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Research articles

Spanning, valuation and options*

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Summary. We model the space of marketed assets as a Riesz space of commodities. In this setting two alterntive characterizations are given of the space of continuous options on a bounded asset, s, with limited liability. The first characterization represents every continuous option on s as the uniform limit of portfolios of calls on s. The second characterization represents an option as a continuous sum (or integral) of Arrow-Debreu securities, with respect to s. The pricing implications of these representations are explored. In particular, the Breeden-Littzenberger pricing formula is shown to be a direct consequence of the integral representation theorem.

I. Introduction

The axiomatic elements of the modern theory of arbitrage are a set of marketed assets and prices which can be combined in a linear fashion to produce portfolios and their prices. The only behavioral assumption is that agents transacting in the given markets prefer more to less and consequently the marketed assets with semipositive pay offs have positive prices. By arbitrage these prices define a positive linear functional on the space of marketed portfolios.

When in addition derivative assets (options) on the marketed assets, such as call options or put options, are also marketed, then some fundamental questions arise:

1. Are certain derivative assets basic in the sense that all other derivative assets in a large class are portfolios of these basic derivative assets?
2. Under what circumstances can prices on the marketed assets or basic derivative assets be uniquely extended by arbitrage to prices on all derivative assets in a large class and when is such an extension unique?

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These two questions are the subject of this paper.

An early answer to the first question was given by Ross in [18] for marketed assets defined on a finite state space. Ross defined an asset as resolving if its returns differed in every state of the world. He then showed that the set of call options on a resolving asset spans the complete space of state-contingent claims, i.e., if the state space is finite and a resolving asset is marketed then every state-contingent claim can be expressed as a portfolio of call options on the given resolving asset.

It is this result which we shall extend to assets which need not be resolving and may be defined on an infinite state space. That is, we will characterize the space of assets spanned by call options on a given asset. In our model we will assume the existence of a bond. Given prices of the stock and bond, we will investigate conditions for extending these prices to calls on the stock and to assets which are the limits of portfolios of calls.

In any formal analysis of markets, the first issue is the nature of the commodity space. In the early work on arbitrage pricing it was only assumed that the commodity space, i.e. the set of marketed portfolios, was a linear vector space. This was the model in Ross's paper. Ross [16, 17] recognized that arbitrage simply exploited agents' preferences for more to less, but it did not influence the space used to model the set of marketed assets.

The first explicit formalization of the essential behavioral assumption of the modern theory of arbitrage, that agents prefer more to less, appears in [9]. In this paper, Kreps models the space of assets as a partially ordered linear vector space. These are simply linear vector spaces with a notion of positivity. The set of positive vectors or the positive cone, E_+, formalizes the notion that in some states of the world an asset x has larger returns than asset y and in all states of the world the returns of x are at least as great as those of y. Under the partial ordering on the space of assets, denoted $\geqq$, this is equivalent to saying $x \geqq y$ or in terms of positive elements that $x-y$ is positive, i.e. that $x-y \in E_+$. To say that agents in these markets prefer more to less is to say that if the agent's utility function is U and $x \geqq y$ then $U(x) > U(y)$.

The partial order is also necessary for formalizing the notion of positive price system. These are linear functionals on the linear space of portfolios such that positive vectors are mapped into positive numbers.

Perhaps the major contribution of this paper is to argue for modeling the commodity space for options markets as a Riesz space (vector lattice). A Riesz space E is a partially ordered linear vector space with the additional structure that every pair of elements $\{x, y\}$ in the space has a least upper bound, $x \vee y$, in the space and greatest lower bound, $x \wedge y$, in the space. If $x \in E$, then $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$. A basic identity is $x = x^+ - x^-$. $x > 0$ is notation for $x \geqq 0$ and $x \neq 0$.

The linear operations for forming portfolios together with the lattice operations of $\vee$ and $\wedge$ allow us to express the basic derivative assets, i.e. calls and puts, as nonlinear functions of the stock, bond, and striking price. That is, if a stock, s, and a bond, b, are elements of a Riesz space E, then a call option on s with striking price λ is given by $(s-\lambda b)^+$ and a put on s with the same striking price is given by $(\lambda b - s)^+$. Notice that the put-call parity relationship follows immediately from the basic identity, $\text{Stock} + \text{Put} = s + (\lambda b - s)^+ = s + (\lambda b - s) + (\lambda b - s)^- = \lambda b + (\lambda b - s)^- = \text{Bond} + \text{Call}$.

Currently in the finance literature, there are four basic models used for the space of assets, E: First, if Ω, the state space, is a measure space with a probability measure μ, then either E is the set of measurable functions on Ω – see (6), (10) – or E is the set of L_p functions on Ω, for $1 \leqq p \leqq \infty$, see [14].

Our paper is concerned primarily with a third class of models, which are independent of any prior notion of probability. We assume that Ω, the state space, is a compact Hausdorff space, and $E = C(\Omega)$, the space of continuous functions on Ω. A special case is when Ω is a finite set with the discrete topology and hence $C(\Omega) = R^n$, for some finite n. This is the model in Ross. Finally some models simply use $E = R^{\Omega}$, the space of real-valued functions on Ω, see [4].

In each case, if we define the linear vector space operations and the lattice operations pointwise, e.g. $(x \vee y)(\omega) = \max\{x(\omega), y(\omega)\}$ for all $\omega \in \Omega$, then each of the above models is a Riesz space.

To illustrate our results, consider the special case where $E = C[0,1]$, the space of continuous functions on the compact interval $[0,1]$. We assume that the stock s has limited liability, i.e. $s \in E_+$ and that the bond b is the constant-one function, $b(\omega) = 1$ for all $\omega \in [0,1]$. Let $[\alpha, \beta]$ be the range of s, i.e. $[\alpha, \beta] = s[0,1]$, then $[\alpha, \beta]$ is a compact subset of the real line. Ross defined a simple option on s as a real-valued function whose domain is the range of s. Hence $C[\alpha, \beta]$ is the space of continuous options on s. Our first result characterizes $C[\alpha, \beta]$ in terms of portfolio of calls.

We define $\mathscr{L}_v\{s, b\}$ as the smallest Riesz subspace in $C[0,1]$, containing $\{s, b\}$. It is known that $C[0,1]$ is a Banach space w.r.t. the sup norm; moreover, the lattice operations ar continuous w.r.t. the norm topology i.e. $C[0,1]$ is a Banach Lattice. We show by the Stone-Weierstrass Theorem that $\overline{\mathscr{L}_v\{s, b\}}$, the norm closure of $\mathscr{L}_v\{s, b\}$, is lattice isometric to $C[\alpha, \beta]$. By lattice isometric we mean that there exist a bi-continuous bi-jection between $\overline{\mathscr{L}_v\{s, b\}}$ and $C[\alpha, \beta]$ which preserves the Riesz space structure. Moreover, we show that $\mathscr{L}_v\{s, b\} = \operatorname{Span}\{s, b, C_\alpha\}$, where $C_\alpha = (s - \alpha b)^+$. Hence $C[\alpha, \beta]$, the space of continuous options on s, $= \overline{\operatorname{Span}\{s, b, C_\alpha\}}$ which answers our first fundamental question.

Given prices of the stock, s, and bond, b, we turn to the second fundamental question and ask can these prices be uniquely extended to $\overline{\mathscr{L}_v\{s, b\}}$. In general, the answer is no. But the degree of indeterminacy follows from our result that $\mathscr{L}_v\{s, b\} = \operatorname{Span}\{s, b, C_\alpha\}$. The prices on $\{s, b\}$ can be uniquely extended, by arbitrage, to a positive linear functional on $\operatorname{Span}\{s, b\}$. But there may be many extensions of these prices to $\operatorname{Span}\{s, b, C_\alpha\}$ – the existence of at least one such extension is guaranteed by a theorem of Kantorovič. Additional assumptions are required to uniquely price $\operatorname{Span}\{s, b, C_\alpha\}$. For example, continuous time trading and the assumption that the price process of the stock is geometric Brownian motion, produces the Black-Scholes pricing formula for each call option, C_α. Hence, by arbitrage, we can price out every asset in $\operatorname{Span}\{s, b, C_\alpha\} = \mathscr{L}_v\{s, b\}$.

Moreover, the prices on $\mathscr{L}_v\{s, b\}$ can be uniquely extended to its norm closure. The given prices on $\mathscr{L}_v\{s, b\}$ define a positive linear functional. Any positive linear functional on a linear subspace of a Banach Lattice is continuous w.r.t. the norm topology. By Kantorovič's Theorem, the positive linear functional on $\mathscr{L}_v\{s, b\}$ has a unique positive extension to $\overline{\mathscr{L}_v\{s, b\}}$. Therefore given prices of the stock, s, and the bond, b, the prices on $\overline{\mathscr{L}_v\{s, b\}}$ are completely determined by the prices on C_α.

The basic elements in the above spanning and valuation theorems are calls, C_α, on the stock. We now give a spanning result in terms of Arrow-Debreu securities. This representation allows us to interpret our prices as state space prices, which in turn can be interpreted as marginal rates of substitution.

We begin with an alternative characterization of $\overline{\mathscr{L}_v\{s,b\}}$, the space of continuous options on s. Each asset $x \in \overline{\mathscr{L}_v\{s,b\}}$ can be represented as a Riesz space valued Riemann-Stieltjes integral. This representation has a number of important implications, one of which is a generalization of the Breeden-Litzenberger (B-L) pricing formula.

As an illustration of our approach, we first consider the case where $E = B[0,1]$, the space of bounded Borel measurable functions on [0, 1]. Any $s \in E_+$ is completely determined by the sets $A_\lambda = \{\omega \in [0,1] \mid s(\omega) < \lambda\}$ or equivalently by the characteristic functions of these sets, χ_λ. The family of partitions of [0, 1] can be used to construct an approximating monotone sequence of step functions which converge uniformly to s, from below. The sets which appear in the definition of these step functions are simply differences of the A_λ, i.e. a typical such set will have a characteristic function $\chi_{\lambda_1} - \chi_{\lambda_2}$ where $0 \leqq \lambda_1 < \lambda_2 \leqq 1$.

This limit of step functions can be interpreted as a Riesz space, E, valued integral. Let $s_b(\lambda) = D^-(\lambda b - s)^+$, the left derivative of the put, $(\lambda b - s)^+$, w.r.t. its striking price λ. It is easy to show that $s_b(\lambda) = \chi_\lambda$, the characteristic function of the set $A_\lambda = \{\omega \in [0,1] \mid s(\omega) < \lambda\}$. Hence we can denote the integral as $\int_0^1 \lambda ds_b(\lambda)$. Freudenthal's Spectral Theorem asserts that $s = \int_0^1 \lambda ds_b(\lambda)$.

Luxemburg shows for any $\varphi \in C[0,1]$ that the Riesz space, E, valued Riemann Stieltjes integral $\int_0^1 \varphi(\lambda) ds_b(\lambda)$ exists. We show that $\overline{\mathscr{L}_v\{s,b\}} = \left\{ \int_0^1 \varphi(\lambda) ds_b(\lambda) \mid \varphi \in C[0,1] \right\}$. This is our second spanning result. Here the basic assets are the $s_b(\lambda)$. $\Delta s_b(\lambda) \equiv s_b(\lambda_2) - s_b(\lambda_1)$ for $\lambda_2 < \lambda_1$ is analogous to an Arrow-Debreu security.

In words, every continuous option on s can be expressed as either the uniform limit of portfolios of calls on s or as a continuous sum of Arrow-Debreu securities.

Given the lattice isometry between $\overline{\mathscr{L}_v\{s,b\}}$ and $C[0,1]$, every positive linear functional Π on $\overline{\mathscr{L}_v\{s,b\}}$ defines a positive linear functional $\hat{\Pi}$ on $C[0,1]$. The Riesz Representation Theorem states that $\hat{\Pi}$ can be respresented as a Riemann-Stieltjes integral w.r.t. a positive monotonic non-decreasing function $\Phi(\lambda)$ on [0, 1], i.e. $\hat{\Pi}[\varphi] = \int^1 \varphi(\lambda) d\Phi(\lambda)$ for every $\varphi \in C[0,1]$.

In this setting, the Breeden-Litzenberger pricing formula simply asserts that $\Phi(\lambda) = D^-(\Pi \cdot (\lambda b - s)^+)$, the left-derivative of the value of a put on s with striking price λ. Hence if $x = \int_0^1 \varphi(\lambda) d\lambda_b(\lambda)$, then $\Pi \cdot x = \hat{\Pi}[\varphi] = \int_0^1 \varphi(\lambda) d\Phi(\lambda)$.

The rest of the paper is organized as follows: The model, definitions, and statements of the propositions on Riesz spaces which are necessary for proving our results are given in Sect. II; the theorems are stated in Sect. III; proofs of the theorems are given in Sect. IV; the final section of the paper is a review of the relevant literature.

II. The Model

Our model for the space of marketed assets is a Riesz space, E.

If $y \in E$ and $y \neq 0$, then the principal ideal generated by $y, A_y, = \{x \in E \mid |x| \leqq \lambda |y|$, for some $\lambda > 0\}$. A_y has a natural norm, $\|\cdot\|_\infty$, where $\|x\|_\infty = \inf\{\lambda > 0 \mid |x| \leqq \lambda |y|\}$ for all $x \in A_y$.

A Riesz space, E, is said to be uniformly complete if $\langle A_y, \|\cdot\|_\infty \rangle$ is a Banach space for all non zero $y \in E$, $C[0, 1]$ is an example of a uniformly complete Riesz space.

We shall assume that the space of assets, E, is a uniformly complete Riesz space; that the stock, s, and the bond, b, belong to E^+; and that $s \in A_b$. The limited liability assumption is made for ease of exposition, but the assumption that the stock is bounded by some multiple of the bond is essential for our analysis.

$\mathscr{L}_v\{s, b\}$ is the Riesz space generated by $\{s, b\}$, i.e. the smallest Riesz subspace of E containing $\{s, b\}$. $\overline{\mathscr{L}_v\{s, b\}}$ is the norm closure of $\mathscr{L}_v\{s, b\}$ in A_b w.r.t. the $\|\cdot\|_\infty$ norm on A_b.

A call option on the stock with striking price λ, C_λ, $= (s - \lambda b)^+$. A put option on the stock with striking price λ, P_λ, $= (\lambda b - s)^+$. The put-call parity identity is $(s - \lambda b) = (s - \lambda b)^+ - (\lambda b - s)^+$.

Span $\{s, b, C_\lambda\}$ is the vector space generated by $\{s, b, C_\lambda\}$, i.e. the smallest vector subspace of E containing $\{s, b, C_\lambda\}$.

y is an upper bound for a subset F of E if $x \leqq y$ for every $y \in F$. A Riesz space, E, is σ-Dedekind complete if every countable subset of E which has an upper bound has a least upper bound. $B[0, 1]$ is a σ-Dedekind complete Riesz space.

Every σ-Dedekind complete Riesz space is uniformly complete, but the converse need not hold, e.g. $C[0, 1]$ is uniformly complete but not σ-Dedekind complete.

A Riesz space, E, is Dedekind complete if every subset of E which has an upper bound has a least upper bound, $L_p[0, 1]$ for $1 \leqq p < \infty$ are examples of Dedekind complete Riesz spaces as is R^Ω. $B[0, 1]$ is σ-Dedekind complete but not Dedekind complete.

A norm $\|\cdot\|$ on a Riesz space E is a lattice norm if $|x| \leqq |y|$ implies $\|x\| \leqq \|y\|$ for all $x, y \in E$.

A Banach lattice is a Riesz space with a lattice norm such that the norm topology is complete.

A Banach lattice is an AM-space if for all $x, y \geqq 0$ we have $\|x \vee y\| = \max\{\|x\|, \|y\|\}$. An AM-space is said to have a unit $e > 0$ whenever $\|x\| = \inf\{\lambda > 0 \mid |x| \leqq \lambda e\}$ holds for all $x \in E$.

If E and F are Banach lattices, then $T: E \rightarrow F$ is a lattice isometry if T is a linear map which preserves the lattice structure, is bi-continuous, and is one-to-one and onto. In this case we shall say that $E \simeq F$.

If E is an AM-space with unit, then $E \simeq C(\Omega)$ for a unique Hausdorff compact topological space, where the unit corresponds to the constant-one function on Ω; see [1, Theorem 12.28, p. 94].

If E is uniformly complete, then A_y is an AM-space with unit; see [1, Example 12, p. 198].

The following material is taken from Chap. 11 in [12].

If E is uniformly complete, then a map $\mu: [0, 1] \rightarrow A_b$ is of bounded variation (w.r.t. b) if there exists a real number M s.t. for every partition

$\pi = [0 = t_0 < t_1 < \ldots < t_n = 1]$ of $[0, 1]$, $\left\| \sum_{k=1}^{n} |\mu(t_k) - \mu(t_{k-1})| \right\|_\infty < M$, where $\|\cdot\|_\infty$ is the norm on A_b, and $b > 0$.

If f is a real valued function on $[0, 1]$, μ is a function of bounded variation on $[0, 1]$, and π is a partition on $[0, 1]$, then for every choice $\tau = \tau(\pi)$ of nodes τ_k satisfying $t_{k-1} \leqq \tau_k \leqq t_k (k = 1, 2, \ldots, n)$ we assign the E-valued Riemann-Stieltjes sum $S(f; \pi, \tau) = \sum_{k=1}^{n} f(\tau_k) |\mu(t_k) - \mu(t_{k-1})|$.

A real valued function f on $[0, 1]$ is called Riemann-Stieltjes integrable w.r.t. an E-valued function μ of bounded variation if there exists an element $x \in E$ s.t. $(\forall 0 < \varepsilon \in R)(\exists \delta > 0)(\forall \pi \in \pi[0, 1])(\forall \tau(\pi))$

$$|\pi| < \delta \Rightarrow \| |S(f; \pi, \tau) - x| \|_\infty < \varepsilon , \quad \text{where } |\pi| = \max_k |t_k - t_{k-1}| .$$

In this case x is uniquely determined and is called the Riemann-Stieltjes integral of f w.r.t. μ and will be denoted by $\int_0^1 f(\lambda) d\mu(\lambda)$.

If μ is an E-valued function of bounded variation on $[0, 1]$ and if f is a real valued continuous function on $[0, 1]$, then $\int_0^1 f(\lambda) d\mu(\lambda)$ exists; see [12, Theorem 1.3, p. 38].

If E is σ-Dedekind complete - hence uniformly complete – $s \in A_b$ and $\|s\|_\infty \leqq 1$, then the map $s_b : [0, 1] \to A_b$, where $s_b(\lambda) = D^-[\lambda b - s]^+$ is the left derivative of $[\lambda b - s]^+$ w.r.t. λ, is of bounded variation; see [13, Exercises 40.10 and 40.11, p. 268]; and [12, Theorem 2.3, p. 44].

$\{S_b(\lambda)\}_{\lambda \in [0, 1]}$ is called the spectrum of s (w.r.t. b).

Freudenthal's Spectral Theorem asserts that $s = \int_0^1 \lambda ds_b(\lambda)$; see [12, Theorem 2.4, p. 46].

It is easy to see that $\int_0^1 ds_b(\lambda) = b$.

Moreover, $T : C[0, 1] \to A_b$ is a lattice homomorphism, i.e. preserves the Riesz space structure, where $T[\varphi(\lambda)] = \int_0^1 \varphi(\lambda) ds_b(\lambda)$; see [12, Theorem 3.1, p. 50].

Finally, if p is a positive linear functional on A_b and $x = \int_0^1 \varphi(\lambda) ds_b(\lambda)$, then $p \cdot x = \int_0^1 \varphi(\lambda) d(p \cdot s_b(\lambda))$ where $\int_0^1 \varphi(\lambda) d(p \cdot s_b(\lambda))$ is a real valued Riemann-Stieltjes integral w.r.t. the real valued function of bounded variation $p \cdot s_b : [0, 1] \to R$; see [12, p. 55].

If Π is a positive linear functional on $C[0, 1]$ then the Riesz Representation Theorem states that there exists a positive monotone nondecreasing function $\Phi(\lambda)$ on $[0, 1]$ such that $\Pi \cdot x = \int_0^1 \varphi(\lambda) d\Phi(\lambda)$ for every $\varphi \in C[0, 1]$, where $\int_0^1 \varphi(\lambda) d\Phi(\lambda)$ is the Riemann-Stieltjes integral of φ w.r.t. Φ; see [20, Theorem 18.6.2, p. 321].

III. Theorems

In all of the following propositions, we will assume that the space of assets, E, is a uniformly complete Riesz space; that the stock, s, has limited liability, i.e. $s \geqq 0$; and that the bond, $b > 0$. Under these assumptions $\langle A_b, \|\cdot\|_\infty \rangle$ is an AM-space with unit b. In addition, we assume $x \in A_b$ and w.o.l.o.g. that $\|s\|_\infty \leqq 1$.

Theorem (1). $\mathscr{L}_v\{s, b\} = \operatorname{Span}\{s, b, C_\lambda\}$.

Theorem (2). $\overline{\mathscr{L}_v\{s, b\}} \simeq C(\theta)$, *where* θ *is a compact subset of* $[0, 1]$.

Theorem (3). *If* Π_0 *is a positive linear functional on* $\operatorname{Span}\{s, b\}$, *then* Π_0 *has a positive extension,* Π, *to* $\mathscr{L}_v\{s, b\}$.

Theorem (4). *If* Π *is a positive linear functional on* $\mathscr{L}_v\{s, b\}$, *then* Π *has a unique positive extension,* $\bar{\Pi}$, *to* $\overline{\mathscr{L}_v\{s, b\}}$.

Theorem (5). *If* E *is* σ*-Dedekind complete, then the map* $T: C[0, 1] \rightarrow A_b$, *where* $T[\varphi(\lambda)] = \int_0^1 \varphi(\lambda)\, ds_b(\lambda)$ *is a lattice isometry between* $C[0, 1]$ *and* $\overline{\mathscr{L}_v\{s, b\}}$.

Theorem (6). *If* E *is a* σ*-Dedekind complete;* $\int_0^1 \varphi(\lambda)\, ds_b(\lambda)$ *and* $\varphi \in C[0, 1]$; *and* Π *is a positive linear functional on* $\overline{\mathscr{L}_v[s, b\}}$. *Then* $\Pi \cdot x = \int_0^1 \varphi(\lambda)\, d\Phi(\lambda)$, *where* $\Phi(\lambda) = D^-(\Pi \cdot (\lambda b - s)^+)$ *is the left-derivative of* $\Pi \cdot (\lambda b - s)^+$ *w.r.t.* λ.

IV. Proofs

Theorem (1). Since E is uniformly complete, $\langle A_b, \|\cdot\| \rangle$ is an AM-space with unit b. Hence $A_b \simeq C(\Omega)$, for some compact Hausdorff space, Ω, let $\hat{s}$ be the image of s under this lattice isometry, V, and $\hat{s}(\Omega) = \theta$, the range of $\hat{s}$. Then θ is a compact subset of the real line.

$\hat{s}$ induces a lattice homomorphism, R, from $C(\theta)$ to $C(\Omega)$, where $R(f) = f \circ \hat{s}$ for all $f \in C(\theta)$; see [19, Theorem 9.1, p. 195]. Notice that $R(e_\theta) = e_\Omega$, where e_Ω and e_θ are the respective units (constant-one functions) of $C(\Omega)$ and $C(\theta)$. Moreover, $R(i_\theta) = \hat{s}$, where $i_\theta(\alpha) = \alpha$ for all $\alpha \in \theta$.

If $x, y \in F$, some arbitrary Riesz space, then $\mathscr{L}_v\{x, y\} =$ Lattice generated by $\operatorname{Span}\{x, y\}$; see [7, Theorem 2.2.11, p. 47]. $\operatorname{Span}\{i_\theta, e_\theta\}$ is the family of linear functions on R, with domain θ. Hence $\mathscr{L}_v\{i_\theta, e_\theta\}$ is the family of piecewise linear functions on R, with domain θ. But the family of piecewise linear functions on R, with domain θ, is identical to $\operatorname{Span}\{i_\theta, e_\theta, (i_\theta - \lambda e_\theta)^+\}$, where $(i_\theta - \lambda e_\theta)^+$ is a call on i_θ with striking price λ; see [5, Sect. 7.2, pp. 371–375].

Consequently, we have the following diagram:

$$\begin{tikzcd} C(\theta) \arrow[r, "R"] \arrow[dr, "T = V \circ R"'] & C(\Omega) \arrow[d, "V"] \\ & A_b \end{tikzcd}$$

T is a lattice homomorphism from $C(\theta)$ to A_b, where $T(e_\theta)=b$ and $T(i_\theta)=s$. Hence $T((i_\theta-\lambda e_\theta)^+)=(s-\lambda b)^+$. Since R is one-to-one and onto w.r.t. calls on i_θ in $C(\theta)$ and calls on $\hat{s}$ in $C(\Omega)$, T is one-to-one and onto w.r.t. calls on i_θ in $C(\theta)$ and calls on $s \in A_b$. Let $\mathscr{L}_v\{i_\theta, e_\theta\}=\operatorname{Span}\{i_\theta, e_\theta, (i_\theta-\lambda e_\theta)^+\}$. Operating with T on both sides, we obtain $\mathscr{L}_v\{T(i_\theta), T(e_\theta)\}=\operatorname{Span}\{T(i_\theta), T(e_\theta), (T(i_\theta)-\lambda T(e_\theta))^+\}$ or $\mathscr{L}_v\{s,b\}=\operatorname{Span}\{s, b, C_\lambda\}$ where $C_\lambda=(s-\lambda b)^+$.

Theorem (2). The proof that $\overline{\mathscr{L}_v\{s,b\}} \simeq C(\theta)$ will follow from the Stone-Weierstrass Theorem, which states that if F is a Riesz subspace of the Riesz space $C(\Omega)$, the space of continuous functions on a compact Hausdorff space Ω, which contains the constant-one function, e, and F separates the points of Ω, then F is dense in $C(\Omega)$ w.r.t. norm topology; see [19, Theorem 7.3, p. 103].

We define $C(\theta)$ as we did in the proof of Theorem (1), then we know that there exist lattice homomorphisms $U: \operatorname{Span}\{i_\theta, e_\theta, (i_\theta-\lambda e_\theta)^+\} \to \overline{\mathscr{L}_v\{s,b\}}$ and $U^{-1}: \mathscr{L}_v\{s,b\} \to C(\theta)$, where $U=T|\operatorname{Span}\{i_\theta, e_\theta, (i_\theta-\lambda e_\theta)^+\}$ and $U^{-1}=T^{-1}|\mathscr{L}_v\{s,b\}$. Here we have used the Stone-Weierstrass Theorem since $\overline{\operatorname{Span}\{i_\theta, e_\theta, (i_\theta-\lambda e_\theta)^+\}}=C(\theta)$ i.e. the piecewise linear functions, with domain θ, are norm-dense in $C(\theta)$. U and U^{-1} are positive transformations of vector subspaces of Banach lattices into Banach lattices, hence they are continuous; see [1, Theorem 12.3, p. 175]. Therefore, U has a unique continuous extension (as a linear map) to the closure of $\operatorname{Span}\{i_\theta, e_\theta, (i_\theta-\lambda e_\theta)^+\}$ denoted $\bar{U}$; see [15, Theorem 4.6.2, p. 55]. Similarly U^{-1} has a unique continuous extension, $\bar{U}^{-1}$, to $\overline{\mathscr{L}_v\{s,b\}}$ (as a linear map). It is easy to show that $\bar{U}^{-1}=(\bar{U})^{-1}$. Hence $\bar{U}$ is one-to-one and onto, but T is a linear map from $C(\theta)$ into $\overline{\mathscr{L}_v\{s,b\}}$ and $T|\operatorname{Span}\{i_\theta, e_\theta, (i_\theta-\lambda e_\theta)^+\}=U$. Therefore $T=\bar{U}$, i.e. T is one-to-one and onto, and is a lattice isometry between $C(\theta)$ and $\overline{\mathscr{L}_v\{s,b\}}$.

Theorem (3). $\operatorname{Span}\{s,b\}$ is a majorizing vector subspace of the Riesz space $\mathscr{L}_v\{s,b\}$, i.e. if $x \in \mathscr{L}_v\{s,b\}$ then for some $\lambda > 0$, $x \leqq \lambda b$, since $s \in A_b$. Hence by a theorem of Kantorovič, Π_0 has a positive extension, Π, to $\mathscr{L}_v\{s,b\}$; see [1, Theorem 2.8, p. 26].

Theorem (4). $\overline{\mathscr{L}_v\{s,b\}}$ is contained in A_b and $\mathscr{L}_v\{s,b\}$ is a Riesz subspace of $\langle A_b, \|\cdot\|_\infty\rangle$; see [19, Corollary 1, p. 84]. $\mathscr{L}_v\{s,b\}$ is a majorizing vector subspace of the Riesz space $\overline{\mathscr{L}_v\{s,b\}}$, hence again by Kantorovič's theorem, Π has a positive extension, $\bar{\Pi}$, to $\overline{\mathscr{L}_v\{s,b\}}$. But since Π is norm-continuous, this extension to the closure of $\mathscr{L}_v\{s,b\}$, $\overline{\mathscr{L}_v\{s,b\}}$, must be unique.

Theorem (5). We know from Theorem 3.1 in [12, p. 50] that T is a lattice homomorphism. Hence we need only show that T is one-to-one and onto $\overline{\mathscr{L}_v\{s,b\}}$. The argument is the same as that in the proof of Theorem 2, and we omit it.

Theorem (6). If $x \in \overline{\mathscr{L}_v\{s,b\}}$, then by Theorem 5 $x=\int_0^1 \varphi(\lambda)\,ds_b(\lambda)$ for some $\varphi(\lambda) \in C[0,1]$. Hence $\Pi \cdot x=\int_0^1 \varphi(\lambda)\,d(\Pi \cdot s_b(\lambda))$, see [12, p. 55]. Let $x=(kb-s)^+$, then $(kb-s)^+=\int_0^1 (k-\lambda)^+\,ds_b(\lambda)$ and $\Pi \cdot (kb-s)^+=\int_0^k (k-\lambda)^+\,d(\Pi \cdot s_b(\lambda)) =\int_0^k (k-\lambda)\,d(\Pi \cdot s_b(\lambda))$. Integrating by parts, $\Pi \cdot (kb-s)^+=\int_0^k \Pi \cdot s_b(\lambda)\,d\lambda$. Let $\Phi(k)=\bar{D}(\Pi \cdot (kb-s)^+)$, for $0 \leqq k<1$, and $\Phi(1)=\Pi \cdot b$, $\Phi(k)=\Pi \cdot s_b(k)$ a.e.

Hence $\int_0^k \Phi(k)d\lambda = \int_0^k \Pi \cdot s_b(\lambda)d\lambda$ for all k, and $\int_0^1 (k-\lambda)^+ d\Phi(\lambda) = \int_0^1 (k-\lambda)^+ d(\Pi \cdot s_b(\lambda))$ for all k, again by integration by parts. Since the positive linear functional on $C[0, 1]$, $\hat{\Pi}[\varphi] = \int_0^1 \varphi(\lambda)d\Phi(\lambda)$ for $\varphi \in C[0, 1]$ agrees with Π on the puts on s, the bond, b, and the stock, s; it follows from Theorem (1) and the put-call identity that $\Pi = \hat{\Pi}$ for all $\varphi \in C[0, 1]$. That is, if $x = \int_0^1 \varphi(\lambda)ds_b(\lambda)$, then $\Pi \cdot x = \int_0^1 \varphi(\lambda)d\Phi(\lambda)$.

V. Relationship to the literature

The work most closely related to the analysis in this paper considers the space of marketed portfolios, M, as a linear subspace of the Riesz space of assets R^{Ω}. In general, the authors impose a measure-theoretic structure on M by endowing Ω with the σ-algebra $\sigma(M)$, the smallest σ-algebra on Ω s.t. each $f \in M$ is $\sigma(M)$-measurable. This model first appears in Green and Jarrow [6]. The basic question is one of spanning, i.e. when are markets complete in the sense that every $\sigma(M)$-measurable $f \in R^{\Omega}$ is a member of M.

Green and Jarrow, in their Theorem (1), give the following necessary and sufficient conditions for markets to be complete:

(i) M is a Riesz subspace of R^{Ω}.
(ii) b, the bond, in M.
(iii) M is closed under pointwise monotone limits of sequences. Hence, Green and Jarrow characterize the set of measurable options on s and we characterize the set of continuous options on s.

Green and Jarrow, in footnote [6], suggest that the methods of Breeden and Litzenberger can be used to show that the first derivative of the value of a call w.r.t. its striking price is the value of the characteristic function of the set $A_{\lambda} = \{\omega \in \Omega \mid s(\omega) < \lambda\}$, where λ is the striking price. In fact, we have shown that the Breeden-Litzenberger pricing formula is a consequence of Freudenthal's Spectral Theorem and the observation that the characteristic function of $A_{\lambda} = \{\omega \in \Omega \mid s(\omega) < \lambda\}$ is the left-derivative of a put option on s with striking price λ.

In [14], Nachman extends Ross's spanning result to infinite state space models, using the theory of Banach Lattices, where the space of assets is a L_p space for some $1 \leqq p \leqq \infty$, say L_2.

Cox and Rubenstein [5] were the first to show for assets defined on the real line, that Span $\{s, b, C_{\alpha}\}$ = Set of piecewise linear options on s. Their result has been recently extended to the model of Green and Jarrow by Lim in [10]. He has shown that if $M = \{x\}$ then every measurable option on x which can be expressed as a portfolio of calls can be identified with a piecewise linear option on x. The Cox-Rubenstein result is used in the proof of our Theorem (1).

The Stone-Weierstrass Theorem is implicit in the work of Cox-Rubenstein where they construct upper and lower bounds of the values of continuous options on s by valuing piecewise linear options on s, this is our second theorem.

Both Jarrow [8] and Bick [2] in models where assets are random variables derive the Breeden-Litzenberger pricing formula under assumptions on the smoothness of the value of a call option, as a function of its striking price, which are weaker than those of Breeden and Litzenberger [3]. The Breeden-Litzenberger pricing formula and its extensions by Jarrow and Bick are special cases of our Theorem 6, for continuous options. Moreover, our pricing formula is exact in both the discrete state space and continuous state space models.

Lim has proven a version of Theorem (6) for $E = B[0, 1]$; see [11].

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