technology was given by Cornet (1982). His theorem was extended by Brown, Heal, Khan and Vohra (1986) to private ownership economies with a single non-convex firm and several convex firms. Independently, Beato and Mas-Colell (1985) proved existence for a private ownership economy with several non-convex firms and several convex firms. Both the theorems of Beato and Mas-Colell and Brown, Heal, Khan and Vohra are special cases of the Bonnisseau–Cornet existence theorem which is discussed in the next section. Khan and Vohra (1987a) have recently extended the notion of a MCP equilibrium, where the price system consists of marginal cost pricing and Lindahl prices to non-convex economies with public goods.

All of these theorems are proven by invoking the Brouwer or Kakutani fixed-point theorems and rely on assumptions which guarantee that the relevant portion of each firm's efficiency frontier is homeomorphic to the simplex. This fact was first exploited in proving the existence of marginal cost pricing equilibria in economies with a single firm by Mantel (1979) and independently by Beato (1982), where the firm's technology is a smooth hypersurface. For smooth technologies, the Clarke normal cone at a point reduces to scalar multiples of the normal vector. Later, Brown and Heal (1982) gave an index-theoretic proof of existence for Mantel's model, using the fixed-point index theorem introduced by Dierker (1972). Their theorem is a special case of Kamiya's general equilibrium existence theorem which he proves using a degree-theoretic argument, reminiscent of Smale's existence proof for extended price equilibria for smooth exchange economies. Kamiya's proof has a number of important implications, such as local uniqueness and computational algorithms, and his theorem is discussed in Section 3.

There is another family of existence theorems for economies where commodities are divided into the classical dichotomy of factors and products. The most important, for the purposes of this chapter, is MacKinnon's article (1979), in which he gives the first extension of Scarf's computational algorithm to general equilibrium models with non-convex production sets. Also see the non-constructive existence proof of Dierker, Guesnerie and Neuefeind (1985).

Hence we now have the beginnings of a significant literature on marginal cost pricing equilibria that extends the equilibrium analysis of the classical Arrow–Debreu model in terms of existence, optimality, computation, finiteness and local uniqueness to general equilibrium models with non-convex production sets.

As with marginal cost pricing, the modern literature on average cost pricing begins with a seminal article on optimality. We have in mind Boiteux's paper on second best Pareto optimality for public utilities. Boiteux (1956) derives the necessary conditions for Pareto optimality in a general equilibrium model, where firms with increasing returns are constrained to break even. Prices solving these first-order conditions are now called Boiteux–Ramsey prices, since Ramsey (1927) derived similar conditions for a single agent economy. The optimal excise taxes which result from Boiteux–Ramsey pricing have the intuitive property, for independent demands, that the taxes are inversely proportional to the elasticities of demand, e.g. inelastic demands are highly taxed – a result anticipated by Ruggles in her discussion of price discrimination as an alternative to marginal cost pricing. Existence of a Boiteux–Ramsey pricing equilibrium was first demonstrated by Dierker, Guesnerie and Neuefeind (1985) in a model with factor and product markets and non-convex technologies. Of course, a Boiteux–Ramsey pricing equilibrium only satisfies the first-order conditions necessary for second best Pareto optimality; analogous to marginal cost pricing equilibria satisfying the first-order conditions necessary for Pareto optimality. Dierker (1989) has extended his analysis of sufficient conditions for a marginal cost pricing equilibrium to be Pareto optimal to include sufficient conditions for a Boiteux–Ramsey pricing equilibrium to be second best Pareto optimal.

An average cost pricing equilibrium, ACP equilibrium, is formally defined as a family of consumption plans, production plans and prices such that households are maximizing utility subject to budget constraints, firms with convex technologies are maximizing profits, firms with non-convex technologies are breaking even, i.e. making zero profits; and all markets clear. The existence of average cost pricing equilibria follows from both Kamiya's theorem and the existence theorem of Bonnisseau–Cornet. The conventional wisdom is that average cost pricing equilibria, since they violate the first-order conditions necessary for Pareto optimality, are Pareto inferior to marginal cost pricing equilibria. This intuition is challenged in an important paper by Vohra (1988a), in which he gives examples of economies possessing second best average cost pricing equilibria that are Pareto superior to marginal cost pricing equilibria. Of course, this could only be true in an economy with non-convex production sets, where marginal cost pricing equilibria may not be Pareto optimal. Kamiya's constructive existence proof provides an algorithm for computing marginal cost pricing equilibria and average cost pricing equilibria for general non-convex technologies. Rutherford (1988) also has constructed computable general equilibrium models with increasing returns to scale, that compute MCP and ACP equilibria, for economies where the utility and production functions can be represented as members of a "nested" family of CES functions.

Both marginal cost pricing and average cost pricing are linear pricing rules, but in markets where resale is impossible, non-linear prices are a viable alternative to linear pricing systems. Non-linear pricing schemes abound, e.g. quantity discounts, bundling of commodities and multipart tariffs, see Phlips for a discussion (1983). In his important contribution to the marginal cost pricing controversy, Coase (1946) proposed discriminating two-part tariffs as a means of covering individual specific overhead costs.

There is an extensive partial equilibrium literature on two-part tariffs [see Oi (1971), Brown and Sibley (1986) and their references]. This literature suggests that if firms with decreasing average costs use a discriminating two-part tariff, where each potential customer is charged a "hook-up" fee for the right to consume the natural monopoly's output (this fee may differ from household to household) and a per unit charge equal to the marginal cost of production, then the resulting equilibrium is Pareto optimal. If we formally define a discriminating (or non-uniform) two-part marginal cost pricing equilibrium (TPMCP equilibrium) as a family of consumption plans, production plans, prices and hook-up fees such that households are maximizing utility subject to their non-convex budget sets, defined by prices and their hook-up charge; firms with convex technologies are marginal cost pricing; firms with non-convex technologies are marginal cost pricing and recovering any losses by charging non-uniform hook-up fees, thus making zero profits, then Vohra (1988b) has shown the partial equilibrium intuition concerning the optimality of discriminating two-part tariffs fails in general equilibrium models, in the sense that the first welfare theorem does not hold even for discriminating two-part marginal cost pricing equilibria. Quinzii (1991), in the first general equilibrium discussion of the optimality of two-part marginal cost pricing, demonstrates that the second welfare theorem fails for this equilibrium notion, if there is insufficient willingness to pay on the part of consumers. It is clear that the work of Dierker and Quinzii on sufficient conditions for optimality of a MCP equilibrium also provides sufficient conditions for a TPMCP equilibrium to be Pareto optimal.

Existence of a non-uniform two-part marginal cost pricing equilibrium has recently been established by Brown, Heller and Starr (1989), using the model of Beato and Mas-Colell (1985). The basic assumption in the Brown–Heller–Starr paper is that the aggregate willingness to pay, in equilibrium, exceeds the losses incurred by pricing the monopoly good at marginal cost. They also show that any Pareto optimal allocation, where the aggregate willingness to pay exceeds the losses incurred by pricing the monopoly good at marginal cost, can be supported as a two-part marginal cost pricing equilibrium. There are no algorithms for computing two-part marginal cost pricing equilibria, to our knowledge, nor are there any results on finiteness or local uniqueness of equilibria.

The existence of equilibria for linear pricing rules other than marginal or average cost pricing is discussed in a recent special issue on increasing returns edited by Cornet (1988). In particular, we recommend Cornet's introductory essay which is a survey of the general equilibrium literature on increasing returns. This completes the introduction and in the next two sections we shall discuss in detail the existence, computation, finiteness and local uniqueness of MCP, ACP and TPMCP equilibria. The optimality of marginal cost pricing is the subject of the final section of this chapter.

2. Existence

One of the first proofs of existence of a MCP equilibrium is due to Mantel (1979) [see also Beato (1982)]. Mantel considers a private ownership economy with l goods, m consumers and a single firm. The production set, Y, of the firm is a subset of $\mathscr{R}^l$. The firm's pricing rule ψ is a mapping from ∂Y, the boundary of Y, into the price simple of $\mathscr{R}^l$, denoted S. The consumption set X_i of the ith consumer is the positive orthant $\mathscr{R}^l_+$. Each consumer has a utility function U_i, an endowment vector ω_i and a share θ_i in the firm. Given a production plan y of the firm and market prices $p \in S$, then the income of household i is $r_i(p, y) = p \cdot \omega_i + \theta_i p \cdot y$.

Mantel makes the following assumptions.

(A1) For all i:

(i) U_i is continuous, strictly quasi-concave and locally non-satiated;

(ii) $\omega_i = \theta_i \omega$, where $\omega \in \mathscr{R}^l_{++}$ and $\theta_i > 0$, $\Sigma_{i=1}^m \theta_i = 1$.

(A2) (i) Y is closed, $0 \in Y$, $Y - \mathscr{R}^l_+ \subset Y$ (free disposal);

(ii) Y is a smooth hypersurface of $\mathscr{R}^l$, i.e. there exists a smooth function, f, from $\mathscr{R}^l$ into $\mathscr{R}$ such that $Y = \{x \in \mathscr{R}^l \mid f(x) \leq 0\}$, 0 is a regular value of f, and $\partial Y = f^{-1}(0)$;

(iii) for all $y \in \partial Y$, $\psi(y) = \nabla f(y)/\|\nabla f(y)\|_1$ where $\nabla f(y)$ is the gradient of f at y and $\|\cdot\|_1$ is the l_1-norm;

(iv) if $\hat{Y} = (Y + \omega) \cap \mathscr{R}^l_+$ and $y + \omega \in \partial \hat{Y}$ then $\psi(y) \in \mathscr{R}^l_{++}$;

(v) $\hat{Y}$ is bounded.

Before proving the existence of a MCP equilibrium, let us discuss these assumptions. A1(i) is standard and guarantees the existence of demand functions, given strictly positive prices and positive income. A1(ii) implies Guesnerie's fixed structure of revenues condition, i.e. $r_i(p, y) = p \cdot \omega_i + \theta_i p \cdot y = \theta_i p \cdot (y + \omega)$. A2(i) is also standard, but notice that we do not assume convexity of Y. A2(ii) and (iii) define the marginal cost pricing rule for a firm with a smooth technology. A2(iv) can be weakened to $\psi(y) \in \mathscr{R}^l_+$ [see Brown and Heal (1982)]. A2(v), in this model, is equivalent to assuming that the set of feasible allocations is compact.

The two central ideas in Mantel's proofs are the basis of most of the subsequent existence proofs of equilibria in economies with non-convex technologies where firms follow pricing rules. The first idea is that $\partial \hat{Y}$ is homeomorphic to the simplex, S. The second idea is to use this homeomorphism to construct a continuous map of ∂Y into ∂Y whose fixed-points are the desired equilibria. We now give Mantel's existence proof for a MCP equilibrium, where the lump sum taxation to cover the losses of the firm is implicit in the formulation of the budget constraint, i.e. $r_i(p, y) = \theta_i p \cdot (y + \omega)$ should be interpreted as "after-tax" income.

Theorem 1 [Mantel (1979)]. *Given assumption* A1 *and* A2 *there exists a MCP equilibrium, i.e. there exist consumption plans x_i, a production plan y and marginal cost prices p such that each consumer is maximizing his/her utility at x_i subject to his/her budget constraint*; $p = \nabla f(y)/\|\nabla f(y)\|_1$ *and* $\Sigma_{i=1}^m x_i = y + \omega$.

Proof. That $\partial \hat{Y}$ is homeomorphic to the simplex is obvious from Figure 36.3, where the homeomorphism is simply the intersection of a ray through the origin and the given $y \in \hat{Y}$ with the simplex S. Given y such that $y + \omega \in \hat{Y}$, let $p = \psi(y)$. Then for each i, $x_i(p, y)$, the demand of household i at prices p and given production plan y, is well defined since $p \in \mathcal{R}^l_{++}$ and $r_i(p, y) > 0$.

The aggregate demand, $x(p, y)$, is $\Sigma_{i=1}^m x_i(p, y)$ and $x(p, y) \in \mathcal{R}^l_+$. Denote by $\hat{x}(p, y)$ the projection of $x(p, y)$ onto $\partial \hat{Y}$ through the origin, i.e. $\hat{x}(p, y)$ is the intersection of the ray through the origin and $x(p, y)$ with $\partial \hat{Y}$. We now define the continuous map $\Gamma : \partial \hat{Y} \to \partial \hat{Y}$ which is a composition of these maps, i.e. $\Gamma(y + \omega) = \hat{x}(p, y)$. By Brouwer's fixed-point theorem, Γ has a fixed-point $\bar{y} + \omega$. By construction, we need only show that $x(\bar{p}, \bar{y}) = \bar{y} + \omega$, where $\bar{p} = \psi(\bar{y})$, to complete the proof. Since $\Gamma(\bar{y} + \omega) = \bar{y} + \omega$, we know that $\hat{x}(\bar{p}, \bar{y}) = \bar{y} + \omega$. Moreover, $\hat{x}(\bar{p}, \bar{y}) = \gamma x(\bar{p}, \bar{y})$ for some positive scalar γ. Hence $\gamma x(\bar{p}, \bar{y}) = \bar{y} + \omega$, but by Walras' law $\bar{p} \cdot x(\bar{p}, \bar{y}) = \Sigma_{i=1}^m \bar{p} \cdot x_i(\bar{p}, \bar{y}) = \Sigma_{i=1}^m r_i(\bar{p}, \bar{y}) = \Sigma_{i=1}^m \theta_i \bar{p} \cdot (\bar{y} + \omega) = \bar{p} \cdot (\bar{y} + \omega)$. Therefore, $\gamma = 1$, completing the proof.

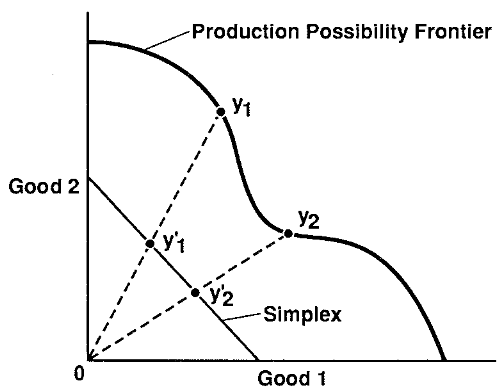

Figure 36.3. The images of the efficient production plans y_1 and y_2 under the homeomorphism between the production possibility frontier and the simplex are the points y_1' and y_2'.

We next prove the existence of an ACP equilibrium in Mantel's model. For each $y \in \partial Y$, we define the average cost pricing correspondence $\mathrm{AC}(y) = \{p \in S \mid p \cdot y = 0\}$. We will need the additional assumption that $\partial Y \cap (-\mathscr{R}^l_+) = \{0\}$. The following lemma is due to Kamiya.

Lemma 1 [Kamiya (1988a, Lemma 12)]. *If* A2(i) *holds, then* $\mathrm{AC} : \partial Y \to S$ *is a lower hemi-continuous correspondence with non-empty, closed convex values for all* $y \in \partial Y \backslash \{0\}$.

Following Kamiya (1988a, Lemma 13), we can invoke the Michael selection theorem, given our assumption that ∂Y is smooth at the origin, to prove the existence of a continuous function $\rho : \partial \hat{Y} \to S$ such that if $y + \omega \in \partial \hat{Y}$, $y \neq 0$ then $\rho(y) \in \mathrm{AC}(y)$. See Hildenbrand and Kirman (1989), Appendix IV, for a discussion of the Michael selection theorem. Given the selection ρ, we now assume that the underlying exchange economy is a regular exchange economy, hence has only a finite number of locally unique equilibrium prices, which are smooth functions of the individual endowments. Then, generically, $\rho(0)$ is not an equilibrium price for the underlying exchange economy. A non-trivial ACP equilibrium is defined as an ACP equilibrium where the equilibrium production plan is not the zero vector.

Theorem 2. *Given assumptions* A1 *and* A2, *if* $\partial Y \cap (-\mathscr{R}^l_+) = \{0\}$, $\omega \in \mathscr{R}^l_{++}$ *and* $Y \cap \mathscr{R}^l_+ = \{0\}$, *then generically there exists a non-trivial ACP equilibrium, i.e. consumption plans* x_i, *a production plan y and average cost prices p such that each consumer is maximizing his/her utility at* x_i *subject to his/her budget constraint,* $p \cdot y = 0$ *and* $\Sigma_{i=1}^m x_i = y + \omega$.

Proof. The argument is exactly the same as in the proof of Theorem 1 with the marginal cost pricing rule $\psi(y) = \nabla f(y) / \|\nabla f(y)\|_1$ replaced by the average cost pricing rule $\rho(y)$. The no free lunch assumption, $Y \cap \mathscr{R}^l_+ = \{0\}$, guarantees that $\rho(y) \neq 0$. Since generically $\rho(0)$ is not an equilibrium price vector, we see that the equilibrium production plan $y \neq 0$.

If there are several non-convex technologies in the economy then a different construction is needed to formulate equivalent fixed-point problems for proving the existence of MCP or ACP equilibria. The first such argument, for MCP equilibria, is due to Beato and Mas-Colell (1985). The intuition underlying their proof is easily explained. Suppose the boundary of the attainable set for each firm is contained in the interior of a compact set, and the boundary of the firm's attainable production set is homeomorphic to the simplex. Also suppose that each firm's boundary is smooth, hence the marginal cost pricing rule is simply the function mapping an efficient production plan into the normalized marginal rates of transformation. Now imagine an auctioneer who announces both market prices and an efficient production plan for each firm. Consumers respond to prices with their utility maximizing consumption plans; and firms respond to demand with their marginal cost prices, i.e. normalized marginal rates of transformation. In this story households are quantity-setting price-takers and firms are price-setting quantity-takers. The auctioneer uses the excess demand to adjust the market prices, which are then normalized to the price simplex, and uses the announced marginal cost prices of the firms to adjust the production plans that are then normalized to their respective simplices, which are homeomorphic to the relevant portions of the boundaries of the production sets. This process defines a continuous map of the $(n+1)$-fold product of the simplex into itself – there are n firms in the economy – and hence has a fixed-point by Brouwer's fixed-point theorem. This fixed-point is shown to be a free-disposal marginal cost pricing equilibrium, by the usual arguments. Of course, for a single firm economy this argument reduces to that of Mantel.

The model of Beato and Mas-Colell is given next along with their existence proof for marginal cost pricing equilibria. The consumption side of their model is represented in reduced form, i.e. aggregate market demand is represented as a continuous function of market prices p, and production plans $y = (y_1, \ldots, y_n)$ into $\mathscr{R}^l_+$, that satisfies Walras' law when aggregate wealth is non-negative. Hence they implicitly assume that every consumption set is a subset of $\mathscr{R}^l_+$. There are n firms in the model characterized by production sets Y_j and pricing rules $g_j : \partial Y_j \to S$, where each g_j is a correspondence. Let $\partial Y = \partial Y_1 \times \cdots \times \partial Y_n$, then a pair $(y, p) \in \partial Y \times S$ is a production equilibrium if $y_j \in \partial Y_j$ and $p \in g_j(y_j)$, for all j. It is a feasible production equilibrium if $\Sigma^n_{j=1} y_j \geq 0$. If $(y, p) \in \partial Y \times S$, then define $M(y, p) = p \cdot \Sigma^n_{i=1} y_j$. We are now ready to state their three basic assumptions.

(H1) For all j:

(i) $Y_j = K_j - \mathscr{R}^l_+$, where K_j is compact;

(ii) let $e = (1, 1, \ldots, 1)$ then $\exists r > 0$ s.t. K_j is in the interior of $[\{-re\} + \mathscr{R}^l_+]$, where $r > 0$.

(H2) For all j:

(i) g_j is upper hemi-continuous and convex-valued;

(ii) if $y_{jh} < -r$ and $p \in g_j(y_j)$ then $p_h = 0$ (see Figure 36.4).

(H3) At every production equilibrium (y, p), $M(y, p) > 0$.

The reader should think of the K_j as the attainable production set of firm j. Also Beato and Mas-Colell have embedded the social endowment into the production sets. It is H1 that allows them to assume $\partial Y_j \cap [\{-re\} + \mathscr{R}^l_+]$ is homeomorphic to the simplex, S. Again, see Figure 36.4. The interpretation of H2 is that g_j is the marginal cost pricing rule for a non-smooth production set. Bonnisseau and Cornet (1988a) have shown in their Lemma 4.2(c) that if g_j is

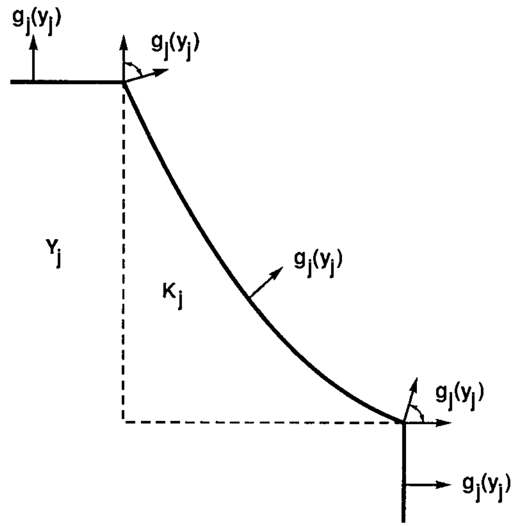

Figure 36.4.

defined by the Clarke normal cone and H1(i) holds, then the boundary condition H2(ii) is satisfied. Unfortunately, this boundary condition need not hold for the average cost pricing correspondence. Hence the Beato and Mas-Colell model does not readily extend to this case. H3 is the important survival assumption and implies that at equilibrium the profits of the competitive sector, i.e. profit maximizing firms with convex technologies, plus the value of the social endowment exceeds the aggregate losses incurred by firms with decreasing average costs, i.e. firms with non-convex technologies, who price at marginal cost. The importance of H3 is underscored by an example of Kamiya (1988b) with three goods, two firms, and an arbitrary number of consumers where the survival assumption does not hold and a marginal cost pricing equilibrium does not exist.

Formally the consumption side of their model is given by a continuous function $f : \partial Y \times S \rightarrow \mathscr{R}^l_+$, where $p \cdot f(y, p) = M(y, p)$ whenever $M(y, p) \geqslant 0$.

A free-disposal equilibrium in the Beato–Mas-Colell model is a pair $(y, p) \in \partial Y \times S$ such that (y, p) is a production equilibrium, $f(y, p) \leqslant \Sigma_{j=1}^n y_j$; and $p \cdot f(y, p) = p \cdot \Sigma_{j=1}^n y_j$, i.e. goods in excess supply have zero price.

Theorem 3 [Beato and Mas-Colell (1985)]. *Given assumptions* H1, H2 *and* H3, *a free-disposal equilibrium exists.*

Proof. Let η_j be the homeomorphism of the simplex S onto $\partial Y_j \cap [\{-re\} + \mathcal{R}^l_+]$ for each j. Unlike in the Mantel model, ∂Y_j need not be smooth. Hence g_j is a correspondence. Moreover, g_j need not be lower hemi-continuous and therefore we cannot invoke the Michael's selection theorem, as did Kamiya for the average cost pricing correspondence. Instead Beato and Mas-Colell rely on the clever trick of using Cellina's theorem on the approximation of an upper hemi-continuous correspondence by the graph of a continuous function; see Hildenbrand and Kirman (1988), Appendix IV, for a discussion of Cellina's theorem and some of its applications. Hence, g_j is assumed to be a function in the proof. To be completely rigorous we would have to show that the limit of "approximate equilibria" is an equilibrium, but these arguments are well known. Beato and Mas-Colell define the continuous map $\varphi : S^{n+1} \to S^{n+1}$ where for $(x, p) \in S^{n+1}$ and $\eta_j(x_j) = y_j$, let

$$\varphi_{jh}(x, p) = \frac{(x_{jh} + \max\{0, p_h - g_{jh}(y_j)\})}{1 + \sum_{h=1}^{l} \max\{0, p_h - g_{jh}(y_j)\}}, \quad \text{for } j \leq n \text{ and } 1 \leq h \leq l,$$

$$\varphi_{n+1,h}(x, p) = \frac{\left(p_h + \max\left\{0, f_h(y, p) - \sum_{j=1}^{n} y_{jh}\right\}\right)}{1 + \sum_{h=1}^{l} \max\left\{0, f_h(y, p) - \sum_{j=1}^{n} y_{jh}\right\}}, \quad \text{for } h \leq l.$$

This map has a fixed point $(\bar{x}, \bar{p})$ by Brouwer's fixed-point theorem. The fixed point of the first family of equations, using the boundary condition H2(ii), $\bar{x}_j = \varphi_j(\bar{x}, \bar{p})$ gives that $\bar{p} = g_j(\bar{y}_j)$, i.e. $(\bar{y}, \bar{p})$ is a production equilibrium. Hence by H3, $\bar{p} \cdot f(\bar{y}, \bar{p}) = \bar{p} \cdot (\Sigma_{j=1}^{n} \bar{y}_j)$; this fact and $\bar{p} = \varphi_{n+1}(\bar{y}, \bar{p})$ yield that $f(\bar{y}, \bar{p}) \leq \Sigma_{j=1}^{n} \bar{y}_j$, completing the proof.

Although the model of Beato and Mas-Colell may not be a natural model for investigating average cost pricing, it is excellent for outlining the recent existence proof of Brown, Heller and Starr (1989) for a two-part marginal cost pricing (TPMCP) equilibrium. In their model, there is a single firm with a non-convex technology that produces a single good (the "monopoly good") which is not produced by any other firm, and the social endowment of this good is zero. The remaining $n-1$ firms in the economy possess convex technologies and comprise the competitive sector of the model, i.e. these firms are price-taking profit-maximizers. We shall view the firm producing the monopoly good as a regulated public monopoly. Regulation takes the form of marginal cost pricing with discriminating (or non-uniform) "hook-up" fees charged for the right to consume the monopoly good. All firms, including the regulated natural monopoly, are privately owned and all shareholdings carry limited liability. Hence losses of the regulated firm can only be recovered through the hook-up fees, thus there are no taxes in this model, lump sum or otherwise. The hook-ups are required to just recover the losses that the regulated firm incurs by marginal cost pricing. Hence in equilibrium, the regulated public monopoly makes zero economic profits.

Brown, Heller and Starr define TPMCP equilibrium as a family of consumption plans $\bar{x}_i$, production plans $\bar{y}_j$, market prices $\bar{p}$ and hook-up fees $\bar{q}_i$, such that consumer i is maximizing his/her utility at x_i subject to his/her budget constraint:

$$\bar{p} \cdot x_i \leq \bar{p} \cdot \omega_i + \sum_{j=1}^{n} \theta_{ij} \bar{p} \cdot \bar{y}_j \quad \text{if } x_{i1} = 0 ,$$

or

$$\bar{q}_i + \bar{p} \cdot x_i \leq \bar{p} \cdot \omega_i + \sum_{j=1}^{n} \theta_{ij} \bar{p} \cdot \bar{y}_j \quad \text{if } x_{i1} > 0 ;$$

$\bar{p} = g_j(\bar{y}_j)$, where the g_j are the marginal cost pricing rules in the Beato–Mas-Colell model, $\Sigma_{i \in \theta} \bar{q}_i = \min(0, -\bar{p} \cdot \bar{y}_1)$, where θ is the set of consumers who purchase the monopoly good; and $\Sigma_{i=1}^{m} \bar{x}_i \leq \Sigma_{j=1}^{n} \bar{y}_j + \omega$, where ω is the social endowment, $\Sigma_{i=1}^{m} \omega_i$.

The basic idea underlying the existence proof of Brown, Heller and Starr is the notion of willingness to pay and the assumption that, in equilibrium, the aggregate willingness to pay exceeds the losses of the regulated monopoly resulting from marginal cost pricing.

More formally, they assume that the set of feasible allocations is compact; hence $\hat{X}_i$, the attainable set of the ith consumer is compact. Let $\check{X}_i$ be a convex compact set which contains $\hat{X}_i$ in its interior. Suppose also, in addition to the standard assumptions on utility functions, that we assume U_i is strictly quasi-concave for all i. Let $r_i(y, p) = p \cdot \omega_i + \Sigma_{j=1}^{n} \theta_{ij} p \cdot y_j$. We can now calculate each household's "reservation level of utility," i.e. the maximum utility level she could obtain if the natural monopoly good were unavailable:

$$V_i(y, p) = \max U_i(x_i) \quad \text{s.t. } p \cdot x_i \leq r_i(y, p),\ x_{i1} = 0,\ x_i \in \check{X}_i .$$

The income necessary to obtain this utility level at prices p if the monopoly good is available is given by

$$E_i(p, V_i(y, p)) = \min p \cdot x_i , \quad U_i(x_i) \geq V_i(y, p),\ x_i \in \check{X}_i .$$

Each household's "willingness to pay" for the monopolist's output, given

(y, p), is $s_i(y, p) = r_i(y, p) - E_i(p, V_i(y, p))$. Notice that s_j is an ordinal concept, i.e. it is independent of the utility representation. $s_j(y, p)$ is the amount of income at given prices, p, that must be subtracted from income, $r_i(y, p)$, to reduce utility to its value, $V_i(y, p)$, when the monopoly good was unavailable. As such, it is akin both to the compensating variation of adding the monopoly good and to Dupuit's notion of benefit arising from the introduction of a public good.

The principal assumption in the Brown–Heller–Starr model is that the aggregate willingness to pay, $s(y, p) = \Sigma_{i=1}^{m} s_i(y, p)$ exceeds the losses of the natural monopoly at every production equilibrium (y, p), i.e. $s(y, p) > -p \cdot y_1$. Given this assumption, they define hook-up fees, $q_i(y, p)$, as continuous functions of (y, p) on the set of production equilibria. The $q_i(y, p)$ have the following properties:

(i) $\Sigma_{i=1}^{m} q_i(y, p) = \min(0, -p \cdot y_1)$;
(ii) if $s_i(y, p) > 0$ then $q_i(y, p) < s_i(y, p)$; and
(iii) if $s_i(y, p) = 0$ then $q_i(y, p) = 0$.

We see that if $s_i(y, p) > 0$ then consumer i will choose to pay the hook-up fee, since it is less than the maximum willingness to pay. If $s_i(y, p) = 0$ then $q_i(y, p) = 0$ and consumer i will not choose to consume the monopoly good. Hence in all cases the consumer's budget set is convex and therefore the demand correspondence is convex-valued. Assuming strict quasi-concavity of the utility function, we define the individual demand function $x_i(y, p)$. Letting $f(y, p) = \Sigma_{i=1}^{m} x_i(y, p)$ be the market demand function and extending it continuously but arbitrarily over $\partial Y \times S$, we now have reduced the TPMCP model to the MCP model of Beato–Mas-Colell. A fixed-point $(\bar{y}, \bar{p})$ of the Beato–Mas-Colell map $\varphi : S^{m+1} \to S^{m+1}$ is a production equilibrium, hence $f(\bar{y}, \bar{p})$ is the true aggregate demand. Moreover, the hook-ups, $q_i(\bar{y}, \bar{p})$ will, by construction, just cover the losses of the monopoly. The remaining step, to show that the resulting allocation is a free-disposal equilibrium, is the same as in Beato and Mas-Colell.

The final topic in this section is the existence theorem of Bonnisseau and Cornet (1988a, Theorem 2.1) where firms follow bounded losses pricing rules. This remarkable theorem provides a general existence result for a wide class of general equilibrium models including the existence of Walrasian equilibria in the classical Arrow–Debreu model, the existence of MCP equilibria in the Beato and Mas-Colell model, and the existence of ACP equilibria in general equilibrium models with several non-convex firms. Unfortunately, their proof is too technical for a survey of this kind. Instead, we will discuss the main ideas and structure of their argument. The model of Bonnisseau and Cornet is defined as follows.

The economy has l goods, m consumers and n firms. The social endowment ω is a vector in $\mathcal{R}^l$. Each firm's production set, Y_j, is a subset of $\mathcal{R}^l$. The consumption set, X_i, of consumer i is also a subset of $\mathscr{R}^l$. Tastes are defined by complete, transitive, reflexive binary relations $>_i$ on X_i. Finally, the wealth of the ith consumer is defined by a function $r_i : \partial Y \times \mathscr{R}^l_+ \to \mathscr{R}$ where $\partial Y = \partial Y_1 \times \cdots \times \partial Y_n$. A special case of this wealth structure is $r_i(p, y_1, \ldots, y_n) = p \cdot \omega_i + \Sigma_{j=1}^n \theta_{ij} p \cdot y_j$ for $\theta_{ij} \geq 0$, $\Sigma_{i=1}^m \theta_{ij} = 1$ which holds for a private ownership economy. The behavioral assumptions are that households are maximizing their preferences subject to the standard budget constraint and that firms are following pricing rules. The pricing rule ψ_j of the jth firm is characterized by a correspondence from ∂Y_j, the boundary of Y_j, to $\mathscr{R}^l_+$ where $\psi_j(y_j)$ is a cone with vertex 0. The jth firm is in equilibrium given (y, p) if $p \in \psi_i(y_j)$ and $y = (y_1, \ldots, y_m)$. Pricing rules subsume profit maximization, since $\mathrm{PM}_j(y_j) = \{p \in \mathscr{R}^l \mid p \cdot y_j \geq p \cdot y_j' \text{ for all } y_j' \in Y_j\}$. Assuming free disposal, $\mathrm{PM}_j(y_j) \subset \mathscr{R}^l_+$. A Bonnisseau–Cornet equilibrium is a family of consumption plans x_i, production plans y_j and prices p, such that consumers are maximizing utility at x_i, subject to their budget constraints; firms are in equilibrium, i.e. for all j, $p \in \psi_j(y_j)$; and all markets clear, i.e. $\Sigma_{i=1}^m x_i = \Sigma_{j=1}^n y_j + \omega$. A free disposal equilibrium is defined in the standard way. Their principal existence theorem, Theorem 2.1, is a consequence of the following assumptions:

(C) (i) Standard assumptions on consumptions sets and preferences, say as in Debreu (1959); (ii) $r_i(y, p)$ is continuous, satisfies Walras' law, i.e. $\Sigma_{i=1}^m r_i(y, p) = p \cdot (\Sigma_{j=1}^n y_j + \omega)$ and is homogenous of degree 1 in prices.

(P) For all j, Y_j is non-empty, closed and $Y_j - \mathscr{R}^l_+ \subset Y_j$ (free disposal).

(B) For every $\omega' \geq \omega$, the set $A(\omega') = \{((x_i), (y_j)) \in \Pi_{i=1}^m X_i \times \Pi_{j=1}^n Y_j \mid \Sigma_{i=1}^m x_i \leq \Sigma_{j=1}^n y_j + \omega')$ is bounded.

Given the homogeneity assumptions on r_i and ψ_j, and the local non-satiation of preferences, the equilibrium prices will lie in the price simplex S. The normalized pricing rule $\bar{\psi}_j$ is the correspondence from ∂Y_j to S defined as $\bar{\psi}_j(y_j) = \psi_j(y_j) \cap S$. The final definition is that of a production equilibrium: (y, p) is a production equilibrium if $y \in \Pi_{j=1}^n \partial Y_j$, $p \in S$ and for all j, $p \in \bar{\psi}_j(Y_j)$. PE, a subset of $\Pi_{j=1}^n \partial Y_j \times S$, is the set of production equilibria. The remaining assumptions are:

(PR) for all j, the normalized pricing rule, $\bar{\psi}_j$, is upper hemi-continuous with non-empty, convex compact values;

(BL) (bounded losses assumption) for all j, there exists a real number α_j such that for all $(y_j, p) \in \partial Y_j \times S$, $p \in \bar{\psi}_j(y_j)$ implies $p \cdot y_j \geq \alpha_j$;

(SA) (survival assumption) $(y, p) \in \mathrm{PE}$ implies $p \cdot (\Sigma_{j=1}^n y_j + \omega) > \inf_{x_i \in X_i} p \cdot \Sigma_{i=1}^m x_i$;

(R) $(y, p) \in \mathrm{PE}$ and $p \cdot (\Sigma_{j=1}^n y_j + \omega) > \inf p \cdot \Sigma_{i=1}^m x_i$ imply $r_i(y, p) > \inf\{p \cdot x_i \mid x_i \in X_i\}$ for all i.

Assumptions (C) and (P) need no discussion. (B) is implied by $A(\Sigma_{j=1}^n Y_j) \cap (-A\, \Sigma_{j=1}^n Y_j) = \{0\}$, where $A(\Sigma_{j=1}^n Y_j)$ is the asymptotic cone of $(\Sigma_{j=1}^n Y_j)$ [see Hurwicz and Reiter (1973)].

(PR), the pricing rule assumption is satisfied by a profit maximizing firm j

with convex technology, if $\psi_j(y_j) = \mathrm{PM}(Y_j)$; a firm following the marginal cost pricing rule where $\psi_j(y_j)$ is the Clarke normal cone at y_j; and a firm following average cost pricing where $\psi_j(y_j) = \mathrm{AC}(y_j)$, if $Y_j \cap \mathcal{R}^l_+ = \{0\}$.

Before discussing the remaining assumptions, we now give the formal definition of the Clarke normal cone, denoted $N_Y(y)$. First, we need the notion of the Clarke tangent cone. For a non-empty set $Y \subseteq \mathcal{R}^l$ and $y \in \partial Y$, the tangent cone of Y at y is $T_Y(y) = \{x \in \mathcal{R}^l \mid$ for every sequence $y^k \in Y$, $y^k \to y$ and every sequence $t^k \in (0, \infty)$, $t^k \to 0$, there exists a sequence $x^k \in \mathcal{R}^l$, $x^k \to x$, such that $y^k + t^k x^k \in Y$ for all $k\}$. For any $A \subseteq \mathcal{R}^l$, the polar cone of A, $A^+ = \{z \in \mathcal{R}^1 \mid x \cdot z \leqslant 0$ for all $x \in A\}$. Then $N_Y(y) = [T_Y(y)]^+$. See Figure 36.5 for examples of the Clarke normal cone.

Assumption (SA) simply states that at a production equilibrium there is sufficient income to cover all losses (through lump sum taxation) and still provide consumers with the necessary income to purchase their subsistence consumption bundle. (R) asserts that aggregate income (net of lump sum taxes to cover any losses of firms) is distributed in such a manner that each consumer has sufficient income for subsistence. As Bonnisseau and Cornet point out, special cases of (SA) and (R) are (i) the private ownership models of Arrow–Debreu and (ii) models with the fixed structure of revenues assumption and positive net social wealth at each production equilibrium.

With the exception of (BL), all of these assumptions – albeit with less generality – have appeared in one guise or another in the work of Debreu (1959), Dierker, Guesnerie and Neuefeind (1985), Brown, Heal, Khan and Vohra (1986) and Kamiya (1986a). Hence the major conceptual innovation of this paper is the notion of bounded losses pricing rules. This condition holds for all of the models cited above, together with the model of Beato and Mas-Colell. Notice that (BL) is not needed for economies with a single firm,

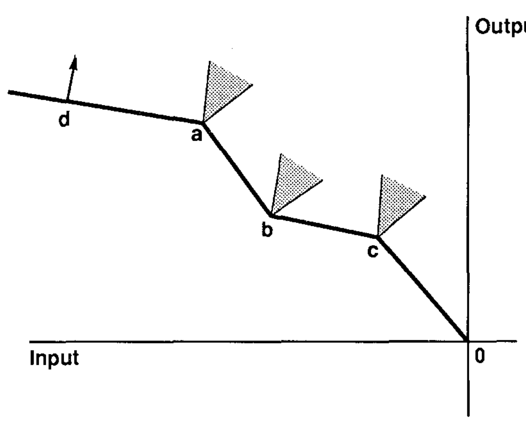

Figure 36.5. The shaded cones at points (a), (b) and (c) are the Clarke normal cones. The Clarke normal cone at (d) consists of all non-negative scalar multiples of the normal vector at (d).

e.g. Mantel (1979) or Cornet (1982). But the most surprising consequence of (BL) is in the case of marginal cost pricing, where the pricing rule is in terms of the Clarke normal cone. In this instance, the (BL) assumption is equivalent to assuming that the production set of the firm is strictly star-shaped; see Lemma 4.2 in their paper.

Star-shaped production sets were introduced by Arrow–Hahn in their discussion of monopolistic competition. These sets are a particularly well-behaved class of non-convex sets, e.g. compact strictly star-shaped sets in $\mathscr{R}^l$ are homeomorphic to the 1-ball in $\mathscr{R}^l$ [see Arrow and Hahn (1971, Appendix B)]. The relevant literature on the properties of these sets for equilibrium analysis appears to be the geometry of numbers; this connection is suggested by the interesting and important work of Scarf (1986) on indivisibilities in production; and non-smooth optimization [see Dem'yanov and Rubinov (1986)].

Subsequent to the paper under discussion, Bonnisseau and Cornet (1988c) were able to drop the (BL) assumption and still prove the existence of a MCP equilibrium. Of course for average cost pricing, (BL) holds trivially.

Returning to Bonnisseau and Cornet (1988a), we see that the existence proof rests on another fact about production sets with free disposal. They show that if a production set Y is a non-empty subset of $\mathscr{R}^l$ such that $Y - \mathscr{R}^l_+ \subset Y$ and $Y \neq \mathscr{R}^l$, then the boundary of Y, ∂Y, is homeomorphic to a hyperplane in $\mathscr{R}^l$ (see Lemma 5.1 in their paper). This lemma, together with (BL) and the compactness of firms' attainable production sets, which follows from (B), allows them to define compact, convex subsets of the hyperplanes corresponding to each ∂Y_j. The interiors of these sets contain the homeomorphic images of the relevant portions of ∂Y_j, analogous to the construction of Beato and Mas-Colell regarding the ∂Y_j. The final step is to use a suitable convex compact ball defined from the sets above: products of the price simplices, one for each firm, as proxies for the range of the pricing rules; a price simplex for market prices; and convex, compact sets which contain the attainable consumption sets in their interior. This ball is then the domain and range of a continuous map, F. The fixed-points of F, which are shown to exist by Kakutani's theorem, constitute free-disposal equilibria. Bonnisseau and Cornet give several extensions of the basic result, Theorem 2.1, but the outline given above conveys the structure of all of their proofs.

In the next section, we consider another method for establishing existence of equilibria in economies with non-convex technologies.

3. Local uniqueness and computation

The non-linear system of equations which define an equilibrium in an economy with non-convex technologies, where firms follow pricing rules, consists of the first-order conditions for utility maximization subject to a budget constraint, the equations defining a production equilibrium, and the market clearing equations. In the previous sections, these equations were shown to have a solution by converting the system into an equivalent fixed-point problem and the existence of a fixed-point was established by using the Brouwer or Kakutani fixed-point theorem. But the arguments used give no indication of the number of equilibria or how to compute an equilibrium. In this section, we consider the path-following or homotopy method for solving a given system of equations; this will allow us to derive both a uniqueness theorem and an algorithm for computing equilibria.

The principal papers in this area are all due to Kamiya [see Kamiya (1986a, b, 1987, 1988a)]. The assumptions in Kamiya's proof of existence [Kamiya (1988a, Theorem 1)] differ from those of Bonnisseau and Cornet in two important respects. Instead of assuming (B), Kamiya assumes (B′): $\mathrm{co}\, A(\Sigma_{j=1}^{n} Y_j) \cap -\mathrm{co}\, A(\Sigma_{j=1}^{n}, Y_j) = \{0\}$, where $\mathrm{co}(D)$ is the convex hull of D; and instead of (BL), he assumes (L): for all sequences $\{(p^\alpha, y_j^\alpha)\} \subset S \times \partial Y_j$, such that $\| y_j^\alpha \|_\infty \to +\infty$ and, for all α, $p^\alpha \in \bar{\psi}_j(y_j^\alpha)$, it is the case that $\lim_{\alpha\to\infty} p^\alpha \cdot (y^\alpha / \| y_j^\alpha \|_\infty) \geqslant 0$. Hence his assumption on the losses incurred by firms is weaker than the bounded losses assumption of Bonnisseau–Cornet, but his assumption on boundedness (B′) is stronger than their assumption (B). Kamiya also assumes that consumption sets are subsets of $\mathscr{R}_+^l$. The essential difference between the models of Kamiya and that of Bonnisseau–Cornet is not the existence theorems, per se. In fact, Bonnisseau (1988) has been able to derive Kamiya's result from his theorem with Cornet by constructing a new economy with different production sets and changing the pricing rule so that it satisfied (BL). The important difference between the two models is the method of proving existence.

Kamiya's path-following or homotopy proof of existence, given the previous work of Dierker (1972), Smale (1987b), Scarf (1973) and Eaves (1972), naturally suggests two results. First, a condition for uniqueness of equilibria and second, an algorithm for computing equilibria. Conditions for local uniqueness and uniqueness can be found in Kamiya (1988a, Theorem 3). Algorithms for computing equilibria (in principle) can be found in Kamiya (1986b, 1987). Unfortunately, all of the arguments in these papers are too technical for this survey. Hence we will illustrate the main idea in his existence theorem by showing that Mantel's model for MCP equilibria has an odd number of equilibria. Our proof will be based on path-following and the homotopy invariance theorem, the essential elements of Kamiya's argument. This result on Mantel's model was originally proved by Brown and Heal (1982), using the fixed-point index introduced by Dierker (1972). As Dierker shows, if each of the finite equilibria in an exchange economy has the same index then the equilibrium is unique. This condition, of course, implies uniqueness of Mantel's model and Kamiya's model and is the condition used by Kamiya to guarantee uniqueness. Roughly, the index at an equilibrium is the sign of the determinant of the excess demand at the equilibrium prices.

The computational algorithm in Kamiya (1986b) is a simplical path-following method based on Scarf's original simplical algorithm for computing equilibrium prices [see Scarf (1973)]. The degree of computational complexity is of the order $(l-1)(n)$, where n is the number of firms and l is the number of goods. This is quite large relative to the degree of computational complexity of Scarf's algorithm applied to classical Arrow–Debreu economies, which is of order $(l-1)$. This increased complexity arises from the need to treat each firm's production possibilities separately in the non-convex case; whereas in the convex case, one can aggregate the technologies or in well-behaved cases only consider market excess demand functions that depend on $l-1$ prices. In Kamiya (1987), using results in differential topology, he gives a second algorithm which "generically" has the same order of computational complexity as in the convex case, i.e. $(l-1)$.

We shall need to make several additional assumptions concerning Mantel's model for marginal cost pricing in order to prove there are an odd number of MCP equilibria. If $z \in \mathscr{R}^l$, let $\tilde{z} \in \mathscr{R}^{l-1}$ be the first $(l-1)$ components of z. For notational convenience, when $y + \omega \in \partial \hat{Y}$ we shall simply say that "$y \in \partial \hat{Y}$". If $y \in \partial \hat{Y}$ then denote $\nabla f(y)/\|\nabla f(y)\|_1$ as $p(y)$. The aggregate demand at these prices will be denoted $x(p(y))$. Finally, we define the homotopy $H : \partial \hat{Y} \times [0,1] \to \mathscr{R}^{l-1}$ where $H(y,t) = (1-t)(\tilde{y}_0 - \tilde{y}) + t(\tilde{x}(p(y)) - \tilde{y})$, $y_0 \in \partial \hat{H}$ and $\tilde{x}(p(y))$ is defined as in the proof of Theorem 1. y_0 is chosen to guarantee assumption A4, the boundary-free condition. In this model, this is not a realistic condition and is only intended to be illustrative. Guaranteeing that the path defined by the homotopy does not run into the boundary for $t<1$ is the crucial part of the path-following methodology. We now assume:

A3 (i) 0 is a regular value of $H(y,t)$,

(ii) 0 is a regular value of $H(y,1)$.

A4 For all $t \in (0,1)$ and all $y \in \partial \hat{Y}$, $(1-t)(\tilde{y}_0 - \tilde{y}) + t(\tilde{x}(p(y)) - \tilde{y}) \neq 0$.

Theorem 3 [Brown and Heal (1982)]. *Given assumptions* A1–A4, *Mantel's model for marginal cost pricing has an odd number of equilibria.*

Proof. The proof is an immediate consequence of the homotopy invariance theorem, which is stated below. First, suppose F is a smooth function from a compact subset of $\mathscr{R}^n$, with non-empty interior, into $\mathscr{R}^n$, i.e. $F : D \to \mathscr{R}^n$. If 0 is a regular value of F and $F^{-1}(0) \cap \partial D = \emptyset$, then we define the degree of F to be the integer, $\deg(F) = \Sigma_{x \in F^{-1}(0)}$ sgn det $F'(x)$, where F' is the Jacobian of F.

Homotopy Invariance Theorem [Garcia and Zangwill (1981, Theorem 3.4.3)]. *Suppose D is a compact subset of $\mathcal{R}^n$, with non-empty interior; $H : D \times [0, 1] \to \mathcal{R}^n$ is a regular homotopy, i.e. 0 is a regular value of H; 0 is a regular value of $H(x, 0)$ and $H(x, 1)$. If H is boundary-free, i.e. if $H(x, t) = 0$ implies $x \not\in \partial D$, then* $\deg(H(x, 0)) = \deg(H(x, 1))$.

Applying this theorem to Mantel's model we see that $H(y, 0)$ has the unique solution y_0; hence at $H(y, 1)$ we must have an odd number of solutions. Since by Walras' law, all solutions of $H(y, 1) = 0$ are marginal cost pricing equilibria, this completes the proof.

Of course, the above theorem proves the existence of a MCP equilibrium, but we now give a third proof of existence which is the basis for the computational algorithm in Kamiya (1986a).

Theorem 4. *Given assumptions* A1–A4, *Mantel's model has a MCP equilibrium.*

Proof. Since 0 is a regular value of H, we see that $H^{-1}(0)$ is a one-dimensional manifold. Because of the boundary-free assumption, A4, and the uniqueness of the solution at $H(y, 0)$, there is a "path" from $y = y_0$ to $y = y_1$, where $H(y_1, 1) = 0$ (see Figure 36.6).

Path-following algorithms are simply numerical schemes for computing this one-dimensional manifold. Garcia and Zangwill give an explicit differential

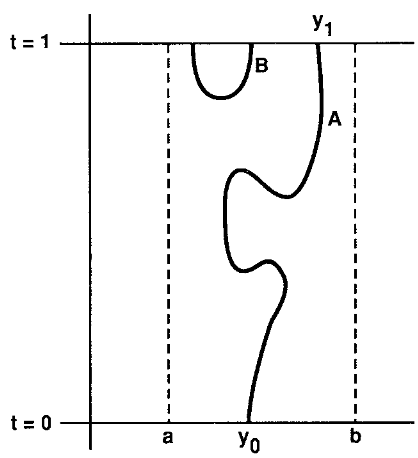

Figure 36.6. In this figure, D is the compact interval $[a, b]$. H^{-1} consists of the two paths A and B. A is the path from y_0, a solution of $H(y, 0) = 0$, to y_1, a solution of $H(y, 1) = 0$.

equation whose solution is the manifold in question [see Garcia and Zangwill (1981, Theorem 2.1.1)]. The path following approach for computing equilibria in economic models consists of solving a differential equation which traces out a one-dimensional manifold to an equilibrium, was introduced into equilibrium analysis by Smale (1976b). Smale's method is not explicitly a homotopy method and is known in the literature as the Global Newton's method.

Returning to Kamiya's work, we ask what makes his proof so complicated? First, there is the issue of several firms which cannot be aggregated by the use of a market supply function and, in addition, there is the difficulty of finding at least one production equilibrium to begin the homotopy. Finally, he must find an economically meaningful boundary condition to guarantee that his homotopy is boundary-free. These problems are resolved in an ingenious fashion and the reader is invited to read the first chapter of Kamiya (1986a) for an informal discussion of his model and proof of existence.

4. Optimality

In this final section of the paper, we present two examples which illustrate the inefficiency of marginal cost pricing. Also we prove the second welfare theorem for marginal cost pricing equilibria in an economy with a single non-smooth technology. That is, we show that every Pareto optimal allocation can be supported as a marginal cost pricing equilibrium where the marginal rates of transformation at each efficient production plan are defined by the Clarke normal cone and households are minimizing expenditure. Of course, our result is a special case of the necessity of marginal cost pricing, in terms of the Clarke normal cone, for Pareto optimality as shown by Quinzii (1991). But the basic intuition that the separation argument depends only on the convexity of the appropriate tangent cone and not the convexity of the production set is due to Guesnerie (1975).

Our first example of inefficiency is taken from Brown and Heal (1979), where they give an example of an economy having only three MCP equilibria, all of which are inefficient. The non-convex production possibility set Y is illustrated in Figure 36.7. There are two households, and only three production plans are candidates for MCP equilibria, i.e. points A, B and C in the figure. But plan C is inefficient since the relevant Scitovsky community indifference curve is clearly below feasible production plans. Hence only A and B are candidates for efficient MCP equilibria. But suppose the Scitovsky community indifference curves through A and B look as they do in Figure 36.7; then points A and B are also inefficient. Another way of making the same point is to draw the Edgeworth boxes for distribution at these points. if we then plot the corresponding contract curves in utility space, we find that A′ and B′ in utility

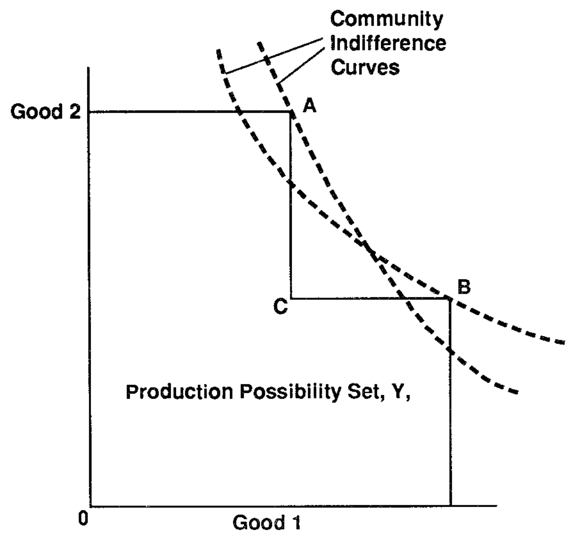

Figure 36.7.

space, corresponding to A and B, lie inside the utility possibility frontier, i.e. are inefficient (see Figure 36.8). The interested reader is referred to Brown and Heal (1979) for a numerical example with these properties. Please note that the first example of this kind is due to Guesnerie (1975).

A more striking example of the inefficiency of MCP equilibria is found in Beato and Mas-Colell (1983). In this example there are only three MCP equilibria, and aggregate production efficiency fails to obtain in each case.

There are two goods in their economy, denoted x and y. x is used as an input to produce y. There are two firms, one with constant returns to scale, i.e. $y_1 = x_1$ and the other with increasing returns, i.e. $y_2 = \frac{1}{16}(x_2)^2$. There are two consumers. One consumer has a utility function $U_1(x_1, y_1) = y_1$, who is endowed with $\omega_1 = (0, 50)$ and owns both firms, i.e. profits and losses are paid by the consumer. The second consumer has a utility function $U_2(x_2, y_2) = \min\{6x_2, y_2\}$ and is endowed with $\omega_2 = (20, 0)$.

See Figure 36.9 for descriptions of the individual technologies and the aggregate technology. Here we give only the intuition for their result, the reader interested in the details should consult either the above cited reference or Beato and Mas-Colell (1985). It is clear that p_y, the price of output, cannot be 0 in equilibrium, since the first consumer's utility function is $U_1(x_1, y_1) = y_1$. Hence we choose output as numeraire and set $p_y = 1$. If the first firm produces in equilibrium then $p_x = 1$, since the first firm produces with constant

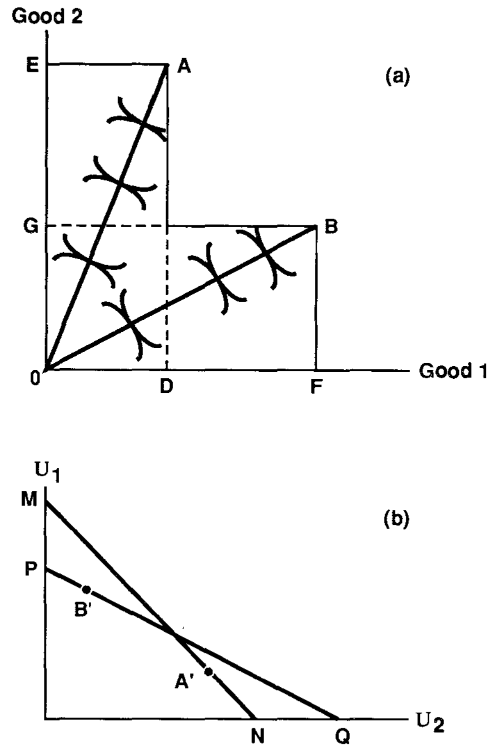

Figure 36.8. The contract curves $0A$ and $0B$ in (a) correspond, respectively, to the curves MN and PQ in (b). The utility possibility frontier is the outer envelope of MN and PQ.

returns to scale, where the constant marginal rate of transformation is 1. In this case, if the second firm is also producing in equilibrium then it must be at a point where the marginal rate of transformation is also 1. Checking the first-order conditions for profit maximization, utility maximization and market clearing, we see that this is a MCP equilibrium. But we see from the figure for the aggregate production possibility set that it is never efficient for both firms to produce.

The other two cases are when one firm produces and the other firm is inactive. In both cases the active firm produces inside the production possibility set.

Finally, we prove the second welfare theorem in an economy with a single non-smooth technology. This model allows us to follow the traditional separating hyperplane argument, but in our proof the convex sets are the (Clarke) tangent cone at the efficient production plan and the sum of the sets of consumption plans that each consumer strictly prefers to her given Pareto

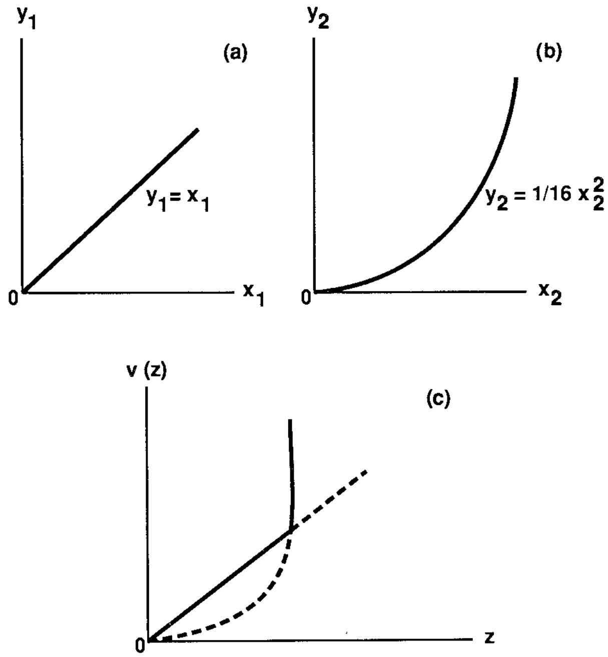

Figure 36.9. Figures (a) and (b) are the technologies of the two firms. $V(z) = \max[f_1(z_1) + f_2(z_2)\colon z_1 + z_2 = z]$ and (c) is the aggregate production function, $V(z)$.

optimal consumption plan. The standard argument, say in Debreu (1959) separates the latter set from the (convex) aggregate production set. First, we recall some notions from non-smooth analysis.

Let Y be a closed non-empty subset of $\mathscr{R}^l$, then the cone of interior displacements at x, denoted $K_Y(x)$, $= \{z \in \mathscr{R}^l \mid \exists \eta > 0,\ \theta > 0, \text{ s.t. } \forall \tau \in [0, \eta], \{x\} + \tau B_\theta(z) \subset Y\}$. Again, $K_Y(x)$ need not be convex.

In contrast, the Clarke tangent cone, $T_Y(x)$, is always convex. For a comparison of these two cones, we consider the interior of $T_Y(x)$: Int $T_Y(x) = \{z \in \mathscr{R}^l \mid \exists \eta > 0,\ \theta > 0,\ l > 0, \text{ s.t. } \forall \tau \in [0, \eta], \forall x' \in \mathrm{Cl}(Y) \cap \mathrm{Cl} B_l(x), \{x'\} + \tau B_\theta(z) \subset Y\}$. Clearly, Int $T_Y(x) \subseteq K_Y(x)$.

We consider a private ownership economy with l goods, m consumers and a single firm. We make the same assumptions on the characteristics of households as we did in our discussion of Mantel's model in Section 2. But we only assume that the firm has a non-smooth technology, Y (this terminology is a bit confusing since smooth technologies are special cases of non-smooth technologies, see Section 1). The pricing rule $\psi(y)$ is the Clarke normal cone (normalized to the price simplex). To guarantee that $\psi(y) \neq 0$ for all $y \in \partial Y$, we assume $0 \in Y$, $Y - \mathscr{R}^l_+ \subset Y$ (free disposal) and Y is closed.

Second Welfare Theorem. *If $\langle x_1, \ldots, y_m, y \rangle$ is a Pareto optimal allocation and $\Sigma_{i=1}^m x_i = y + \omega \in \mathscr{R}_{++}^l$, then there exists a $p \in \bar{\psi}(y)$ such that x_i is the expenditure minimizing consumption plan for agent i at prices p and utility level $\bar{U}_i = U_i(x_i)$.*

Proof. Let $B_i = \{z \in R_+^l \mid U_i(z) > U_i(x_i)\}$, then B_i is non-empty and convex for each i. Let $B = \Sigma_{i=1}^m B_i$, then B is also convex. Let $x = \Sigma_{i=1}^m x_i$, then $x \in \bar{B}$, the closure of B. Define $B' = \Sigma_i B_i - x$. Now consider B' and Int $T_Y(y)$. These are non-empty, convex sets (the non-emptiness of Int $T_Y(y)$ following from free disposal) and 0 belongs to the boundary of both sets. Moreover, Int $T_y(y) \cap B' = \emptyset$. Suppose not, i.e., there exists $z \in B' \cap$ Int $T_Y(y)$. The fact that $z \in$ Int $T_Y(y)$ implies that there exists a $t \in (0, 1)$ such that $y + tz \in Y$. Let $y' - y + tz$ and $x' = x + tz$. Since $x = y + \omega$, we have $x' = y' + \omega$, i.e., (x', y') is feasible. The fact that $z \in B'$ implies that there exist $\bar{x} \in B$ such that $z = (\bar{x} - x)$. Thus $x' = x + t(\bar{x} - x) = (1 - t)x + t\bar{x}$. By local non-satiation and convexity, this yields an allocation which Pareto dominates (x, y)—a contradiction. Now we know that 0 belongs to the boundary of the convex sets Int $T_Y(y)$ and B' and these sets have an empty intersection. By the separating hyperplane theorem, there exists $p \neq 0$ such that $p \cdot z \leqslant 0$ for all $z \in$ Int $T_Y(y)$ and $p \cdot z \geqslant 0$ for all $z \in B'$. The first condition yields $p \in N_Y(y)$ and the second one yields expenditure minimization.

This proof is due to R. Vohra.

References

Arrow, K.J. (1951) 'An extension of the basic theorems of classical welfare economies', in: J. Neuman, ed., *Proceedings of the 2nd Berkeley symposium on mathematical statistics and probability*. Berkeley: University of California Press.

Arrow, K.J. and G. Debreu (1954) 'Existence of an equilibrium for a competitive economy', *Econometrica* 22: 265–290.

Arrow, K.J. and F.H. Hahn (1971) *General competitive analysis*. San Francisco, CA: Holden-Day.

Beato, P. (1982) 'The existence of marginal cost pricing equilibria with increasing returns', *The Quarterly Journal of Economies*, 389: 669–688.

Beato, P. and A. Mas-Colell (1983) 'Gestion au cout marginal et efficacité' de la production aggrégée: un example', *Annales de l'INSEE*, 51: 39–46.

Beato, P. and A. Mas-Colell (1985) 'On marginal cost pricing with given tax-subsidy rules', *Journal of Economics Theory*, 37: 356–365.

Boiteux, M. (1956) 'Sur la gestion des monopoles publics astreints à l'equilibre budgétaire', *Econometrica*, 24: 22–44; English translation: (1971) 'On the management of public monopolies subject to budgetary constraints', *Journal of Economic Theory*, 3: 219–240.

Bonnisseau, J.M. (1988) 'On two existence results of equilibria in economies with increasing returns', *Journal of Mathematical Economics*, 17: 193–208.

Bonnisseau, J.M. and B. Cornet (1988a) 'Existence of equilibria when firms follow bonded losses pricing rules', *Journal of Mathematical Economics*, 17: 103–118.

Bonnisseau, J.M. and B. Cornet (1988b) 'Valuation of equilibrium and Pareto optimum in nonconvex economies', *Journal of Mathematical Economics*, 17: 293–315.

Bonnisseau, J.M. and B. Cornet (1988c) 'Existence of marginal cost pricing equilibria: the nonsmooth case', CERMSEM Working Paper, Univsersité Paris I, Paris.

Brown, D.J. and G.M. Heal (1979) 'Equity, efficiency, and increasing returns', *Review of Economic Studies*, 46: 571–585.

Brown, D.J. and G.M. Heal (1982) 'Existence, local uniqueness, and optimality of a marginal cost pricing equilibrium with increasing returns', Social Science Working Paper 415, California Institute of Technology, Pasadena, CA.

Brown, D.J. and G.M. Heal (1983) 'The optimality of regulated pricing: A general equilibrium analysis', in: C. Aliprantis and Burkinshaw, eds., *Advances in equilibrium theory*. Berlin: Springer-Verlag.

Brown, D.J. and G.M. Heal, M. Ali Khan and R. Vohra (1986) 'On a general existence theorem for marginal cost pricing equilibria', *Journal of Economic Theory*, 38: 371–379.

Brown, D.J., W.P. Heller and R.M. Starr (1990) 'Two-part marginal cost pricing equilibria: Existence and efficiency', mimeo.

Brown, S.J. and D.S. Sibley (1986) *The theory of public utility pricing*. Cambridge: Cambridge University Press.

Clarke, F. (1983) *Optimization and nonsmooth analysis*, New York: Wiley.

Coase, R.H. (1946) 'The marginal cost controversy', *Economica*, 13: 169–189.

Cornet, B. (1987) 'Regularity properties of open tangent cones', *Mathematical Programming Study* 30, 30: 17–33.

Cornet, B. (1988) 'General equilibrium theory and increasing returns: presentation', *Journal of Mathematical Economics*, 17: 103–118.

Cornet, B. (1990) 'Existence of equilibrium in economies with increasing returns', in: B. Cornet and H. Tulkens, eds., *Contributions to Economics and Operations Research, The XXth Anniversary of CORE*. Cambridge, MA: MIT Press.

Debreu, G. (1951) 'The coefficient of resource utilization', *Econometrica*, 19: 273–292.

Debreu, G. (1959) *Theory of value*. New York: Wiley.

Debreu, G. (1970) 'Economies with a finite set of equilibria', *Econometrica*, 40: 387–392.

Debreu, G. (1972) 'Two remarks on the number of equilibria of an economy', *Econometrica*, 40: 951–953.

Dem'yanov, V.F. and A.M. Rubinov (1986) *Quasidifferential calculus*. New York: Optimization Software.

Dierker, E. (1972) 'Two remarks on the number of equilibria of an economy', *Econometrica*, 40: 951–953.

Dierker, E. (1986) 'When does marginal cost pricing lead to Pareto-efficiency?', *Zeitschrift für Nationalökonomie*, Suppl. 5: 41–66.

Dierker, E. (1989) 'The optimality of Boiteux-Ramsey pricing', Discussion paper no. A-159, University of Bonn, Bonn.

Dierker, E., R. Guesnerie and W. Neuefeind (1985) 'General equilibrium where some firms follow special pricing rules', *Econometrica*, 53: 1369–1393.

Dubovickii, A.J. and A. Miljutin (1965) 'Extremum problems in the presence of restrictions', *Zh. Vychisl. Mat. Fix.* 5: 395–453; and *USSR Computational Mathematics and Mathematical Physics* 5: 1–80.

Eaves, B.C. (1972) 'Homotopies for computation of fixed points', *Mathematical Programming*, 3: 1–22.

Gale, D. (1955) 'The law of supply and demand', *Mathematica Scandinavica*, 3: 155–169.

Garcia, C.B. and W.I. Zangwill (1981) *Pathways to solutions, fixed points, and equilibria*. Englewood Cliffs, NJ: Prentice Hall.

Guesnerie, R. (1975) 'Pareto-optimality in nonconvex economies', *Econometrica*, 43: 1–29.

Hildenbrand, W. and A. Kirman (1988) *Equilibrium analysis*. Amsterdam: North-Holland.

Hotelling, H. (1938) 'The general welfare in relation to problems of taxation and of railway and utility rates', *Econometrica*, 6: 242–269.

Hurwicz, L. and S. Reiter (1973) 'The boundness of the feasible set without convexity assumption', *International Economic Review*, 14: 580–586.

Jerison, M. (1984) 'Aggregation and pairwise aggregation of demand when the distribution of income is fixed', *Journal of Economics Theory*, 33: 1–31.

Kamiya, K. (1986a) 'Existence, uniqueness, and computation of equilibria with increasing returns', Ph.D. Dissertation, Yale University.

Kamiya, K. (1986b) 'Computation of equilibria in an economy with increasing returns to scale technologies', CORE discussion paper no. 8648, Université Catholique Louvain, Louvain-La-Neuve.

Kamiya, K. (1987) 'The decomposition method for systems of nonlinear equations', CORE discussion paper no. 87214, Université Catholique de Louvain, Louvain-La-Neuve.

Kamiya, K. (1988a) 'Existence and uniqueness of equilibria with increasing returns', *Journal of Mathematical Economics*, 17: 149–178.

Kamiya, K. (1988b) 'On the survival assumption in marginal (cost) pricing', *Journal of Mathematical economics*, 17: 261–274.

Kahn, M. Ali and R. Vohra (1987a) 'On the existence of Lindahl–Hotelling equilibria', *Journal of Public Economics*.

Kahn, M. Ali and R. Vohra (1987b) 'An extension of the second welfare theorem to economies with nonconvexities and public goods', *Quarterly Journal of Economics*, 102: 223–241.

Mantel, R. (1979) 'Equilibrio con rendimiento crecientes a escala', *Anales de la Asociation Argentine de Economia Politica*, 1: 271–283.

MacKinnon, J. (1979) 'Computing equilibria with increasing returns', *European Economic Review*, 12: 1–16.

McKenzie, L.W. (1954) 'On equilibrium in Graham's model of world trade and other competitive system', *Econometrica*, 22: 147–161.

Nikaido, H. (1956) 'On the classical multilateral exchange problems', *Metroeconomica*, 8: 135–145.

Oi, W. (1971) 'A Disneyland dilemma: two-part tariffs for a Mickey Mouse monopoly, *Quarterly Journal of Economics*, 85: 77–90.

Phlips, L. (1983) *The economics of price discrimination*. Cambridge: Cambridge University Press.

Quinzii, M. (1991) 'Efficiency of marginal cost pricing equilibria', in: W. Brock and M. Majumdar, eds., *Equilibrium and dynamics: essays in honor of David Gale*. New York: Macmillan.

Ramsey, F. (1927) 'A contribution of the theory of taxation', *Economic Journal*, 37: 47–61.

Rockafellar, R.T. (1981) *The theory of subgradients and its applications to problems of optimization: convex and nonconvex functions*. Berlin: Heiderman.

Ruggles, N. (1949) 'The welfare basis of the marginal cost pricing principle', *Review of Economic Studies*, 17: 29–46.

Ruggles, N. (1950) 'Recent developments in the theory of marginal cost pricing', *Review of Economics Studies*, 17: 107–126.

Rutherford, T. (1988) 'General equilibrium modeling with MPS/GE', manuscript: University of Western Ontario.

Scarf, H. (1967) 'The approximation of fixed-points of a continuous mapping', *Siam Journal of Applied Mathematics*, 15: 1328–1343.

Scarf, H. (with collaboration of T. Hansen (1973)) *Computation of economics equilibria*. New Haven, CT: Yale University Press.

Scarf, H. (1986) 'Neighborhood systems for production sets with indivisibilities', *Econometrica*, 54: 507–532.

Scarf, H. and J.B. Shoven, eds. (1984) *Applied general equilibrium, analysis*. Cambridge: Cambridge University Press.

Shoven, J.B. and J. Whalley (1984) 'Applied general equilibrium models of taxation and international trade', *Journal of Economics Literature*, 22: 1007–1051.

Smale, S. (1974a) 'Global analysis and economies IV: finiteness and stability of equilibria with general consumption sets and production', *Journal of Mathematical Economics*, 1: 119–128.

Smale, S. (1974b) 'Global analysis and economies II A: extension of a theorem of Debreu', *Journal of Mathematical Economics*, 1: 1–14.

Smale, S. (1976a) 'Global analysis and economies VI: geometric analysis of Pareto optima and price equilibria under classical hypotheses', *Journal of Mathematical Economics*, 3: 1–14.

Smale, S. (1976b) 'A convergent process of price adjustment and global Newton methods', *Journal of Mathematical Economics*, 3: 107–120.

Vohra, R. (1988a) 'Optimal regulation under fixed rules for income distribution, *Journal of Economics Theory*, 45: 65–84.

Vohra, R. (1988b) 'On the inefficiency of two part tariffs', Working Paper No. 88-25, Brown University, Providence.

Wald, H.P. (1945) 'The classical indictment of indirect taxation', *Quarterly Journal of Economics*, LIX: 577–596.