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COMPUTING EQUILIBRIA WHEN ASSET MARKETS ARE INCOMPLETE

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Existence of equilibrium with incomplete markets is problematic because demand functions are typically not continuous. Discontinuities occur at prices for which a marketed asset suddenly becomes redundant. We show that this discontinuity disappears if we allow an agent in the economy to introduce a new asset when such redundancies occur. This enables us to prove existence with incomplete markets using a standard path-following argument. Hence, available algorithms for path-following in R^K can be applied to compute equilibria in the GEI case. We demonstrate this by computing equilibrium for a numerical example.

KEYWORDS: General equilibrium theory, incomplete markets, path following, homotopy.

1. INTRODUCTION

IN THIS PAPER WE INVESTIGATE the existence and computation of general equilibrium for economies with incomplete asset markets. The general equilibrium with incomplete markets (GEI) model follows the standard Arrow-Debreu model in most of its methodological assumptions: agents optimize based on rational expectations, there is perfect competition, and markets clear. But the GEI model is much richer in its ability to describe and analyze important economic phenomena, such as financial innovation, the relevance of corporate control and financial policy, and the potential positive role for government intervention. Moreover, this generality is achieved by dropping perhaps the most disturbing and patently invalid assumption of the Arrow-Debreu framework—that agents can trade today assets that allow for every possible contingency that might occur tomorrow.

There are many reasons why asset markets may be incomplete, including asymmetric information, moral hazard, and transactions costs (see Geanakoplos (1990) and Magill and Shafer (1991) for an excellent discussion). Of the many consequences of this incompleteness, perhaps one of the most important is that general equilibrium outcomes are no longer Pareto efficient.[2] Not only does this

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[2] In fact, Geanakoplos and Polemarchakis (1986) show they are no longer even *constrained* Pareto efficient in the sense that a social planner restricted to the same set of available assets can achieve a Pareto improvement.

result illustrate the fact that a perfectly competitive market process need not imply market efficiency, it also calls into question the usefulness of standard representative agent and/or single commodity models in policy analysis.

These limitations, together with the empirical failure of the representative agent model, have recently led economists in these areas to consider heterogeneous agent models with incomplete markets—see Den Haan (1993) for a recent discussion of this literature. In both the public finance literature (e.g., Newberry and Stiglitz (1981)) and the trade literature (see Eaton and Grossman (1985)) there has been a discussion of the role of taxes in improving efficiency when markets are incomplete. The GEI model and its extensions to stochastic exchange economies and stock market economies offer a class of models rich enough to accommodate the concerns of macroeconomists studying short-term interest rates in incomplete asset markets; trade theorists investigating commercial policy when markets are incomplete; and public finance economists considering commodity price stabilization or optimal commodity taxation in the presence of market incompleteness, as in Diamond and Mirlees (1992).

Unfortunately, despite its importance there has until very recently been relatively little attention given to the computation of equilibria in these models.[3] Hence economists have had to rely on highly parameterized models with restrictive functional forms for their comparative statics analyses. Such analyses, because of the strong assumptions needed for analytical tractability, fail to capture the rich complexity of dynamic stochastic models with rational expectations and heterogeneous agents. Not only are computational techniques needed for those macro/finance models but they are also needed for the counterfactual analysis necessary for evaluating tax policy in the public finance or trade literature. In this paper we present a computational existence proof for equilibria in the GEI model.

In the Arrow-Debreu model, equilibrium can be formulated as the zero of the market excess demand function on the interior of the price simplex. Importantly, the market excess demand function is smooth (or at least continuous) on this domain. By contrast, in the GEI model market excess demand need not be continuous since for some prices the assets can have redundant returns, causing the agents' budget sets to shrink suddenly. This problem of the drop in rank of the asset returns matrix was first pointed out by Hart in his seminal paper (1975), in which he showed that equilibria may fail to exist.

Duffie and Shafer (1985) overcome this problem and establish the generic existence of equilibria in the GEI model by reformulating the equilibrium notion and expanding the domain to include both prices and the Grassmannian manifold of N dimensional subspaces of R^S. The Grassmannian is used to represent the span of the returns of the N available assets. This approach gives rise to a smooth market excess demand, but its domain is no longer convex, nor a Euclidean space. Thus, the current proofs of existence are based on abstract

[3] See, however, Den Haan (1993), Judd (1991) and Lucas (1994) for computational methods for somewhat different models than considered here.

degree-theoretic arguments which do not require convexity of the domain of the equilibrium equations; e.g., see Husseini, Lasry, and Magill (1990), Geanakoplos and Shafer (1990), and Hirsch, Magill, and Mas-Colell (1990).

In this paper we propose an alternative approach. The difficulty of the GEI model stems from the discontinuity which occurs when assets become redundant and the return matrix consequently drops rank. We show that a solution to this problem is to allow one agent in the economy to introduce a new asset for trade when such redundancies occur. Intuitively, this "auxiliary asset" maintains the rank of the asset returns matrix available for trade, and thus enlarges the set of prices for which aggregate demand is continuous. We then show formally that this is sufficient to establish the generic existence of an equilibrium via a standard homotopy argument in Euclidean space.

Additionally, we then use this homotopy argument to construct a "path following" algorithm for computing equilibrium.[4] The algorithm can be described intuitively as follows: We begin with an equilibrium for a single agent economy. We then gradually increase the relative size of the remainder of the economy, adjusting prices to maintain equilibrium. If we begin to approach prices for which one of the assets is redundant, that asset is removed and replaced by a new asset chosen by our original agent. Once we have "passed by" the potential redundancy, we then switch back to the original set of assets. We continue this procedure until we reach an equilibrium for the full economy.

Of course, no claim of a computational algorithm is complete without an example. We therefore test our method on an economy for which the discontinuity problem is a serious one. In addition to successful computation of an equilibrium, the numerical results also reveal several insights into the nature of these models.

The paper is organized as follows. Section 2 presents a basic description of the GEI model and the definition of equilibrium. Section 3 gives an intuitive presentation of the difficulty in computing an equilibrium for the GEI model, and our solution to it. This intuition is formalized in Section 4, in which we define the homotopies that describe our solution path. Finally, Section 5 discusses the computational implementation of our results.

2. THE GEI MODEL

The basic model consists of a two period exchange economy with uncertainty. In each period a finite number of commodities are available and uncertainty in the second period is characterized by a finite number of possible states. There are a finite number of real assets marketed, where in each state an asset's payoff is a bundle of available commodities. Each state defines a real spot market and agents may transfer income between periods by purchasing portfolios of assets

[4] Continuation or path following methods for solving nonlinear systems of equations defined on open subsets of R^K are well known and have been used to solve for equilibria in the Arrow-Debreu model—see Kehoe's survey (1991).

in the first period and realizing returns in the second period spot markets. Markets are said to be incomplete if the number of marketed assets is less than the number of states.

We adopt the following notation:

H = number of households,
S = number of states in the second period,
N = number of assets,
L = number of commodities (per state),
M = number of prices/commodities ($M = L(S+1)$),

and we assume $S \geq N$. In addition, let $\Delta^M_{++} = \{p \in R^M\colon\ p > 0,\ \Sigma_i p_i = 1\}$ be the interior of the price simplex in R^M. Also let $B^K = \{\theta \in R^K\colon\ \theta \cdot \theta = 1\}$, the set of unit vectors in R^K.

Given prices $p \in \Delta^M_{++}$, let p_0 be the L vector of prices for time 0 consumption, and p_{1s} be the L vector of prices for time 1 consumption contingent on state s. Additionally, we will define

$$P_1 = \begin{bmatrix} 0 & p_{11} & 0 & \cdots & 0 \\ 0 & 0 & p_{12} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_{1S} \end{bmatrix} \in R^{S \times M}.$$

The assets in the economy are represented by the M by N matrix A, where the payoffs of the jth marketed asset in each commodity in each state are the elements of the jth column. Given our definition of P_1, the state-by-state nominal return of asset j is given by $P_1 A_j$, an S vector. Thus, we define the nominal return matrix for the economy as

$$R(p) = P_1 A \in R^{S \times N}.$$

Additionally, the time 0 dividend of each asset is given by $R_0(p) = [p_0\ \ 0]A \in R^N$. We suppose these assets are available for trade at time 0 at some prices $\pi \in R^N$.

The economy can be characterized via the excess demand function of each household. An agent's consumption plan consists of a consumption vector for the first period and a state contingent consumption vector for the second period. We assume that agents choose their plans to maximize utility over their budget sets where agents face a budget constraint in each state of the world. Given utility function U_h and endowment e_h, household h chooses excess demand z and a portfolio ϕ to solve:

$$(1) \qquad \max_{z,\phi} U_h(e_h + z), \quad \text{subject to}$$

$$p_0 \cdot z_0 + \pi\phi \leq R_0(p)\phi,$$

$$P_1 \cdot z \leq R(p)\phi.$$

Of course, under standard montonicity assumptions, this problem will have a solution only if the assets prices π do not admit an arbitrage opportunity. Because the absence of arbitrage is equivalent to (see Ross (1977))

$$\pi = R_0(p) + qR(p) \quad \text{for some} \quad q \in R^S_{++}, \tag{2}$$

we could equivalently define excess demand in terms of commodity prices p and "state prices" q. Since the budget constraint in (1) is homogeneous in p_s, however, it is even simpler to embed the state prices q in the price vector p by rescaling each p_s by q_s. Using this notion of prices it is possible to restate (1) as follows:

$$\begin{aligned} &\max_z U_h(e_h + z), \quad \text{subject to} \\ &p \cdot z = 0, \\ &P_1 \cdot z = R(p)\phi \quad \text{for some } \phi. \end{aligned} \tag{3}$$

We let $Z^h(p, R(p))$ in R^M denote the excess demand of household h which solves (3). We do not derive, but merely state the following properties of these excess demand functions under standard preference assumptions:[5]

1. Walras' Law: $pZ^h(p, R(p)) = 0$.
2. Market Span Constraint: $P_1 Z^h(p, R(p)) \in \text{ span } R(p)$.
3. Smoothness: Z^h is a smooth function of p when $R(p)$ has full rank N.

Walras' law follows as usual from the agent's budget constraint. The market span constraint is the consequence of our incomplete markets model; only certain income transfers across states are feasible given the available assets. The set of feasible transfers are those which can be financed by an appropriate portfolio of assets. This feasible set is equal to the span of the return matrix $R(p)$, which is referred to in this literature as the "market span." Finally, the smoothness condition follows from the fact that when $R(p)$ has full rank, it has full rank in a neighborhood of p, and thus in this neighborhood the market span is an N-dimensional subspace which varies smoothly with p. If some of the assets' returns become redundant with the others and $R(p)$ drops rank, however, the market span suddenly collapses by a dimension (or more), resulting in a discontinuity in the budget set and hence in excess demand.

Unfortunately, the rank of the asset returns matrix is not a continuous function of the spot prices. Hence, individual excess demands and consequently aggregate excess demands are, in general, not a continuous function of market prices for commodities. It is this lack of continuity that distinguishes the problem of existence of equilibria in incomplete market economies from the problem of existence of equilibria in complete contingent market economies.

Given the excess demand functions (Z^h) and the parameters for the economy $(e, A) \in R^{M \times H}_{++} \times R^{M \times N} = W$, we define an equilibrium for the economy

[5] We also remark that Z^h is easily computed as the solution to a concave optimization problem with linear constraints.

(Z^h, e, A) as a price vector such that aggregate excess demand equals zero:

$$\sum_h Z^h(p, R(p)) = 0. \tag{4}$$

This definition of equilibrium, while correct, is inconvenient for two reasons. First, it exhibits indeterminancy in prices: one can change the price levels across states of the world without changing budget sets or excess demand.[6] Second, aggregate excess demand need not be proper in this formulation—reducing a price in state s to zero does not necessarily lead excess demand to diverge. Properness (the divergence of excess demand when prices converge to the boundary of the simplex) is typically an important property in establishing existence of an equilibrium.

Fortunately, both of these difficulties can be resolved via an alternative definition of equilibrium. The so-called "Cass trick" is to allow one agent in the economy to be unconstrained by the available assets; that is, we allow this agent to act as though s/he were facing complete markets. Call this unconstrained agent u. Then define

$$Z^u(p) = Z^u(p, I_{S\times S}) \quad \text{and} \quad Z^c(p, R(p)) = \sum_{h \neq u} Z^h(p, R(p)).$$

That is Z^u is the excess demand of the unconstrained agent facing prices p (and complete markets), and Z^c is the aggregate excess demand of the remaining constrained agents. Given this definition, we rewrite the equilibrium condition (4) as follows:

$$Z^u(p) + Z^c(p, R(p)) = 0. \tag{5}$$

To see that (4) and (5) are equivalent, note that when (4) holds, (5) holds if we rescale price levels in each state according to the marginal rates of substitution for the unconstrained agent. Second, when (5) holds, the fact that Z^c is in the market span implies that Z^u is also in the market span, so that (4) holds as well. Moreover, this definition solves the two problems previously mentioned. Price levels are now determined by the marginal rates of substitution of the unconstrained agent. Properness follows from the fact that the unconstrained agent's excess demand diverges at the boundary of the simplex, and the excess demand of the constrained agents is bounded below.[7] The remainder of the paper explores the existence and computation of a solution to (5).

3. AN INTUITION

Homotopy, or path following, methods offer one natural approach to finding a price vector solving the equilibrium condition (5). This method involves defining

[6] Note that the indeterminancy is strictly nominal. Essentially, the problem stems from the fact that with incomplete markets, there exist multiple state prices that satisfy (2).

[7] See Geanakoplos (1990) for a further discussion of the "Cass trick." A referee points out that the trick and the associated concept of equilibrium were introduced by Magill and Shafer in 1984–85 and subsequently published in Magill and Shafer (1990).

a family of equations $H(p,t)$ indexed by t such that when $t=0$ there exists a known, unique solution to $H(p,0)=0$, and which is smoothly deformed until $t=1$ into the equilibrium condition of interest. Under appropriate regularity conditions, standard path following techniques can then be used to follow the path from the known solution to an equilibrium of the original problem.

Given our problem, a natural candidate homotopy on which to apply path following is the following:

$$H(p,t)=Z^u(p)+tZ^c(p,R(p)). \tag{6}$$

Note that for $t=0$, this system reduces to a single agent economy, whose equilibrium price is uniquely given by the supporting prices p^u at the first agent's endowment. Starting from this initial solution $(p^u,0)$, a path of solutions to $H(p,t)=0$ (i.e., a path in $H^{-1}(0)$) can be computed by numerically solving the basic differential equation (BDE) defined by

$$H_p\dot{p}+H_t\dot{t}=0. \tag{7}$$

If one tries to follow this approach for the GEI model, however, one quickly realizes that it is inadequate. The problem, of course, is that the excess demand of the constrained agents is discontinuous at prices for which the return matrix $R(p)$ is singular. Thus, the homotopy H defined above can only be smooth on the domain of prices for which $R(p)$ is nonsingular.

Nevertheless, one could pursue this approach and hope that singular prices do not arise in practice. Indeed, it is not difficult to show the following:

PROPOSITION: *For generic economies, $H^{-1}(0)$ contains a smooth path from $(p^u,0)$ to either an equilibrium $(p^*,1)$ or a point (p,t) such that $R(p)$ is singular.*

We will not prove this result here, but simply discuss it to provide some intuition for the approach we will take subsequently. On the domain of prices for which $R(p)$ is nonsingular (an open subset of the price simplex), one can show $H^{-1}(0)$ is (generically) a smooth one-dimensional manifold via an application of the implicit function theorem. Thus the path containing $(p^u,0)$ must terminate on some other boundary of the domain. Since the solution for $t=0$ is unique, the path cannot return to this boundary. Also, if prices converge to the boundary of the simplex, excess demand for some good must diverge, ruling out this possibility. Thus, the only remaining alternatives are that the path converges to a "bad" price for which $R(p)$ is singular, or to the boundary $t=1$, yielding an equilibrium for the original economy. See Figure 1.

Naturally, then, this approach is practical only if there is some way of guaranteeing that hitting a "bad" price is nongeneric. Though it might appear that a small perturbation of the economy would perturb the path away from a bad price, this is in fact not the case. The reason for this is that when $R(p)$ is singular, nearby points imply drastically different market spans, and hence lead to large changes in the excess demand of the constrained agents. Thus, it will in

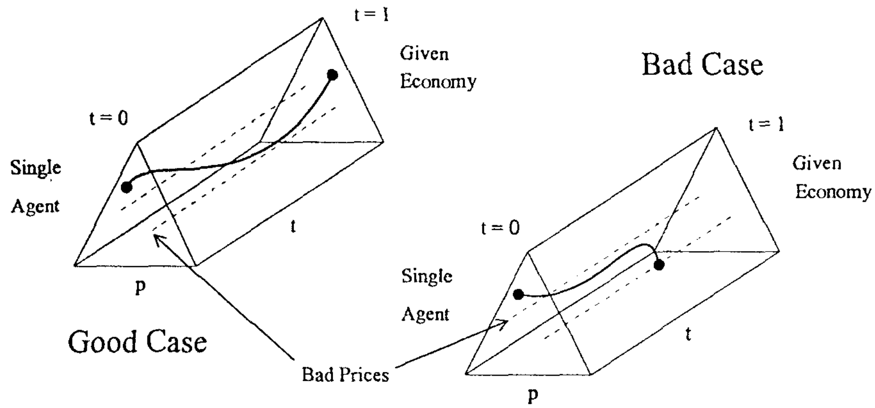

FIGURE 1.—Path following with the "naive" homotopy.

general not be possible to perturb the path "around" a bad price. We shall elaborate on this point in the context of an example.

3.1. *A Numerical Example*

In order to understand the nature of the problem, and our solution to it, it is useful to examine a simple example. For clarity, we take the simplest parameterization necessary to produce an interesting discontinuity "problem" for the algorithm to resolve. In particular, we suppose that there are three future states ($S = 3$), there are two available assets ($N = 2$), and there are two commodities ($L = 2$). In addition, there are two types of agents, A and B, each with utility of the following form ($h = A, B$):

$$U_h(x) = -\sum_{s=0}^{S} \lambda_s \left(K - \prod_{l=1}^{L} (x_{sl})^{\alpha_l^h} \right)^2 .$$

Types A and B differ in terms of their relative preferences for each commodity (determined by α^h) and their endowments (e^h). Specifically, we let $K = 5.7$, $\lambda = [1, 1/3, 1/3, 1/3]$, $\alpha^A = [1/4, 3/4]$, $\alpha^B = [3/4, 1/4]$, and endowments $e^A = [2,2; .5,1; 1,1; 1.5,1]$ and $e^B = [1,1; 2.5,2; 2,2; 1.5,2]$.

The two assets are simply chosen as forward contracts for each good. That is, asset 1 delivers one unit of good 1 in each state, and asset 2 delivers one unit of good 2 in each state. Thus, these two assets become redundant if and only if relative prices are constant across states.

Finally, we suppose that in the actual economy of interest there are twice as many agents of type B as there are of type A. Note that if there are equal numbers of agents of each type, aggregate endowments are constant and symmetric, so that if markets were complete relative prices would indeed be constant in equilibrium.

We compute the path for this economy defined by the BDE (7). In this case it is natural to let the homotopy parameter represent the relative number of type

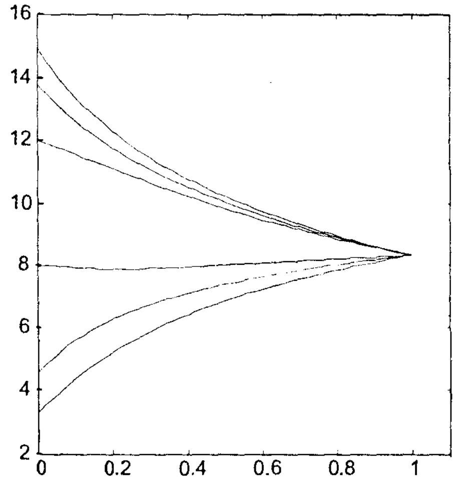

FIGURE 2.—Period 2 prices (p_{1s}) versus T ($\Sigma p = 100$).

B in the economy; hence we let type A be unconstrained and allow $t \in [0,2]$. Figure 2 plots period 2 prices versus t along the computed path.

In this case, the path converges to a bad price at $t=1$. At this point, with constant period 2 prices, unconstrained type A would like to "insure" against state 1. In fact, if we compute A's desired income transfers at these prices, A would like to purchase a security with payoffs proportional to $[1.5, 1.0, 0.5]$ across the 3 states. On the other hand, because prices are constant, the constrained type B is restricted to trade only *riskless* assets. Thus, excess demand jumps discontinuously and H is suddenly far from zero.

For another view of this scenario, let $Y^u \equiv P_1 Z^u$ and $Y^c \equiv P_1 Z^c$ denote the desired income transfers of the unconstrained and constrained agents. Then along the path, $Y^u + tY^c = 0$ and $Y^c \in \mathcal{M}$, the market span defined by the return matrix $R(p)$ for assets 1 and 2.

Consider what happens as we approach $t=1$. Almost everywhere along the path the market span $\mathcal{M}$ is the two-dimensional plane within which the asset returns lie (Figure 3a). As the returns of the two assets cross each other and become redundant, however, the asset return matrix is singular and the market

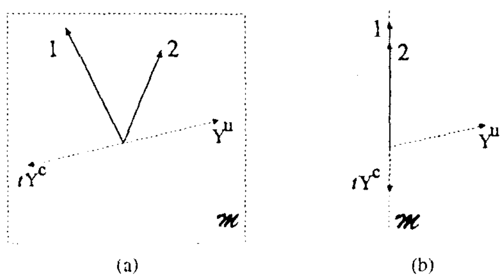

FIGURE 3.—Example configurations along a path.

span suddenly drops from the two-dimensional plane to a one-dimensional line (see Figure 3b). At this point, the constrained agents' expenditure vector is forced to lie in this line. Thus aggregate excess demand is suddenly far from zero.

This example also makes clear the fact that a simple perturbation is not sufficient to solve the problem. One could perturb the two asset returns so that rather than crossing each other, they go around each other (by leaving the plane of the page). The trouble, of course, is that as they go around each other, the market span rotates from the two-dimensional plane of the page up to a two-dimensional plane perpendicular to the page and then back down to the plane of the page. Because the expenditures of the constrained agents must follow the market span as it leaves the page, whereas Z^u is unaffected, aggregate excess demand again necessarily moves far from zero. This suggests that the problem is robust to our particular parameterization.[8]

Thus our attempt at path following gets "stuck" at $t = 1$. Since our target economy has $t = 2$, we have failed to compute an equilibrium.

3.2. *A Proposed Solution*

The approach of the prior literature on incomplete markets has been to introduce the Grassmannian manifold of N-dimensional subspaces of $\boldsymbol{R}^s$ as a means of representing the market span separately from the asset returns themselves.[9] Constrained excess demand is then taken as $Z^c(p, \mathscr{L})$, a function of prices p and the market span, $\mathscr{L}$, an element of the Grassmannian. Excess demand is thus a smooth function on its domain, which now includes the Grassmannian manifold. Finally, in order to be sure that $\mathscr{L}$ remains consistent with the true asset returns, it is necessary to add an additional set of equations that ensures the span of $R(p)$ is contained in $\mathscr{L}$. Thus $R(p)$ can drop rank without affecting the market span. But while this approach has proved successful in establishing the existence of an equilibrium, it is not directly amenable to computation.[10] Thus we seek an alternative approach that avoids the computational complexity of the Grassmannian.

[8] This may seem surprising since it may seem as though we exploited symmetry in our example to get constant relative prices at $t = 1$. The intuition provided here is correct, however. We will verify this robustness in Section 5.

[9] See the special issue of the *Journal Of Mathematical Economics* on GEI (1990).

[10] This is not to say that computing on the Grassmannian is impossible, only substantially more complicated; the current paper demonstrates that this added complexity is not necessary in the GEI model. However, readers interested in the issues which arise when computing with the Grassmannian may wish to see Brown, DeMarzo, and Eaves (1993), where we present a computational existence proof of the subspace fixed-point theorem. This theorem states, approximately, that a continuous map of the Grassmannian in $\boldsymbol{R}^K$ has a fixed point, i.e., there exists a subspace in the Grassmannian such that the image of that subspace under the given map lies in the subspace. (Both the Brouwer fixed-point theorem and the Borsuk-Ulam theorem are corollaries of the subspace fixed-point theorem.) A discussion of the relationship between the existence of equilibria in the GEI model and the subspace fixed-point theorem can be found in Husseini et al. (1990) or Hirsch et al. (1990). See also DeMarzo and Eaves (1994) for a computational implementation applied to the GEI model.

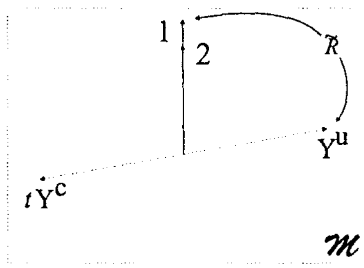

FIGURE 4.—Adding Y^u as a new asset.

The intuition of our approach can be seen by returning to Figure 3. What is needed is a way of recovering the two-dimensional market span at the bad price. Just before and just after the two assets cross, the market span is equal to the plane of the page. If we could recover this plane at the bad price, excess demand would not jump away from zero. Our observation is that this market span can be recovered by selectively introducing a new asset for agents to trade. In particular, note that if we replace one of the assets with the excess demand of the unconstrained agent, the market span again becomes the full plane of the page. See Figure 4.

Thus, if we think of the market span as being formed by the return matrix $\tilde{R}$ of assets and 1 and Y^u rather than by R, the price above is no longer "bad." Excess demand with respect to this alternative set of assets is smooth in this neighborhood. (Recall in our example that at the bad price both securities are riskless but the unconstrained agent would like to trade a risky security with payoffs [1.5, 1, .5].)

Naturally, defining the market span in terms of asset 1 and excess demand Z^u is only satisfactory as long as these two vectors remain independent. If they become dependent it is necessary to switch back to defining the market span via assets 1 and 2.

More formally, one can think of the condition $H = 0$ as equivalent to the following combination of conditions:

1. $R(p)$ determines the market span $\mathscr{M}$.
2. Expenditures Y^u and Y^c lie in $\mathscr{M}$.
3. $Z^u + tZ^c = 0$.

We propose the following modest generalization of these conditions as follows: Let $R_{-i}(p)$ be $R(p)$ with the ith column deleted. Then these conditions could alternatively be written:

1. $R_{-i}(p)$ and Y^u determine the market span $\mathscr{M}$.
2. $R_i(p)$ and Y^c lie in $\mathscr{M}$.
3. $Z^u + tZ^c = 0$.

One can see that most of the time these sets of conditions are equivalent (see, for example, the first configuration in Figure 3). They differ precisely when one has reached a "bad" price with respect to one of the definitions of the market span. At such points it is convenient and natural to resort to a definition of the market span for which this is not a "bad" price; i.e., choose the definition that leads to a market span of full dimension N.

Introducing the unconstrained agent's excess demand as a new asset when other assets are redundant is a means of smoothly moving through a bad price. The final issue that must be considered is whether this one potential new asset is sufficient. What happens if even after adding the new asset, the asset return matrix is still singular? We must argue that such cases are sufficiently rare; that is, that such cases are nongeneric and can be avoided by a small perturbation of the economy.

In Figure 5, case (a) depicts the problematic configuration in which even adding Y^u is not sufficient to restore the two dimensional market span $\mathscr{M}$. Illustration (b) demonstrates, however, that it is possible to perturb the payoffs of the assets slightly so that they do not both cross Y^u simultaneously. Unlike the perturbation contemplated in the case of Figure 3, this perturbation does not require the market span to leave the plane of the page. Thus, at least intuitively, it should be possible to avoid such "doubly bad" prices.

This completes the intuitive description of our results. In the next section, we formally define the family of homotopies corresponding to this intuition. We will define $N+1$ different homotopies, each corresponding to a different choice of N asset returns from the set of $N+1$ available asset returns $[R(p), Y^u(p)]$. Each of these homotopies is smooth on the domain of prices for which its asset return matrix has full rank. We then show that each of these homotopies defines a one dimensional path in (p,t), and that these paths coincide except at bad points. Finally, we show that generically, this path never converges to a point which is bad with respect to every homotopy. This implies that, by switching homotopies when necessary, this path must eventually lead to $t=1$ and an equilibrium price vector. We then apply this technique to resolve the discontinuity problem of our previous example, and compute a true equilibrium for the economy.

4. A FORMAL PROOF

As discussed in the previous section, we wish to consider the excess demand of the unconstrained agent as a potential marketed asset in the economy. Thus

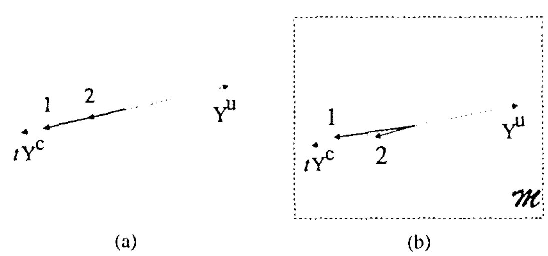

FIGURE 5.—Perturbing a "doubly bad" point.

we define an augmented asset return matrix as follows:

$$R^*(p) = P_1[A, Z^u(p)] = [R(p), Y^u(p)] \in R^{S\times(N+1)},$$

and let $R^*_{-i}(p)$ be the S by N matrix formed by deleting the ith column of R^*. It is natural to define $u = N+1$ so that we can write $R^*_{-u} \equiv R^*_{-(N+1)} \equiv R(p)$.

We will now construct a family of homotopies indexed by i for $i = 1, 2, \ldots, u = N+1$. Each homotopy H_i is defined based on the market span determined by R^*_{-i}, when that matrix has full rank. In this case we can reformulate the homotopy (6) as

$$Z^u(p) + tZ^c(p, R^*_{-i}(p)).$$

Recall that one of our equilibrium constraints is that the deleted asset return R_i, should also lie in the market span. Equivalently, we require that the augmented return matrix R^* have at most rank N. This is represented by the constraint

$$R^*(p)\theta = 0,$$

for some unit vector $\theta \in B^{N+1} \equiv \{\theta \in R^{N+1}: \theta \cdot \theta = 1\}$.

Let $X = \Delta^M_{++} \times [0,1] \times B^{N+1}$, a smooth manifold of dimension $M+N$. Next we define the manifold E_i of "good" points with respect to R^*_{-i} as follows:

$$E_i = \{(p, t, \theta, w) \in X \times W: R^*_{-i}(p) \text{ has full rank } N\},$$

where $w = (e, A)$ in W represents the parameterization (endowments and assets) of the economy.

Note that E_i is an open (dense) subset of $X \times W$ (and thus is a smooth submanifold of Euclidean space). Finally, we can then define the family of homotopies H_i: $E_i \to R^{M+S}$ by

$$H_i(p, t, \theta, w) = \begin{bmatrix} Z^u(p) + tZ^c(p, R^*_{-i}(p)) \\ R^*(p)\theta \end{bmatrix}. \tag{8}$$

Each H_i is well-defined and smooth on its domain E_i. We refer to the first set of equations, which depends on i, as H_{i1}, and the second set, which is the same for all i, as H_2.

4.1. *Existence of Equilibrium in the GEI Model*

Our method of proof proceeds in several steps. First, we show that for generic w and for each i, the solution set $H_i^{-1}(0)$ defines a one-dimensional manifold on its domain. Then we show that these manifolds can be overlaid to form a path in X. Finally, we argue that given the unique solution for $t = 0$, this path must continue until it reaches $t = 1$, yielding an equilibrium price vector.

We begin with a characterization of $H_i^{-1}(0)$. As usual, such a characterization relies on the implicit function theorem. In the usual application of this theorem, one is faced with a set of k independent equations, and (subject to regularity conditions) concludes that the solution set has co-dimension k in the domain.

The problem with a direct application of this theorem to general equilibrium theory is that the equilibrium conditions as usually formulated are not independent. In particular, excess demand always satisfies Walras' Law, implying that the excess demand for each good is not independent. The GEI model exacerbates this, since in addition we know that expenditures must lie in the market span. These constraints imply further dependency across our equilibrium equations. Because this situation is prevalent in economics, we first develop a useful generalization of the implicit function theorem to deal with Walrasian type dependencies across equations. Our result is as follows:

THEOREM I: *Let* $F: U \to R^m$ *be a smooth function on the manifold* $\mu \in U$. *Let* $\Pi: U \to R^{t \times m}$ *and* $G: U \to R^{t \times n}$ *be smooth functions such that, for all* $\mu \in U$, $\Pi(\mu)$ *and* $G(\mu)$ *have constant rank* t *and* n *respectively. If*:

1. $F = 0$ *implies rank* DF *is at least* $k = m + n - t$, *and,*
2. $\Pi(\mu)F(\mu) \in$ *span* $G(\mu)$ *for all* μ,

then $F^{-1}(0)$ *is a smooth submanifold of* U *with co-dimension* k. *In addition, rank* $DF = k$ *on this set.*

PROOF: Condition 2, together with the rank conditions on Π and G, imply that F lies in a linear subspace of R^m of dimension k. Thus, $F \in$ span B for some full rank $m \times k$ matrix B. Moreover, it is easy to show that $B(\mu)$ can be constructed smoothly on a neighborhood V of μ.[11] Define $\rho = (B'B)^{-1}B'F \in \mathbf{R}^k$. Then $F(\mu) = B(\mu)\rho(\mu)$ on V. Note that $F = 0$ if and only if $\rho = 0$. Also, when $F = 0$, by the chain rule $DF = DB\rho + BD\rho = BD\rho$. Hence condition 1 implies rank $DF =$ rank $D\rho = k$. The proof is complete since we can now apply the Implicit Function Theorem (see Appendix) to show that $\rho^{-1}(0) = F^{-1}(0)$ is a smooth submanifold of V of co-dimension k. *Q.E.D.*

This theorem adjusts for the dependencies in the system in equations implied by the restriction that $\Pi F \in$ span G. Note that since n is less than or equal to t, this always has the effect of reducing the co-dimension (the system is less restrictive). Finally, note that Walras' Law is represented by the case $\Pi = p$ and $G = 0$.

This result allows us to establish the following characterization:

THEOREM II: $H_i^{-1}(0)$ *is a smooth submanifold of* E, *of dimension* $W + 1$.

PROOF: We consider separately the cases $i \leq N$ and $i = u = N + 1$.

Case $i = 1, 2, \ldots N$:

[11] The matrix B can be constructed as follows. Let $\Pi^{\perp}$ be a full rank $(m - t) \times m$ matrix orthogonal to Π. Note that $\Pi^{\perp}$ can be constructed smoothly on a neighborhood V. (To see this, suppose w.l.o.g. $\Pi = [A \;\; C]$ where A is $t \times t$ and full rank; then $\Pi^{\perp} = [-(A^{-1}C)' \;\; I]$ is smooth in the neighborhood on which A maintains full rank.) Thus we can write $F = \Pi'\phi + \Pi^{\perp\prime}\varphi$. But then condition 2 implies $\Pi F = \Pi\Pi'\phi - Gz$ for some z. Hence, $\phi = (\Pi\Pi')^{-1}Gz$ and therefore $F = \Pi'(\Pi\Pi')^{-1}Gz + \Pi^{\perp\prime}\varphi$. Clearly, then, we can take $B = [\Pi'(\Pi\Pi')^{-1}G \;\; \Pi^{\perp\prime}]$, which is smooth on V.

Step 1: $H_i = 0$ implies that rank DH_i is at least $M + N - 1$.

First note that $H_i = 0$ implies that θ_i is nonzero by the definition of E_i. This plus $p > 0$ implies that $D_{A_i} H_2$ has full rank S. Note also that $D_{A_i} H_{i1} = 0$. Thus, to complete this step it is sufficient to show that $D_e H_{i1}$ has at least rank $M + N - S - 1$.

Suppose the M-vector v is such that $pv = 0$ and $P_1 v \in \text{span } R^*_{-i}(p)$. Then consider perturbing the unconstrained agent's endowment to $e_u + \lambda v$ for some scalar λ. Because the agent's income is not changed, demand is also unchanged, and so the agent's excess demand Z^u is reduced by λv. Because $P_1 \lambda v$ is in the current market span, this change to R^*_u does not change the market span for λ small. Thus, excess demand Z^c is not affected by this perturbation. Hence.

$$H_{i1}(p, t, \theta; e_u + \lambda v) = -\lambda v.$$

This implies that $D_{e_u} H_{i1} v = -v$. Because there are $M - (S - N) - 1$ independent vectors v, this implies that rank $D_e H_{i1}$ is at least $M + N - S - 1$.

Step 2: Construction of Π_i and G_i.

Define

$$\Pi_i = \begin{bmatrix} p & 0 \\ P_1 & 0 \end{bmatrix} \in R^{(S+1)\times(M+S)} \quad \text{and} \quad G_i = \begin{bmatrix} 0 \\ R^*_{-i} \end{bmatrix} \in R^{(S+1)\times N}.$$

Note that these matrices have full rank $S + 1$ and N respectively. By Walras' Law plus the fact that

$$P_1 H_{i1} = R^*_u + tP_1 Z^c(p, R^*_{-i}) \in \text{span } R^*_{-i},$$

we have $\Pi_i H_i \in \text{span } G_i$.

Step 3: Apply Theorem 1 with $m = M + S$, $n = N$, $t = S + 1$, and $k = M + N - 1$.

Case $i = u$:

Step 1: $H_u = 0$ implies that rank DH_u is at least $M + N - 1$.

Because $R(p)$ has full rank N, $D_\theta H_2$ also has rank N. Next note that $D_{e_u} H_{u1}$ has rank $M - 1$, since any change v in the endowment of the unconstrained agent that does not affect income ($pv = 0$) leads to a direct change in the agent's excess demand by the amount $-v$. This confirms the fact that DH_u has at least rank $M + N - 1$.

Step 2: Construction of Π_u and G_u.

Note that by definition, $P_1 H_{u1} = R^*_u + tP_1 Z^c(p, R(p)) = R^*_u + R(p)\alpha$ for some vector α. Also, $H_2 = R^*_u \theta_u + R(p)\theta_{-u}$. This suggests a restriction on the value of H_u in addition to that imposed by Walras' Law. In particular, define

$$\Pi_u = \begin{bmatrix} p & 0 \\ \theta_u P_1 & -I \end{bmatrix} \in R^{(S+1)\times(M+S)} \quad \text{and} \quad G_u = \begin{bmatrix} 0 \\ R(p) \end{bmatrix} \in R^{(S+1)\times N}.$$

These have full rank and from the above discussion, $\Pi_u H_u \in \text{span } G_u$.

Step 3: Apply Theorem 1 with $m = M + S$, $n = N$, $t = S + 1$, and $k = M + N - 1$.

Finally, we remark that the same arguments apply on the boundary of E_i ($t = 0$ and $t = 1$). Thus the boundary of $H_i^{-1}(0)$ coincides with its intersection with the boundary of E_i. *Q.E.D.*

Thus far, we have taken the underlying parameters of the economy, w, as variables in defining the homotopy and its solution set. At this point, however, we would like to consider the characteristics of the solutions to $H_i = 0$ for a fixed economy w. First define the following notation for a set $Q \subset X \times W$:

$$Q^w = \{x \in X : (x, w) \in Q\}.$$

We can then state the following characterization for fixed w:

THEOREM III: *For generic w, $\sigma_i^w \equiv H_i^{-1}(0)^w$ is a smooth one-dimensional submanifold of E_i^w.*

PROOF: Consider the projection map ρ: $H_i^{-1}(0) \to W$. By Sard's Theorem (see Appendix) and the properness of ρ, the set of regular values w of ρ is open with full measure. For such a regular value w, the Implicit Function Theorem implies that $\rho^{-1}(w)$ is a smooth submanifold of $H_i^{-1}(0)$ of co-dimension W. Therefore, $H_i^{-1}(0)^w$ is a one-manifold. *Q.E.D.*

This establishes that each homotopy in the family generically defines a path within its domain. The next step is to show that the paths associated with alternative homotopies in the family coincide in the overlap of their respective domains. This implies that these paths can in fact be overlaid to form a single one-dimensional manifold.

THEOREM IV: *For generic, w, $\sigma^w = \bigcup_i \sigma_i^w$ is a smooth one-dimensional submanifold of $E^w = \bigcup_i E_i^w$.*

PROOF: We show that $H_i^{-1}(0)$ and $H_j^{-1}(0)$ coincide on $E_i \cap E_j$. Suppose $(p, t, \theta, w) \in E_i \cap E_j \cap H_i^{-1}(0)$. Then $H_2 = 0$ by definition. But this implies R^* has rank N, so that span $R^* =$ span $R^*_{-i} =$ span R^*_{-j}. Therefore, $H_{j1} = H_{i1} = 0$. Hence, $(p, t, \theta, w) \in H_j^{-1}(0)$. *Q.E.D.*

Note that E^w is the set of points for which the augmented return matrix has at least rank N. The above proposition states that the homotopies we have defined determine a path in this set. The path must terminate on some boundary of E^w. But the boundary of this set includes not only the boundary of X but also those "doubly bad" points for which R^* has rank less than N. Our next result establishes that generically, these doubly bad points will not lie on the path, which can therefore only terminate on a boundary of X.

THEOREM V: *For generic w, σ^w is a smooth, compact one-dimensional submanifold of X. Moreover, the boundary of σ^w coincides with its intersection with the boundary of X.*

PROOF: We need to show that the closure of σ^w is in E^w. Suppose there exists a sequence $(p,t,\theta)\in\sigma_i^w$ converging to a point (p^*,t^*,θ^*) not in E^w. Because excess demand diverges at the boundary of the price simplex, this implies that $p^*>0$. Thus, it must be the case that rank $R^*(p^*)<N$. We will now show that this does not occur generically.

Because R^* has rank N along the sequence, its span must converge to some N-dimensional subspace of R^s. Therefore,

$$\operatorname{span} R^*(p)\to\operatorname{span}\Phi\begin{bmatrix} I\\ V\end{bmatrix},$$

for some permutation matrix Φ and $V\in R^{(S-N)\times N}$. Then from the continuity of Z we have

$$(9)\qquad Z^u(p^*)+t^*Z^c\left(p^*,\Phi\begin{bmatrix} I\\ V\end{bmatrix}\right)=0.$$

Because $R^*_{-i}(p^*)$ has rank less than N, there exists at least one further column $j\neq i$ that is redundant. Consider the case $i,j\neq u$. Then the following equations must also hold:

$$(10)\qquad [-V\quad I]\Phi^{-1}R_{-ij}(p^*)=0,$$

$$(11)\qquad R^*_{-i}(p^*)\alpha=0,$$

$$(12)\qquad R^*_{-j}(p^*)\beta=0,$$

for some vectors $\alpha,\beta\in B^N$ with α_j and β_i nonzero. Together, (9)–(12) are a system of $M-1+(S-N)(N-2)+2S$ equations that the unknowns (p,t,V,α,β) must satisfy, where we have dropped the last equation of (9) by Walras' Law. These unknowns belong to a manifold of dimension $M+(S-N)N+2(N-1)$. Taking the derivative of equations (11) and (12) with respect to A_j and A_i respectively shows them each to have rank S. Taking the derivative of equation (10) with respect to A_{-ij} yields rank $(S-N)(N-2)$. Finally, taking the derivative of equation (9) with respect to e_u shows it to have rank $M-1$. Thus this system of equations has rank $M-1+(S-N)(N-2)+2S$. Applying the parametric version of Sard's Theorem (see Appendix), the set of solutions is generically a manifold of dimension

$$[M+(S-N)N+2(N-1)]-[M-1+(S-N)(N-2)+2S]=-1.$$

That is, generically these equations have no solution and this scenario does not occur.

The case in which one of the redundant columns i or j equals u (the augmented column) is handled similarly. Without loss of generality let $j=u$. In this case the following equations must be satisfied in addition to equation (9):

$$(13)\qquad [-V\quad I]\Phi^{-1}R_{-i}(p^*)=0,$$

$$(14)\qquad R^*_{-i}(p^*)\alpha=0,$$

$$(15)\qquad R(p^*)\beta=0.$$

Again, $\alpha, \beta \in B^N$ with α_u and β_i nonzero. We will show that the system (9), (13)–(15) is also overdetermined and hence cannot occur generically. First, the derivative of equation (15) with respect to A_i yields rank S. Also, the derivative of (9) with respect to e_c has rank $M-1+N-S$ (due to Walras' Law and the market span constraint—see Theorem II for a similar calculation). Next, the derivative of (14) with respect to e_u yields rank S. Finally, the derivative of (13) with respect to A_{-i} has rank $(S-N)(N-1)$. Thus the entire system of equations has rank $M-1+(S-N)(N-2)+2S$ as before. This system therefore also has no solution generically.

Thus we have examined all of the cases in which $(p^*, t^*, \theta^*) \notin E^w$ and have shown that they cannot occur generically. *Q.E.D.*

Having established that our homotopies define a path in X, existence of an equilibrium follows naturally. We simply need to show that generically, there exists a unique starting point $(p^u, 0, \theta^u)$ for this path at $t=0$. Then, since the path cannot return to this boundary, nor escape at a boundary of the price simplex (excess demand would diverge), it must be the case that the path continues until $t=1$. The price vector at this terminal point is an equilibrium for the economy.

THEOREM VI: *For generic w, there exists an equilibrium price vector p^* for the GEI model. Moreover, if multiple equilibria exist, they are locally unique and odd in number.*

PROOF: The manifold σ^w is a collection of "paths" in X. From the homogeneity of H_2, note that $(p,t,\theta) \in \sigma^w$ implies $(p,t,-\theta) \in \sigma^w$. Thus, for any path in σ^w there is a corresponding "mirror image" path in σ^w with the sign of θ reversed. Also, since θ is a unit vector, $\theta \neq -\theta$ and therefore each path and its mirror image are distinct. We can therefore partition σ^w into σ_+^w and σ_-^w, each containing one of each mirror image pair.

When $t=0$, $H_{i1}=0$ implies $p=p^u$ such that $Z^u(p^u)=0$. For generic w, $R(p^u)$ has full rank N. Define $\theta^u=[0,\ldots,0,1]'$. Then $H_2=0$ implies $\theta=\pm\theta^u$. Without loss of generality suppose $(p^u,0,\theta^u) \in \sigma_+^w$. This point is then the unique intersection of σ_+^w with the boundary $t=0$. By the Classification Theorem for one-dimensional manifolds (see Appendix), this path must continue until another boundary of the domain. Because excess demand diverges at the boundary of the simplex, this implies the path must terminate at a point $(p^*, 1, \theta^*)$. This is by definition an equilibrium for the economy as long as it is in E_u^w. To see that this is the case generically, note that if it were not true we would have a solution to (9), (13), and (15) with the additional restriction $t^*=1$. An argument similar to that in the proof of Theorem V establishes that this cannot occur generically.

Finally, local uniqueness follows from the fact that all equilibria are endpoints of the manifold defined by the σ_+^w. Oddness follows from the fact that any other

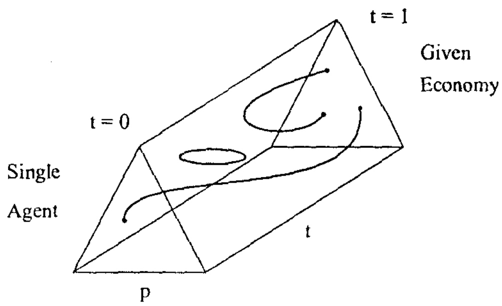

FIGURE 6.—Generic picture of the manifold $\sigma^w = \bigcup_i H_i^{-1}(0)^w$.

equilibrium must belong to a path which both begins and ends on the boundary $t = 1$. See Figure 6 for a generic picture of the solution manifold in (p, t) space. *Q.E.D.*

5. COMPUTATIONAL IMPLEMENTATION

In this section we give a brief discussion of how standard path following techniques can be adapted to solve for a solution of our model. Various numerical procedures for path following in $\boldsymbol{R}^K$ have been developed (see Allgower and Georg (1990, 1992)). Rather than develop the detailed theory of such algorithms here, we instead focus on the practical issues involved in implementing a path following algorithm for the GEI problem.

The basic idea of numerical path following is to think of the solution to the homotopy, H_i as defining a path which is a solution to the following differential equation on E_i^w:

$$D_{p,t,\theta} H_i(x(s))\dot{x}(s) = 0, \tag{16}$$

$$\|\dot{x}(s)\| = 1, \tag{17}$$

with the initial condition $x(0) = (p^u, 0, \theta^u)$ and $i = u$. Equation (16) has a one-dimensional family of solutions, since $D_{p,t,\theta}H_i$ has rank $M + N - 1$, whereas x has dimension $M + N$. Equation (17) normalizes the parameterization to arc length. For a given s and $x(s)$, the system thus has two solutions, $\pm\dot{x}(s)$. At $s = 0$, we choose the solution with $\dot{t} > 0$ and move interior to X (this is always possible generically). From then on, we continue moving in the same direction along the path. In practice this cannot be done exactly, but methods such as Predictor-Corrector proceed by taking small but discrete steps in the tangent direction, and then correcting for error to relocate the path.

One difference between our model and standard applications is that we have redundant equations in the system H_i. This plus the fact that the required derivatives are not analytically tractable in general makes solving the above system computationally tedious. Thus, rather than solve (16) exactly, we instead approximate the path's direction based on the secant defined by the previous two points. We then correct this approximation using a standard Gauss-Newton algorithm to find a solution to $H_i = 0$ in the plane perpendicular to the predicted direction. See Figure 7.

Of course, the above procedure thus far ignores the critical distinction between our model and the usual path following problem: our path is defined by a set of $N + 1$ homotopies rather than a single homotopy. Therefore, we need to outline a method by which a path following algorithm can decide which homotopy to follow. Most of the time, the choice of homotopies is irrelevant since they define coincident paths. The choice only becomes relevant when the path is approaching a singular point with respect to one of the homotopies. In that case, that homotopy becomes numerically unstable, and an alternative homotopy should be used.

Recall that our path starts based on H_u. If we are lucky, this path will continue until $t = 1$. In general, however, this path may converge to a point for which $R(p)$ drops rank. In this case we now know that the path can be continued by switching to another homotopy H_i. The question is determining when and how to switch.

In our intuitive presentation in Section 3, we spoke of switching the homotopy after reaching a "bad" point. This, of course, is not suitable from a computational perspective, since any numerical path following procedure would likely be unstable in the vicinity of a discontinuity. Thus, we would like to have a means of anticipating a bad point, and switching to a good homotopy before the bad point is reached. Fortunately, there is a convenient method of anticipating a bad point built into the homotopy itself.

THEOREM VII: *If* $(p, t, \theta) \in \sigma^w$, *then* $(p, t, \theta) \in E_i^w$ *if and only if* $\theta_i \neq 0$.

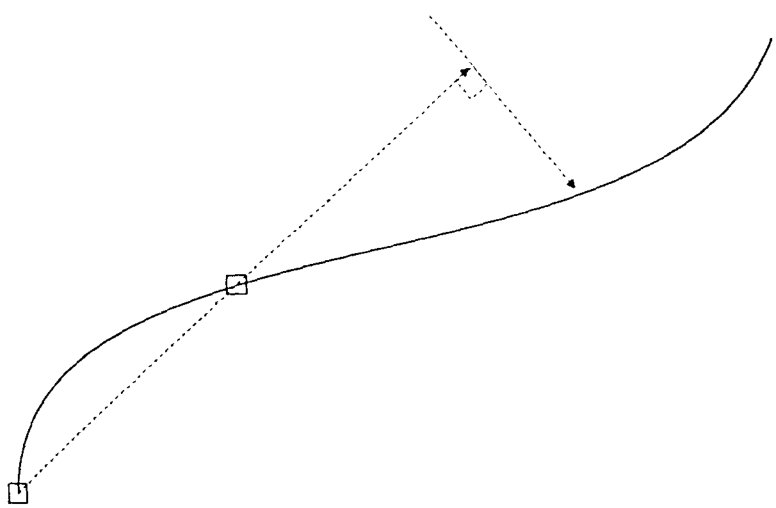

FIGURE 7.—A simple method for path following.

PROOF: Suppose (p, t, θ) is in E_i^w. Then by definition $R^*_{-i}(p)$ has full rank N. But then $R^*(p)\theta = 0$ implies θ_i cannot be equal to zero. Next suppose θ_i is nonzero. By assumption (p, t, θ) is in E^w and $R^*\theta = 0$, so R^* has rank N. Moreover, we can write $R_i^* = R^*_{-i}\theta_{-i}/\theta_i$. But this implies span $R^* =$ span R^*_{-i}. Therefore, R^*_{-i} has rank N and (p, t, θ) is in E_i^w. *Q.E.D.*

Thus, $|\theta_i|$ can be used as a "measure" of how far the path is from a bad point in E_i^w. Thus we can continue to use the homotopy H_i until θ_i becomes too small. At that point, we switch to a homotopy H_j for which θ_j is large. In order to define "small" and "large" in this context, note that θ is chosen from B^{N+1}. Thus there always exists j such that $\theta_j^2 \geq 1/(N+1)$. To avoid the possibility of switching infinitely often, we therefore wait to switch until $\theta_i^2 < \delta/(N+1)$, where $0 < \delta < 1$. We then switch to homotopy j, with j having the largest $|\theta_j|$. The parameter δ is chosen to tradeoff the computational inefficiency of switching with the potential for numerical instability near a bad point.

We illustrate this procedure in Figure 8. This path passes through a singularity with respect to $i = u$ (indicated by the small circle). The approaching singularity is detected several steps earlier when the path enters the ellipse defined by the critical value for θ_u. The algorithm then switches to using H_j for some suitable j for remaining computations. Of course, H_j may also have singularities, but these are outside this neighborhood. Finally, once the path leaves the critical area, the algorithm can again switch back to $i = u$, and terminate at $t = 1$ with a solution.

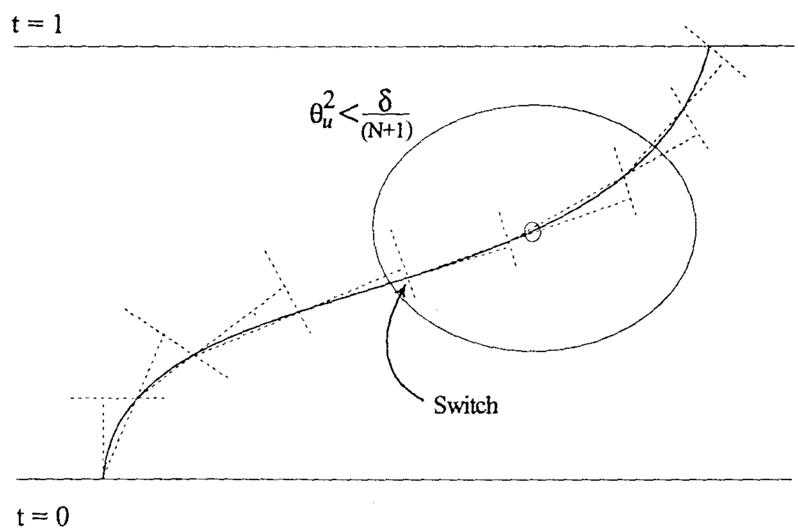

FIGURE 8.—Example of a solution path.

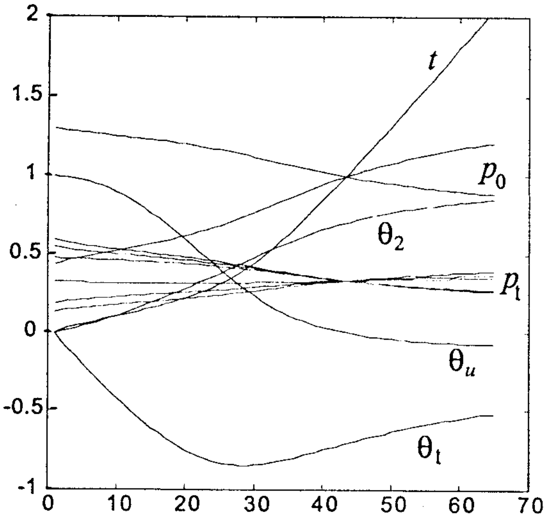

FIGURE 9.—(p, t, θ) per iteration ($\Sigma p = 4$).

5.1. *The Example Revisited*

Recall the example of Section 3. In that example we attempted to follow the homotopy path defined by equation (6), but the path got stuck at a singularity for $t = 1$. This was uninformative since for the example the actual economy of interest has $t = 2$. We now compute an equilibrium for this economy using the algorithm described above. The computed path is plotted in Figure 9, where we show the values of (p, t, θ) after each Predictor-Corrector step. The algorithm stops with $t = 2$ after 65 iterations. Note that the singularity encountered at $t = 1$ causes no difficulty for the algorithm—the path evolves smoothly through this point. At termination, the computed values[12] are $p^* = [30.1, 22.2; 8.7, 6.8; 9.2, 6.6; 9.8, 6.5]$, and $\theta^* = [-0.52, 0.85, -0.08]$. Since $\theta_u^* = -.08 \neq 0$, the return matrix $R(p^*)$ is nonsingular and this is a true equilibrium for the economy.

Below we look at several features of the computation of the path. First note from Figure 10 that the approaching singularity at $t = 1$ is indeed revealed by θ. That is, θ_u starts at 1 for $t = 0$ (when $Z^u = 0$), but then drops to zero as t approaches 1. The graph also plots the critical value at which the algorithm switches homotopies (corresponding to $\delta = .25$). The algorithm switches from $i = u$ to $i = 1$ at approximately $t = 0.6$.

Figure 10 also reveals the robustness of the singularity encountered in this example. A small perturbation of the economy should result in a small perturbation to the path. Note, however, that because the path begins with θ_u strictly positive and ends with θ_u strictly negative, along the perturbed path we must

[12] Note that prices are subject to an arbitrary normalization; for convenience the stated values are normalized to sum to 100. In Figure 9 they sum to 4.

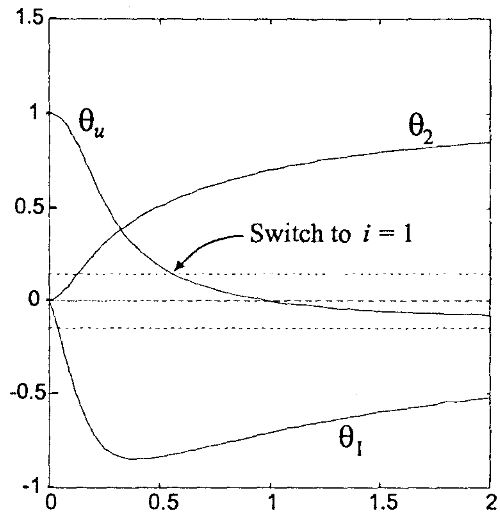

FIGURE 10.—θ versus t along the path.

still cross $\theta_u = 0$. That is, the return matrix $R(p)$ drops rank along the homotopy path for any nearby economy as well.

Another view of this system can be given by computing $\det(R^{*\prime}_{-i} R^*_{-i})$ for i, plotted in Figure 11a. This determinant equals zero if and only if an asset in the return matrix is redundant. Again we note that for $t = 0$, the true return matrix $R = R^*_{-u}$ has full rank, but is singular at $t = 1$. Prior to reaching $t = 1$, however, it is possible to switch to either of the other nonsingular asset return matrices, R^*_{-1} or R^*_{-2}.

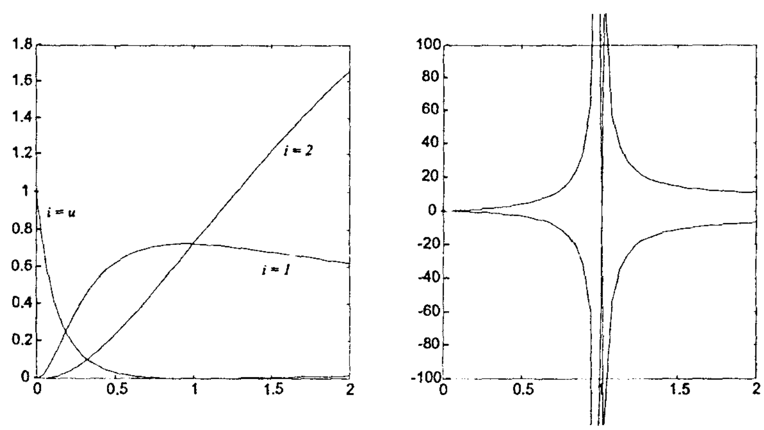

FIGURE 11.—(a) $\det(R^{*\prime}_{-i} R^*_{-i})$ vs. t. (b) Trade in assets 1 and 2 vs. t.

Finally, Figure 11b reports the trades of the actual assets 1 and 2 by the constrained agents along the path. Note again the effect of the singularity at $t = 1$ as agents inflate their portfolios to maintain their desired consumption patterns. Clearly, an algorithm based on the conventional approach would become numerically unstable in this region.

6. CONCLUSION

We have provided in this paper an alternative approach to proving existence of equilibrium in economies with incomplete markets. The fundamental discontinuity which occurs at prices for which asset returns become redundant can be smoothed by allowing an agent in the economy to introduce a new asset. This technique enables us to prove existence via a standard homotopy argument in Euclidean space.

Additionally, we translate this proof into a path following algorithm for computing equilibrium. The essence of the idea is as follows: We begin with an equilibrium for a single agent from the economy. We then gradually increase the relative size of the remainder of the economy, adjusting prices to maintain equilibrium. If we begin to approach prices for which one of the assets is redundant, that asset is removed and replaced by a new asset chosen by our original agent. Once we have "passed by" the potential redundancy, we then switch back to the original set of assets. We continue this procedure until we reach an equilibrium for the full economy.

Finally, we demonstrate this technique in the context of a numerical example. First we show that for this economy, the standard approach for computing with complete markets is inadequate and converges to a singularity. We also show that this singularity is robust to small perturbations to the economy. Our method of switching assets avoids the singularity, however, and allows us to compute equilibrium without difficulty.

We remark that our approach may also be useful for computing comparative statics for GEI economies. Typically, once an equilibrium is found, it is of interest to know how the equilibrium may change with changes in the underlying parameters in the economy, such as preferences, endowments, or asset structures. Conceptually, one can view such comparative statics as following a path in the equilibrium manifold. Naturally, we might again expect this path to pass near or through singularities for the return matrix, so that a technique such as ours will be necessary.

There are several natural extensions of this work to be considered. One important extension is to the case of a multiperiod economy. In this case, even with a single commodity, discontinuities can occur since first period asset returns depend on second period asset prices. At the theoretical level, a straightforward generalization of the current proof should suffice. From a computational standpoint, for short enough horizons the proposed algorithm is feasible. For sufficiently long horizons, however, the dimensionality of the state space grows too large for efficient computation. Hopefully, our ideas may potentially be combined with other techniques for infinite horizon models when potential asset redundancies are a concern.

Another extension of the current model would be to incorporate production. Again we anticipate that the theory and algorithm of this paper should extend to that case as well. Of course, introducing production raises the issue of the objective of the firm when markets are incomplete (see, for example, DeMarzo (1988, 1993) for an analysis of alternative objectives and decision mechanisms within the firm). A computational algorithm would allow for a comparison of share prices and welfare corresponding to different decision mechanisms within the firm. Additionally, a model with incomplete markets and production would be useful for addressing various policy concerns in a more realistic environment then the standard complete markets framework of the applied general equilibrium literature.

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APPENDIX

Below we state the following mathematical results for completeness:

DEFINITION: We say that y is a regular value of $f: U \to R^k$ if rank $Df(u) = k$ whenever $f(u) = y$.

IMPLICIT FUNCTION THEOREM: *Let $f: U \to R^k$ be a smooth function. If $u \in U$ is a regular value of both f and $f|\partial U$ (i.e., the restriction of f to the boundary of U), then $f^{-1}(y)$ is a smooth manifold with co-dimension k. Moreover, $\partial f^{-1}(y) = f^{-1}(y) \cap \partial U$.*

REFERENCE: See Mas-Colell ((1985), page 38, H.2.2).

SARD'S THEOREM: *Let $r > \dim U - \dim V$. Then the set of critical values of the C^r function $f: U \to V$ has measure zero in V.*

REFERENCE: See Mas-Colell ((1985), page 42, I.1.1).

PARAMETRIC SARD'S THEOREM (Transversality Theorem): *Suppose U and W are C^r manifolds and $f: U \times W \to R^k$ is a C^r function with $r > \dim U - k$. If y is a regular value of f, then for almost every w, $f(\cdot, w)$ has y as a regular value.*

REFERENCE: See Mas-Colell ((1985), page 45, 1.2.2).

CLASSIFICATION THEOREM: *Any compact one-dimensional manifold with boundary is equal, up to a diffeomorphism, to a finite union of circles and closed segments. The boundary is formed by the union of the endpoints of the segments, and is thus a finite set of even cardinality.*

REFERENCE: See Mas-Colell ((1985), page 35, H.1.vi).

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