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The Structure of Neutral Monotonic Social Functions*

J. H. Blau and D. J. Brown**

Department of Economics, Yale University, P.O. Box 1972, New Haven, CT 06520, USA

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Abstract. In this paper, we show that neutral monotonic social functions and their specializations to social decision functions, quasi-transitive social decision functions, and social welfare functions can be uniquely represented as a collection of overlapping simple games, each of which is defined on a nonempty set of concerned individuals. Moreover, each simple game satisfies certain intersection conditions depending on the number of social alternatives; the number of individuals belonging to the concerned set under consideration; and the collective rationaly assumption.

0. Preface

I first met Julian Blau at the 1977 Public Choice Meetings in New Orleans. As I recall he chaired the session where I presented an earlier version of what was to become the joint paper presented here.

At those same meetings, John Ferejohn and Peter Fishburn presented their joint paper on the representation of social decision functions, see [4], possibly in the same session as my paper.

I remember several long walks with Julian where we discussed extensions of my paper in the direction of the Ferejohn-Fishburn paper, but emphasizing the role of neutrality. Ferejohn and Fishburn had not assumed neutrality, but the importance of neutrality in social choice theory had been a dominant theme in Julian's earlier researches, see [2]. It was during these conversations that our collaboration began. Over the next year, we corresponded and talked over the phone, I am sorry now that I didn't save those letters. Julian was a perfectionist and we argued long and hard over definitions – he didn't like the term direct sum of games – and proofs.

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When the paper was essentially done, it was decided that I would send it off for publication. That was my last conversation with Julian. I submitted the paper to Review of Economic Studies and a year later received two excellent referees' reports. By then I was actively at work on increasing returns and never got around to making the suggested revisions and sending it back to Review of Economic Studies.

I am, therefore, quite pleased to have this opportunity to share with Julian's friends and colleagues one of his last contributions in his chosen field of research. We shall miss both his clarity and insight.

1. Introduction

The axiomatic analysis of the aggregation of individual preferences, initiated by Arrow [1], has led to partial characterizations of special classes of social functions, collective choice rules which aggregate profiles of weak orderings into asymmetric social preferences.

The most celebrated result is Arrow's Possibility Theorem, where it is shown that any social welfare function, a social function whose range is the family of weak orderings, which satisfies the independence of irrelevant alternatives condition and the weak Pareto principle must be dictatorial.[1] That is, under these conditions, there exists some individual who if he prefers the social alternative a over the social alternative b, can ensure that the social preference is a over b.

Blau and Deb [2] have shown that any social decision function, a social function whose range is the family of acyclic preferences, which is neutral and monotonic has a veto hierarchy.[2] A veto hierarchy is a finite partition $V_1, V_2, \ldots, V_t$ of the set of individuals such that: each V_i is nonempty; each member of V_1 has a veto; for $r \geq 2$, each member of V_r has a veto when all members of $\bigcup_{i=1}^{r-1} V_i$ are indifferent.

Guha [6] has given a complete characterization of quasi-transitive social decision functions, a social function whose range is the family of quasi-transitive preferences, which satisfy the independence of irrelevant alternatives condition, the strong Pareto principle, and UII, i.e., if there is unanimous indifference between a and b, then a and b are socially indifferent. Under these conditions, he has shown that each nonempty set of concerned individuals contains an oligarchy. An individual is concerned about the pair of alternatives $\{a, b\}$ if he is not indifferent between them. A subset of a concerned set of individuals is an oligarchy if each person in the oligarchy has a veto, i.e., if he prefers a to b then society does not prefer b to a, and if everyone in the oligarchy prefers a to b then the social preference is a over b. Guha also established the converse of this result.

In this paper, we extend Guha's characterization to the class of neutral monotonic social functions.[3] We show that neutral monotonic social functions and their specializations to quasi-transitive social decision functions, social decision

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[1] In addition, Arrow assumed that $|A| \geq 3$ and $|I| < \infty$, where A is the set of social alternatives and I is the set of individuals in society.

[2] Blau and Deb also assume that $|I| \leq |A|$ and $|I| < \infty$.

[3] Our Theorem (2), characterizing neutral monotonic quasi-transitive social decision functions, was first proved by Guha, under the stronger hypotheses mentioned in the text.

functions, and social welfare functions can be uniquely represented as a collection of overlapping simple games, each of which is defined on a nonempty set of concerned individuals.

Moreover, each simple game satisfies certain intersection conditions depending on the number of social alternatives; the number of individuals belonging to the concerned set under consideration; and the collective rationality assumption. These results are given in Theorems (1) through (4).

Characterizations of simple majority rule and several of its variants, e.g., relative and absolute special majority rules, can also be found in the social choice literature. Here the classic result is due to May [10] who shows that, for two alternatives, a social function is simple majority rule iff it is strictly positively responsive, neutral, and anonymous.

As is well known, if there are at least three social alternatives then simple majority rule need not be a social decision function, this failure is known as the "paradox of voting". Ferejohn and Grether [5] proved that if the number of individuals exceeds the number of alternatives, then the relative special majority rule defined by θ, where society prefers a over b iff the fraction of concerned individuals who prefer a over b exceeds θ, is a social decision function iff $\theta \geq \frac{M-1}{m}$, where m is the number of social alternatives. This result was first shown by Craven [3] in the special case where individual preferences are strict orderings.

In Theorem (5), we extend the analysis of Ferejohn and Grether, pertaining to relative and absolute special majority rule, to neutral monotonic social decision functions, where the number of individuals exceeds the number of alternatives, and there is a finite number of alternatives.

The remainder of our paper consists of four propositions which, in conjunction with the theorems, are intended to make clear the relationships between our approach and previous partial characterizations of social decision functions that have appeared in the literature.

Decisive sets have played a prominent role in the analysis of social decision functions.[4] In fact, the dictator in Arrow's Theorem is an instance of a minimal decisive set. Therefore in Proposition (1), we identify the simple game in our representation which corresponds to the family of decisive sets.

In Proposition (2), we describe the veto-hierarchy in our representation of social decision functions, quasi-transitive social decision functions, and social welfare functions which are neutral and monotonic. This proposition shows the connection between our approach and the work of Blau and Deb, cited earlier.

Social decision functions are often assumed to be strictly positively responsive or anonymous. These conditions are investigated in Propositions (3) and (4), respectively.

Proposition (3), together with Theorem (4), gives a characterization of strictly positively responsive and neutral social decision functions which is comparable to the Possibility Theorem of Mas-Colell and Sonnenschein [9].

Proposition (4) generalizes the previously cited papers of May and Ferejohn and Grether, where we have weakened May's strict positive responsiveness assumption

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4 Readers interested in this literature should see [4], [7], or [8].

and the positive responsiveness of Ferejohn and Grether to monotonicity and retained their assumptions of neutrality and anonymity.

II. Definitions and Notation

The set of social alternatives will be denoted as A.

P is a *preference relation* on A if P is an asymmmetric binary relation on A, i.e., xPy and yPx cannot both be true. If they are both false, we write $x \sim y$, while xRy means that xPy or $x \sim y$. A preference relation is called *acyclic* if for all $x_1, x_2, \ldots, x_n \in A$; $x_1 Px_2, x_2 Px_3, \ldots, x_{n-1} Px_n \Rightarrow x_1 R x_n$. A preference relation is called *quasi-transitive* if P is transitive. A preference relation is called a *weak ordering* if R is transitive. A preference relation is called a *strict ordering* if it is quasitransitive and for all $x, y \in A$, $x \neq y$ implies either xPy or yPx.

$\mathscr{B}$ is the set of all preference relations on A.

$\mathscr{O}$ is the set of all acyclic preferences on A.

$\mathscr{Q}$ is the set of all quasi-transitive orderings on A.

$\mathscr{R}$ is the set of all weak orderings on A.

$\mathscr{S}$ is the set of all strict orderings on A.

I is the set of individuals in society. A *profile* is a function mapping I into $\mathscr{R}$, hence (by definition) a member of $\mathscr{R}^I$.

A *social function* is a mapping of $\mathscr{R}^I$ into $\mathscr{B}$. A *social decision function* is a mapping of $\mathscr{R}^I$ into $\mathscr{O}$. A *quasi-transitive social decision function* is a mapping of $\mathscr{R}^I$ into $\mathscr{Q}$. A *social welfare function* is a mapping of $\mathscr{R}^I$ into $\mathscr{R}$.

If σ is a social function and p a profile, then $\sigma(p)$ is a preference relation. Moreover, if p is a profile then $p(i)$ is the preference relation of the i^{th} individual. If p is a profile and $a, b \in A$, denote by $p(a>b)$ the set of individuals who prefer a to b, i.e., $ap(i)b$ iff $i \in p(a>b)$. If p is a profile and $a, b \in A$, then $p(a>b) \cup p(b>a)$ is the set of *concerned individuals* (with respect to the profile p and the pair of alternatives $\{a, b\}$.

If σ is a social function and $J \subseteq I$, then J is said to be *decisive* (with respect to σ) if for all profiles $p: J \subseteq p(a>b) \Rightarrow a\sigma(p)b$, for all $a, b \in A$.

Let p, q be profiles and $a, b, x, y \in A$.

A social function, σ, is *neutral* if $p(a>b) = q(x>y)$ and $q(y>x) = p(b>a)$, then $a\sigma(p)b \Rightarrow x\sigma(q)y$.

A social function, σ, is *monotonic* if $p(a>b) \subset q(x>y)$ and $q(y>x) \subset p(b>a)$, then $a\sigma(p)b \Rightarrow x\sigma(q)y$.

A social function, σ, is *neutral* and *monotonic* if $p(a>b) \subseteq q(x>y)$ and $q(y>x) \subseteq p(b>a)$, then $a\sigma(p)b \Rightarrow x\sigma(q)y$.

A social function, σ, is *anonymous* if for any permutation Π of $(1, 2, \ldots, |I|)$, $\sigma(p_1, p_2, \ldots, p_{|I|}) = \sigma(p_{\Pi(1)}, p_{\Pi(2)}, \ldots, p_{\Pi(|I|)})$ where $p = (p_1, p_2, \ldots, p_{|I|})$ is a profile.

Let $(R_1, R_2, \ldots, R_{|I|})$ and $(R_1', R_2', \ldots, R_{|I|}')$ be two profiles of weak orders. Let $\sigma(R_1, R_2, \ldots, R_{|I|}) = R$ and $\sigma(R_1', R_2', \ldots, R_{|I|}') = R'$, the associated social preference orders.

A social function, σ, is *positively responsive* if $(\forall_i)$: $(xP_i y \Rightarrow xP_i' y)$ and $(xI_i y \Rightarrow xR_i' y) \Rightarrow [xPy \Rightarrow xP'y]$ and $[xIy \Rightarrow xR'y]$ for all $x, y \in A$.

A social function, σ, is *strictly positively responsive* if xRy and $(\forall i)\, i \neq j$: $xP_i y \Leftrightarrow xP'_i y$ and $xI_i y \Leftrightarrow xI'_i y$ and $[xI'_j y$ and $yP_j x]$ or $[xP'_j y$ and $xI_j y] \Rightarrow xP'y$. In words, σ is strictly positively responsive if when a single individual changes her mind and ranks x higher than y then if society was previously indifferent, it now prefers x to y.

A social function σ is said to be *null* if for each profile p, $\sigma(p)$ is universal indifference.

If J is a nonempty subset of I, then $\underline{2}^J$ is the family of subsets of J.

If J is a nonempty subset of I, then a *simple game* on J is a collection of subsets of J, Γ_J, such that:

(a) $A \in \Gamma_J,\ A \subseteq B \Rightarrow B \in \Gamma_J$,

(b) $A \in \Gamma_J \Rightarrow A^c \notin \Gamma_J$, where A^c is the complement of A in J. Simple games having property *(a)* are called *monotonic*. Simple games having property *(b)* are called *proper*. The *null* (simple) *game* on J is the empty collection of subsets of J. If Γ_J is a simple game on J, then $\Gamma_J^* = \{E \subseteq J \mid E^c \in \Gamma_J\}$ where E^c is the complement of E in J.

Γ_J is often referred to as the family of *winning coalitions for J* and Γ_J^* the family of *losing coalitions for J*. Note that under our definitions, for a given simple game Γ_J, on J that some coalition $E \subseteq J$ may be neither winning or losing.

If A is a finite set, then an *acyclic game* on $J \subseteq I$ is a simple game, Γ_J, on J such that any empty intersection of winning coalitions in Γ_J has at least $|A|+1$ members.

A *prefilter* on J is a simple game, Γ_J, on J such that $\bigcap_{E \in \Gamma_J} E \neq \emptyset$ and Γ_J is not the null game.

A *filter* on J is a prefilter on J, Γ_J, such that $E, F \in \Gamma_J \Rightarrow E \cap F \in \Gamma_J$.

An *ultrafilter* on J is a filter on J, Γ_J, such that for all $E \subseteq J$, either $E \in \Gamma_J$, or $E^c \in \Gamma_J$.

We now present the central notion of our paper, the direct sum of simple games.

A *direct sum of simple games* is an indexed family of simple games $\{\Gamma_J\}_{J \in 2^I}$ such that:

(i) Γ_J may be the null simple game on J.

(ii) For all $K, L \in 2^I$; if $K \subset L$, then $\Gamma_L \cap 2^K \subseteq \Gamma_K$.

(iii) For all $K, L \in 2^I$; if $K \subset L$ then $\Gamma_K^* \subseteq \Gamma_L^*$.

Every direct sum of simple games, $\Gamma = \{\Gamma_J\}_{J \in 2^I}$ generates a social function $\underline{\mu_\Gamma}$ where for every profile p and alternatives $a, b \in A$: $a\mu_\Gamma(p)b$ iff $\underline{p(a > b)} \in \Gamma_{p(a>b) \cup p(b>a)}$. That is, the set of individuals who prefer a over b is a winning coalition in the simple game defined on the set of individuals concerned about the pair of alternatives $\{a, b\}$.

III. Theorems[5]

Theorem 1. *(a) If Γ is a direct sum of simple games, then μ_Γ, the social function generated by Γ, is a neutral monotonic social function.*

(b) If σ is a neutral monotonic social function, then there exists a unique direct sum of simple games, Γ, such that $\sigma = \mu_\Gamma$.

[5] We shall assume throughout the paper that $|A| \geq 3$, with the exception of Proposition (4), where we allow $|A| \geq 2$.

Proof. (a) If Γ is a direct sum of simple games then it is obvious that μ_Γ is a neutral social function. Hence we need only show that μ_Γ is monotonic. Let $p, q \in \mathscr{R}^I$ and $a, b \in A$. There are three cases: (i) $q(a>b) \supset p(a>b)$, $q(b>a) = p(b>a)$, and $a\mu_\Gamma(p)b$. Let $K = p(a>b) \cup p(b>a)$ and let $L = q(a>b) \cup q(b>a)$, then $K \subset L$, $q(b>a) = p(b>a) \in \Gamma_K^*$. Hence $q(b>a) \in \Gamma_L^*$, i.e., $q(a>b) \in \Gamma_L$. Therefore $a\mu_\Gamma(q)b$. (ii) $q(a>b) = p(a>b)$, $q(b>a) \subset p(b>z)$, and $a\mu_\Gamma(p)b$. Let $K = q(a>b) \cup q(b>a)$ and $L = p(a>b) \cup p(b>a)$, then $K \subset L$. $q(a>b) = p(a>b) \in \Gamma_L \cap 2^K$. Hence $q(a>b) \in \Gamma_K$. Therefore $a\mu_\Gamma(a)b$. (iii) $q(a>b) \supset p(a>b)$, $q(b>a) \subset p(b>a)$, and $a\mu_\Gamma(p)b$. There exists a profile $r \in \mathscr{R}^I$ such that $br(i)a$ iff $bq(i)a$ and $ar(i)b$ iff $ap(i)b$. Then by case (ii); $a\mu_\Gamma(r)b$. We can then apply case (i) to profiles r and q. Hence $a\mu_\Gamma(q)b$.

(b) Suppose σ is a neutral monotonic social function, then $\sigma : \mathscr{R}^I \to \mathscr{B}$. Given any profile $p \in \mathscr{R}^J$, we can extend it to a profile $q \in \mathscr{R}^I$ where $q(i) = p(i)$ for all $i \in J$ and $q(i)$ is universal indifference for all $i \in I/J$. Hence $\sigma : \mathscr{R}^J \to \mathscr{B}$ is well defined. If we further restrict σ to profiles of strict orders, i.e., members of $\mathscr{S}^J$, then $\sigma : \mathscr{S}^J \to \mathscr{B}$, the restriction of σ to $\mathscr{S}^J$ which we denote as $\sigma | \mathscr{S}^J$, is completely determined by its decisive sets. This follows from the observation that if Γ_J is the family of decisive sets of $\sigma | \mathscr{S}^J$, then $E \in \Gamma_J$ iff there exists some profile $p \in \mathscr{S}^J$ and $a, b \in A$ such that $a\sigma(p)b$ and $E = \{i \in J | ap(i)b\}$. If $\Gamma = \{\Gamma_J\}_{J \subseteq I}$, then we shall show that $\sigma = \mu_\Gamma$; that each Γ_J is a simple game; and that Γ is a direct sum of the Γ_J.

Suppose $q \in \mathscr{R}^I$ and $a, b \in A$ and let $J = q(a>b) \cup q(b>a)$. There exists $p \in \mathscr{S}^J$ such that $ap(i)b \Leftrightarrow aq(i)b$. Hence $a\mu_{\Gamma_J}(p)b \Leftrightarrow a\mu_\Gamma(p)b \Leftrightarrow a\mu_\Gamma(q)b$. But $a\sigma(q)b \Leftrightarrow a\sigma(p)b \Leftrightarrow a\mu_{\Gamma_J}(p)b$. Therefore, $a\sigma(q)b \Leftrightarrow a\mu_\Gamma(a)b$.

Suppose for some $J \subseteq I$, that Γ_J is not proper. Then there exists $K \subset J$ such that K and J/K, the relative complement of K with respect to J, are both in Γ_J. For some pair of alternatives $a, b \in A$, we consider the profile $q \in \mathscr{S}^J$ where $q(a>b) = K$ and $q(b>a) = J/K$. Then $a\sigma(q)b$ and $b\sigma(p)a$, which contradicts social asymmetry. Γ_J is monotonic for every J, since by definition families of decisive sets are monotonic. To show that Γ is the direct sum of the Γ_J, we suppose $K \subset L$ and consider some $E \in \Gamma_L \cap 2^K$. There exists a profile $p \in \mathscr{S}^L$ such that $p(a>b) = E$ and $p(b>a) = L/K$ for some $a, b \in A$. Hence $a\sigma(q)b$ where $q \in \mathscr{R}^I$ and $p(a>b) = q(a>b)$, $p(b>a) = q(b>a)$. Consider the profile $r \in \mathscr{R}^I$ where $r(a>b) = E$ and $r(b>a) = K/E$, then by monotonicity, $a\sigma(r)b$. Let s be the profile in $\mathscr{S}^K$ where $r(a>b) = s(a>b)$ and $r(b>a) = s(b>a)$, then $a\mu_{\Gamma_K}(s)b$. That is, $E \in \Gamma_K$. Now we consider some E in Γ_K^*. Hence there exists some $F \in \Gamma_K$ such that $E = K/F$. Consider the profile $r \in \mathscr{R}^I$ where $r(a>b) = F$ and $r(b>a) = E$. Then $a\sigma(r)b$. Let s be the profile in $\mathscr{R}^I$ where $s(a>b) = F \cup L/K$ and $s(b>a) = E$, then by monotonicity $a\sigma(s)b$. Let q be the profile in $\mathscr{S}^L$ where $s(a>b) = q(a>b)$ and $s(b>a) = q(b>a)$. Then $a\mu_{\Gamma_L}(q)b$. That is, $E \in \Gamma_L^*$.

Suppose $\sigma = \mu_{\tilde{\Gamma}}$ where $\tilde{\Gamma} \neq \Gamma$. Then for some $J \subset I$, $\tilde{\Gamma}_J / \Gamma_J \neq \emptyset$. If $E \in \tilde{\Gamma}_J / \Gamma_J$, then we construct a profile $s \in \mathscr{R}^I$ where everyone in E prefers x to y, everyone in J/E prefers y to x, and everyone in I/J is indifferent between x and y. Then $x\mu_{\tilde{\Gamma}}(s)y$ and $\neg x\mu_\Gamma(s)y$.

Proposition 1. *Let $\{\Gamma_J\}_{J \subseteq I}$ be a direct sum of simple games and μ_Γ the neutral monotonic social function generated by Γ.*

(a) μ_Γ is a null social function iff Γ_I is the null game.

(b) Γ_I is the family of decisive sets for μ_Γ.

Proof. (a) If μ_Γ is not the null social function, then there exists $a, b \in A$ and a profile $p \in \mathscr{R}^I$ such that $a\mu_\Gamma(p)b$. Since $p(b>a) \in \Gamma_J^*$ where $J = p(a>b) \cup p(b>a)$, $p(b>a) \in \Gamma_I^*$ by *(iii)*. Hence Γ_I is not empty. The converse is immediate.

(b) If μ_Γ is not the null social function, then there exists $E \in \Gamma_I$, by part *(a)*. By monotonicity, any such E is decisive. Suppose some $E \subset I$ is decisive, then consider the profile $q \in \mathscr{S}^I$ where for some $a, b \in A$; $E = \{i \in I | aq(i)b\}$ and for all $i \in I/E$, $bq(i)a$. In this case, $a\mu_\Gamma(q)b$ and therefore $E \in \Gamma_I$.

If μ_Γ is the null social function, then μ_Γ has no decisive sets. But by part *(a)*, in this case $\Gamma_I = \emptyset$.

Lemma. *Let $R = P \cup I$ be a weak ordering on A and let C be a minimal cycle on A, i.e., $C = \{(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1)\}$ where $x_i \neq x_j$ if $i \neq j$. If $C/I \neq \emptyset$, then $Q = P \cup (I \cap C)$ can be extended to a strict ordering, $\tilde{Q}$, over A.*

Proof. $Q \subset P \cup I = R$ and $Q \subset P \cup C$. Any Q-cycle is an R-cycle and therefore an I-cycle, since R is a weak ordering. But it is also a $(P \cup C)$-cycle. Since the Q-cycle is a subset of I, it is disjoint from P, and therefore is a subset of C. Since C is minimal, the cycle is C. But then C is a subset of I, which is false, since $C/I \neq \emptyset$. Therefore Q is acyclic. I defines an equivalence relation on A and we denote by $\hat{A}$ the family of indifference or equivalence classes defined by I. R induces a strict order, $\hat{P}$, on $\hat{A}$. Since Q is acyclic on A, it is acyclic on any subset of A. Hence Q is acyclic on every equivalence class $[b]$. Therefore Q, restricted to $[b]$, can be extended to a strict order on $[b]$, call this extension $Q_{[b]}$. $\hat{P}$ and the family of strict orders $\{Q_{[b]}\}_{b \in A}$ generate a strict order, $\tilde{Q}$, over A which extends Q, where for all $a, b \in A$; $a\tilde{Q}b$ iff aPb or if aIb and $aQ_{[b]}b$.

Theorem 2. *(a) If Γ is a direct sum of filters, then μ_Γ, the social function generated by Γ, is a quasi-transitive social decision function.*

(b) If σ is a neutral monotonic quasi-transitive social decision function, then there exists a unique direct sum of filters, Γ, such that $\sigma = \mu_\Gamma$.

Proof. (a) μ_Γ is a neutral monotonic social function by part *(a)* of Theorem (1). Suppose for some profile $p \in \mathscr{R}^I$, there exists distinct x_1, x_2, x_3 such that $x_1\mu_\Gamma(p)x_2$ and $x_2\mu_\Gamma(p)x_3$. Let $\mathscr{C}$ by the cycle $\{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$ and J_l the set of concerned individuals for the l^{th} pair in the cycle $\mathscr{C}$. If $J = \bigcup_{l=1}^{3} J_l$, then for each $i \in J$ we apply the Lemma and obtain a strict ordering $\tilde{Q}_i$ over A. Let q be the profile in $\mathscr{R}^I$ where $q(i) = \tilde{Q}_i$ for $i \in J$ and $q(i) = p(i)$ for $i \in I/J$. Since μ_Γ is monotonic, $x_1\mu_\Gamma(q)x_2$ and $x_2\mu_\Gamma(q)x_3$. If $s \in \mathscr{S}^J$ where $s(i) = q(i)$ for all $i \in J$, then $x_1\mu_\Gamma(s)x_2$ and $x_2\mu_\Gamma(s)x_3$. Hence $s(x_1 > x_2) \in \Gamma_J$ and $s(x_2 > x_3) \in \Gamma_J$. Since Γ_J is a filter, $E = s(x_1 > x_2) \cap s(x_2 > x_3) \in \Gamma_J$. If $F = \{i \in J | x_1 s(i) x_3\}$, then $E \subset F$ and $F \in \Gamma_J$. That is, $x_1\mu_{\Gamma_J}(s)x_3$, hence $x_1\mu_\Gamma(q)x_3$. By monotonicity, $x_1\mu_\Gamma(p)x_3$.

(b) By part *(b)* of Theorem (1), we know that $\sigma = \mu_\Gamma$ where Γ is a unique direct sum of simple games, $\{\Gamma_J\}_{J \subseteq I}$. Suppose $\Gamma_L \neq \emptyset$ and $E \in \Gamma_L$, $F \in \Gamma_L$. Let $G = E \cap F$ and $K = E \cup F$. If x, y, z are distinct elements of A, then consider the acyclic profile over $\{x, y, z\}$ given by:

$E/G: zxy$

$G: xyz$

$F/G: yzx$

$L/K: zyx$

We extend the acyclic preferences of each individual in L to a strict order over A. This defines a profile $s \in \mathscr{S}^L$. Then extend s to a profile $p \in \mathscr{R}^I$ by making individuals in I/L universally indifferent. But $s(x > y) = E$ and $s(y > z) = F$, hence $x\mu_{\Gamma_L}(s)y$ and $y\mu_{\Gamma_L}(s)z$. This implies that $x\mu_\Gamma(p)y$ and $y\mu_\Gamma(p)z$. Since $\mu_\Gamma = \sigma$ and σ is a quasi-transitive social decision function, $x\mu_\Gamma(p)z$. That is, $x\mu_{\Gamma_L}(s)z$. But $G = s(x > z)$ and therefore $G \in \Gamma_L$.

Theorem 3. *(a) If Γ is a direct sum of ultrafilters, then μ_Γ, the social function generated by Γ, is a social welfare function.*

(b) If σ is a neutral monotonic social welfare function, then there exists a unique direct sum of ultrafilters, Γ, such that $\sigma = \mu_\Gamma$.

Proof. (a) μ_Γ is a path independent social decision function by part *(a)* of Theorem (2). Suppose for some profile $p \in \mathscr{R}^I$, there exists distinct x_1, x_2, x_3 such that $\neg x_1 \mu_\Gamma(p) x_2$ and $\neg x_2 \mu_\Gamma(p) x_3$. Let $\mathscr{C}$ be the cycle $\{(x_3, x_2), (x_2, x_1), (x_1, x_3)\}$ and J_l the set of concerned individuals for the l^{th} pair in the cycle $\mathscr{C}$. If $J = \bigcup_{l=1}^{3} J_l$, then for each $i \in J$ we apply the Lemma and obtain a strict ordering $\tilde{Q}_i$ over A. Let q be the profile in $\mathscr{R}^I$ where $q(i) = \tilde{Q}_i$ for $i \in J$ and $q(i) = p(i)$ for $i \in I/J$. Since μ_Γ is monotonic, $\neg x_1 \mu_\Gamma(q) x_2$ and $\neg x_2 \mu_\Gamma(q) x_3$. If $s \in \mathscr{S}^J$ where $s(i) = q(i)$ for all $i \in J$, then $\neg x_1 \mu_{\Gamma_J}(s) x_2$ and $\neg x_2 \mu_\Gamma(s) x_3$. Hence $s(x_1 > x_2) \notin \Gamma_J$ and $s(x_2 > x_3) \notin \Gamma_J$. But Γ_J is an ultrafilter and therefore $s(x_2 > x_1) \in \Gamma_J$ and $s(x_3 > x_2) \in \Gamma_J$, since they are respectively the complements of $s(x_1 > x_2)$ and $s(x_2 > x_3)$. Hence $s(x_2 > x_1) \cap s(x_3 > x_2) \in \Gamma_J$, which implies that $x_3 \mu_{\Gamma_J}(s) x_1$, or $\neg x_1 \mu_{\Gamma_J}(s) x_3$. Hence $\neg x_1 \mu_\Gamma(q) x_3$, by monotonicity $\neg x_1 \mu_\Gamma(p) x_3$.

(b) By part *(b)* of Theorem (2), we know that $\sigma = \mu_\Gamma$ where Γ is a unique direct sum of filters, $\{\Gamma_J\}_{J \subseteq I}$. Suppose $\Gamma_L \neq 0$ and $L = E \cup E^c$. Consider the acyclic profile:

$E: yxz$

$E^c: xzy$

We extend the acyclic preferences of each individual in L to a strict order over A. This defines a profile $s \in \mathscr{S}^L$. Then extend s to a profile $p \in \mathscr{R}^I$ by making individuals in I/L universally indifferent. Since Γ_L is a filter, $L \in \Gamma_L$. Hence $x\mu_{\Gamma_L}(s)z$ which implies that $x\mu_\Gamma(p)z$. If either $x\mu_\Gamma(p)y$ or $y\mu_\Gamma(p)x$, then $E^c \in \Gamma_L$ or $E \in \Gamma_L$. If $\neg x\mu_\Gamma(p)y$ and $\neg y\mu_\Gamma(p)x$, then $y\mu_\Gamma(p)z$ and $E \in \Gamma_L$. This follows from the transitivity of $\mu_\Gamma(p)$, i.e., μ_Γ is a social welfare function.

Theorem 4. *(a) If $\{\Gamma_J\}_{J \subseteq I}$ is a direct sum of prefilters, then μ_Γ, the social function generated by Γ, is a neutral monotonic social decision function.*

(b) If σ is a neutral monotonic social decision function and $|A| \geq |I|$, then there exists a unique direct sum of prefilters, Γ, such that $\sigma = \mu_\Gamma$.

Proof. (a) μ_Γ is a neutral monotonic social function by part *(a)* of Theorem (1). Suppose for some profile $p \in \mathscr{R}^I$, that $\mu_\Gamma(p)$ has a minimal social cycle $C = \{(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1)\}$. Let J_l be the set of concerned individuals for the l^{th} pair in the cycle C and $J = \bigcap_{l=1}^{n} J_l$. For each $i \in J$, we apply the Lemma and obtain the strict ordering $\tilde{Q}_i$ over A. Let q be the profile in $\mathscr{R}^I$ where $q(i) = \tilde{Q}_i$ for $i \in J$ and $q(i) = p(i)$ for $i \in I/J$. Since μ_Γ is monotonic, C is also a social cycle for $\mu_\Gamma(q)$. Hence C is a social cycle for $\mu_{\Gamma_J}(s)$ where $s \in \mathscr{S}^J$ and $s(i) = q(i)$ for all $i \in J$. Therefore, $s(x_i > x_{i+1}) \in \Gamma_J$ for all i, where $x_{n+1} \equiv x_1$. Since Γ_J is a prefilter there exists some individual $i_0 \in \bigcap_{i=1}^{n} s(x_i > x_{i+1})$. Hence i_0 has cyclic preferences, which is a contradiction.

(b) By part *(b)* of Theorem (1), we know that $\sigma = \mu_\Gamma$ where Γ is a unique direct sum of simple games. Suppose for some J that $\Gamma_J \neq \emptyset$ and Γ_J is not a prefilter. Then there exists a minimal finite family of $\{E_l\}_{l=1}^n$, where $E_l \in \Gamma_J$ for all l, and $\bigcap_{l=1}^{n} E_l = \emptyset$. Then consider the acyclic latin square profile below:

$E_1 : x_1 x_2$

$E_2 : x_2 x_3$

$\vdots$

$E_n : x_n x_1$

where the x_i are distinct. $\bigcap_{l=1}^{n} E_l = \emptyset$ and $n \leq |I|$; but $|I| \leq |A|$ and therefore $n \leq |A|$. We extend the acyclic profiles of each individual in $E = \bigcap_{l=1}^{n} E_l$ to a strict order over A. This defines a profile $s \in \mathscr{S}^E$. Then we extend s to a profile $p \in \mathscr{R}^I$ by making individuals in I/E universally indifferent. Since Γ is a direct sum and $E \subseteq J$, $E_i \in \Gamma_J \cap 2^E$ for all i, we conclude that each E_i is in Γ_E. But for all i, $E_i \subseteq s(x_i > x_{i+1})$. Hence $x_i \mu_{\Gamma_E}(s) x_{i+1}$. Therefore $x_i \mu_\Gamma(p) x_{i+1}$ and C is a social cycle for $\mu_\Gamma(p)$, which contradicts the assumption that μ_Γ is a social decision function.

Proposition 2. *Let I be finite; Γ a direct sum of simple games; μ_Γ the neutral monotonic social function generated by Γ.*

(a) If Γ is a direct sum of prefilters, then $\{V_r\}_1^t$ is a veto hierarchy for μ_Γ, where $V_1 = \cap \Gamma_I$ and $V_j = \cap \Gamma_{I/\bigcup_1^{j-1} V_r}$ for $j = 2, \ldots, t$.

(b) If Γ is a direct sum of filters, then $\{V_r\}_1^t$ is a hierarchy of oligarchies for μ_Γ.

(c) If Γ is a direct sum of ultrafilters, then $\{V_r\}_1^t$ is a hierarchy of dictators for μ_Γ.

Proof. (a) Suppose $i \in V_1$ and i does not have a veto, then for some $a, b \in A$ and profile $q \in \mathscr{R}^I$; $aq(i)b$ and $b\mu_\Gamma(q)a$. By monotonicity $b\mu_\Gamma(p)a$ where $p \in \mathscr{S}^I$ and $ap(i)b$, $p(b > a) = I/\{i\}$. But $b\mu_\Gamma(p)a$ implies $b\mu_{\Gamma_I}(p)a$ which means $I/\{i\} \in \Gamma_I$, contradicting the assumption that $i \in V_1 = \cap \Gamma_I$. Hence i has a veto.

Suppose $i \in V_2$ and i does not have a veto when all members of V_1 are indifferent. Then using the same argument as above, we get a contradiction. Therefore members of V_2 have a veto when everyone in V_1 is indifferent. Proceeding in this fashion we see that $\{V_r\}_1^t$ is a veto hierachy for μ_Γ.

(b) Since I is finite and the Γ_J are filters, if $\Gamma_J \neq 0$, we see that $V_1 = \cap \Gamma_I \in \Gamma_I$. By Proposition (1), Γ_I is the family of decisive sets for μ_Γ. Hence V_1 is an oligarchy. Suppose everyone in V_1 is indifferent. If $\Gamma_{I/V_1} \neq 0$, then $V_2 = \cap \Gamma_{I/V_1} \in \Gamma_{I/V_1}$. Hence by monotonicity, V_2 is an oligarchy when everyone in V_1 is indifferent. Proceeding in this fashion, we see that $\{V_r\}_1^t$ is a hierarchy of oligarchies for μ_Γ.

(c) Since I is finite and the Γ_J are ultrafilters, if $\Gamma_J \neq 0$, we see that each $\cap \Gamma_J$ is a singleton. That is, the oligarchies exhibited in part (b) are dictators.

Proposition 3. *Let $\Gamma = \{\Gamma_J\}_{J \subseteq I}$ be a direct sum of simple games and μ_Γ the neutral monotonic social function generated by Γ. If μ_Γ is strictly positively responsive and $|I| < \infty$, then for every J such that Γ_J is a prefilter and $|J| \geq 3$, $\cap \Gamma_J = \{i_1\}$ for some $i_1 \in J$.*

Proof. If Γ_J is a prefilter, then $\cap \Gamma_J \neq \emptyset$. Let $i_1 \in \cap \Gamma_J$ and $i_2, i_3 \in J$. By strong monotonicity, $\{i_1, i_2\} \in \Gamma_J$, hence no other members of J belong to $\cap \Gamma_J$. But, again by strong monotonicity, $\{i_1, i_3\} \in \Gamma_J$, hence no other members of J belong to $\cap \Gamma_J$. Thus only $i_1 \in \cap \Gamma_J$.

Theorem 5. *Let $|A| < |I|$ and A be finite.*

(a) If $\{\Gamma_J\}_{J \subset I}$ is a direct sum of acyclic games, then μ_Γ, the social function generated by Γ, is a neutral monotonic social decision function.

(b) If σ is a neutral monotonic social decision function, then there exists a unique direct sum of acyclic games, Γ, such that $\sigma = \mu_\Gamma$.

Proof. (a) Suppose $|A| = m$ and for some profile $p \in \mathscr{R}^I$, that $\mu_\Gamma(p)$ has a minimal social cycle $C = \{(x_1, x_2), (x_2, x_3), \ldots, (x_n, x_1)\}$. (Note $n \leq m$.) Let J_l be the set of concerned individuals for the l^{th} pair in the cycle C and $J = \bigcup_{l=1}^{n} J_l$. For each $i \in J$, we apply the Lemma and obtain the strict ordering $\tilde{Q}_i$ over A. Let q be the profile in $\mathscr{R}^I$ where $q(i) = \tilde{Q}_i$ for $i \in J$ and $q(i) = p(i)$ for $i \in I/J$. Since μ_Γ is monotonic, C is also a social cycle for $\mu_\Gamma(q)$. Hence C is a social cycle for $\mu_\Gamma(s)$ where $s \in \mathscr{S}^J$ and $s(i) = q(i)$ for all $i \in J$. Therefore, $s(x_i > x_{i+1}) \in \Gamma_J$ for all i, where $x_{n+1} \equiv x_1$. If there exists some individual $i_0 \in \bigcap_{i=1}^{n} s(x_i > x_{i+1})$, then i_0 has cyclic preferences, a contradiction. If $\bigcap_{i=1}^{n} s(x_i > x_{i+1}) = \emptyset$, then Γ_J is not an acyclic game. Therefore μ_Γ is a social decision function.

(b) By part *(b)* of Theorem (1), we know that $\sigma = \mu_\Gamma$ where Γ is a unique direct sum of simple games. Suppose for some J that $\Gamma_J \neq \emptyset$ and Γ_J is not an acyclic game. Then there exists a finite family of $\{E_l\}_{l=1}^{n}$ where $E_l \in \Gamma_J$ for all l; $\bigcap_{l=1}^{n} E_l = \emptyset$; and $n \leq m$. Consider the acyclic latin square profile below:

$E_1 : x_1 x_2$

$E_2 : x_2 x_3$

$\vdots$

$E_n : x_n x_1$

where the x_i are distinct. We extend the acyclic profiles of each individual in

$E=\bigcup_{i=1}^{n} E_i$ to a strict order over A. This defines a profile $s \in \mathscr{S}^E$. Then we extend s to a profile $p \in \mathscr{R}^I$ by making individuals in I/E universally indifferent. Since Γ is a direct sum of simple games and $E_i \in \Gamma_J \cap 2^E$ for all i; we conclude that each $E_i \in \Gamma_E$. But for all i, $E_i \subseteq s(x_i > x_{i+1})$. Hence $x_i \mu_{\Gamma_E}(s) x_{i+1}$. Therefore $x_i \mu_\Gamma(p) x_{i+1}$ and C is a social cycle for $\mu_\Gamma(p)$, which contradicts the assumption that μ_Γ is a social decision function.

Proposition 4. *Let $|A|<|I|<\infty$ and A be finite. Let $\{\Gamma_J\}_{J\subseteq I}$ be a direct sum of acyclic games and μ_Γ the neutral monotonic social decision function generated by Γ. If $|A|=m$ then μ_Γ is anonymous iff*

(a) For each J such that $\Gamma_J \neq 0$ there exists an integer r such that $E\in\Gamma_J$ iff $|E|\geq r$ and $r>\frac{m-1}{m}|J|$.

(b) If $|J|\leq|A|$ and $\Gamma_J\neq 0$, then $\Gamma_J=\{J\}$.

Proof. Suppose μ_Γ is anonymous and $\Gamma_J \neq \emptyset$. Clearly all the minimal decisive sets in Γ_J have the same cardinality, which we will call r. If $r\leq\frac{m-1}{m}|J|$ and $|J|>|A|$, then there exists $p\in\mathscr{S}^J$ such that $\mu_\Gamma(p)$ has a social cycle. This profile is constructed in Craven's paper [3]. Hence if $|J|>|A|$, then $r>\frac{m-1}{m}|J|$. If $|J|\leq|A|$ and $\Gamma_J\neq 0$, then Γ_J is a prefilter. But the only anonymous social function defined by a prefilter on J is the prefilter $\{J\}$.

The converse is immediate.

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