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COMPUTING ZEROS OF SECTIONS OF VECTOR BUNDLES USING HOMOTOPIES AND RELOCALIZATION

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An algorithm is described for computing fixed points on a Grassmannian manifold. The method can be applied in the more general setting of solving equations on abstract smooth manifolds into vector bundles. This development is part of a project to compute economic equilibria in the presence of incomplete markets.

1. Introduction. There are many existence theorems for systems of equations where the domain and range spaces are smooth manifolds as opposed to Euclidean space. These theorems are cast in the vocabulary of a mathematical level where the results are conveniently summarized, for example, with fixed points, index theory, intersection theory, or homology. However, it appears to the present authors that such results are invariably proved using homotopies though the homotopies are long since buried in the development by the time the result is summarized as a theorem. Given the ever increasing importance of computation these buried homotopies are of greater interest. There seems to be considerable merit in bringing the homotopy to the forefront to emphasize the avenue of computation. When homotopies are used in an existence proof in differential (smooth) topology there seems to be a rather evident avenue for computation. Namely, follow the route of zeros of the homotopy by localizing the domain of the homotopies to Euclidean space wherein the computation can actually take place. Of course, localization of Euclidean space to obtain global results is the basic vehicle in the subject of differential topology. Herein we shall outline a relocalization method for route following to solve a system of equations on the Grassmannian manifold, and then we show that the approach is a special case of computing a zero of a section of a vector bundle of an abstract smooth manifold.

Let G_k^n be the set of k-planes passing through the origin in R^n. The set G_k^n, endowed with the obvious topology, is known as the Grassmannian manifold of k-planes in R^n. The manifold is smooth, has no boundary, is compact, and has dimension $k(n-k)$. Let $f_i\colon G_k^n \to R^n$ for $i = 2, \dots, k$ be continuous functions. It is a fact that there is a fixed point of the functions f_i in the sense that there exists a τ in G_k^n where

$$f_i(\tau) \in \tau \qquad i = 1, \dots, k.$$

It is our purpose in this paper to indicate how a fixed point for the f_i's can be computed. For the smooth methods to be employed herein, we shall require the additional assumption that the f_i's are smooth. A homotopy is constructed so that a one-dimensional manifold of zeros $\tilde{Z}$ of the homotopy leads to a fixed point of the

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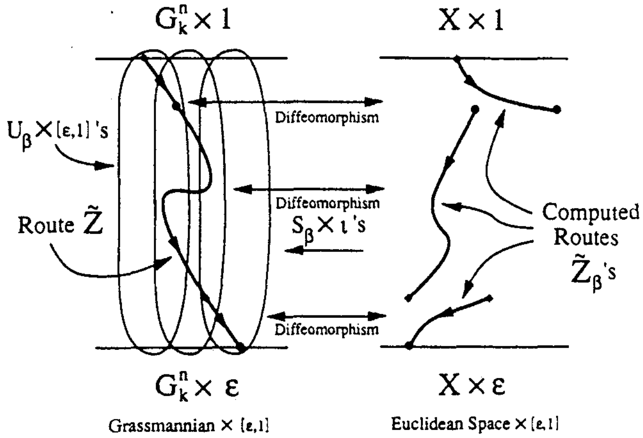

FIGURE 1. Relocalization.

f_i's. To follow this one-dimensional manifold of zeros on the cylinder $G_k^n \times [\epsilon, 1]$, we sequentially localize the computation to Euclidean space. Diffeomorphic segments in $R^{k(n-k)+1}$ of the one-dimensional manifold $\tilde{Z}$ are followed, and thereby, we are able, in effect, to follow $\tilde{Z}$ in the cylinder to a fixed point of the f_i's. We refer to this process of sequentially following diffeomorphic copies of pieces of $\tilde{Z}$ as relocalization, see Figure 1.

The fixed point theorem considered in this paper is the essence of the proof of generic existence of an equilibrium of an economy with incomplete markets. How does the Grassmannian manifold arise in this economic problem? Each economic agent optimizes a utility while maintaining budget conditions; in particular, certain budget vectors must be kept in a linear subspace of asset returns $L(p)$ which has the form

$$L(p) \equiv \text{column span} \begin{pmatrix} p_1 A_1 \\ p_2 A_2 \\ \vdots \\ p_n A_n \end{pmatrix} \subseteq R^n,$$

where $p = (p_1, p_2, \ldots, p_n)$, each p_i is a row vector of prices, and the matrices A_i represent the traded assets. Generically $L(p)$ has a constant dimension, say k; however, for critical prices p the dimension of $L(p)$ drops, which in turn, transmits discontinuities to the optimal responses to prices. This caveat delayed knowledge of generic existence of equilibria in this market for many years; see Arrow (1953) and Hart (1975). The impasse was circumvented by Duffie and Shafer (1985) where the subspace $L(p)$ in R^n was replaced by k-dimensional planes τ in R^n, that is, by introduction of the Grassmannian manifold, and by adjoining the equation $L(p) \subseteq \tau$. The fixed point of the resulting system yields an economic equilibrium only when $L(p) = \tau$, which occurs generically. In the paper DeMarzo and Eaves (1993) the algorithm given herein is fully extended for the computation of economic equilibria in the presence of incomplete markets; a price simplex must be adjoined to the Grassmannian manifold. This computational scheme together with that of Brown, DeMarzo, and Eaves (1993) offers the first methods for computation of equilibria of economic models with incomplete asset markets. For further discussion of the relation of this fixed point problem on the Grassmannian to the economic problem, see Hirsch, Magill and Mas-Colell (1990), Chichilnisky and Heal (1993), and Geanakoplos (1990). For $k = 1$ the Grassmannian G_k^n specializes to n-dimensional real projective space, and the fixed point theorem herein specializes to the Borsuk-Ulam Theorem. Wright (1985) used relocalization on projective spaces in the computation of zeros of polynomials; however, the method was not proved and relocalization was not suggested as a device with broad applicability.

In §2 needed projections are briefly discussed. In §3 the homotopy to be used is introduced and the route of zeros of the homotopy to be followed is indicated. In §4 the atlas of the Grassmannian employed is described. In §5, using the atlas, the equations are moved locally from the Grassmannian to Euclidean space. In §6 generic conditions are given for zero to be a regular value of the homotopy. In §7 the existence of the route to be followed is settled. In §8 the sequential localization for following the route of zeros to the solution is established. In §9 a broader perspective is taken; indeed, the methods employed for the Grassmannian manifold are shown to be a special case of computing a zero of a section of a vector bundle. Fundamental terminology and results from differential topology as in Guillemin and Pollack (1974), Hirsch (1976), and Milnor (1969) are adopted herein.

2. Projections. For each plane τ in R^n define the projection operator Π_τ: $R^n \to \tau$. For x in R^n the point $\Pi_\tau(x)$ is the point in τ nearest to x and, in particular, we have

$$x - \Pi_\tau(x) \in \tau^\perp$$

where $\tau^\perp$ is the orthogonal complement of τ in R^n, that is $\tau + \tau^\perp = R^n$ and $\tau \cdot \tau^\perp = 0$.

Define $\tilde{f}_i$: $G_{n-k}^n \to R^n$ by

$$\tilde{f}_i(\tau) = \Pi_\tau(f_i(\tau^\perp))$$

for $k = 1, \ldots, k$. Observe that $\tilde{f}_i(\tau) \in \tau$ for all τ. We also observe that τ is a fixed point of the f_i's if and only if $\tau^\perp$ is a zero of the $\tilde{f}_i$'s. Further for τ in G_k^n and x in R^n the functions $\Pi_\tau(x)$ are computable and smooth; this statement will be elaborated on later. If the f_i's can be computed and differentiated the $\tilde{f}_i$'s can be computed and differentiated also.

With these facts in mind we formulate, without loss of generality, a more convenient but equivalent problem. First replace k by $n - k$. Let f_i: $G_k^n \to R^n$ for $i = 1, \ldots, n-k$ be smooth functions with $f_i(\tau) \in \tau$ for all τ in G_k^n. We seek a τ which is a zero of the f_i, that is, $f_i(\tau) = 0$ for $i = 1, \ldots, n-k$. Define f: $G_k^n \to R^{n(n-k)}$ to be the vector of functions $f = (f_1, \ldots, f_{n-k})$. Let τ^{n-k} denote $\tau \times \tau \times \cdots \times \tau$ where τ is repeated $n-k$ times. We have $f(\tau) \in \tau^{n-k}$ for all τ in G_k^n. Our task is to find a zero of f. An example of such f and zero τ is g_α and τ_α given in the next paragraph.

Let α_i for $i = 1, \ldots, n-k$ be vectors in R^n and let $\alpha = (\alpha_1, \ldots, \alpha_{n-k})$ be the vector in $R^{n(n-k)}$. Let A be the set of α such that $\alpha_1, \ldots, \alpha_{n-k}$ are linearly independent. The set A is open in $R^{n(n-k)}$. Define g: $A \times G_k^n \to R^{(n(n-k)}$ by

$$g(\alpha, \tau) = (\Pi_\tau(\alpha_1), \ldots, \Pi_\tau(\alpha_{n-k})).$$

For all (α, τ) in $A \times G_k^n$ we have $g(\alpha, \tau)$ in τ^{n-k}. The function g is smooth. Let $g_\alpha(\cdot) = g(\alpha, \cdot)$. Let τ_α be the orthogonal complement of the span of the vectors $\alpha_1, \ldots, \alpha_{n-k}$. Clearly τ_α is the unique zero in G_k^n of g_α, that is, $\tau_\alpha = \tau$ is the only solution of $g_\alpha(\tau) = 0$. We refer to g as the auxiliary function; it will be used to define the top of the homotophy, to initiate the computation, to assure that there is a route to follow, and to assure that the route leads to a zero of f at the bottom of the homotopy.

3. The homotopy and route. Given the smooth function f: $G_k^n \to R^{n(n-k)}$ with $f(\tau) \in \tau^{n-k}$ for all τ in G_k^n we seek a solution of $f(\tau) = 0$. In our quest for a zero of f our plan is to begin with the unique zero τ_α of the auxiliary function g_α and follow the route of zeros as g_α is deformed, perhaps with retrogressions, to our function of interest f. To this purpose we introduce a homotopy.

Select ϵ as a small fixed positive constant in the open interval (0, 1). The domain of our homotopy is the cylinder $G_k^n \times [\epsilon, 1] = \{(\tau, \theta): \tau \in G_k^n, \theta \in [\epsilon, 1]\}$. This cylinder is a compact smooth manifold with boundary $G_k^n \times \{\epsilon, 1\}$. We refer to $G_k^n \times 1$ and $G_k^n \times \epsilon$ as the top and bottom of the cylinder, respectively.

Define the homotopy

$$H: G_k^n \times [\epsilon, 1] \to R^{n(n-k)}$$

by

$$H(\tau, \theta) = \theta g_\alpha(\tau) + (1 - \theta) f(\tau).$$

As $g_\alpha(\tau) \in \tau^{n-k}$ and $f(\tau) \in \tau^{n-k}$ we have $H(\tau, \theta) \in \tau^{n-k}$ for all τ in G_k^n and θ in $[\epsilon, 1]$. The homotopy restricted to the top of the cylinder is g_α and restricted to the bottom is almost f,

$$H(\cdot, 1) = g_\alpha,$$

$$H(\cdot, \epsilon) = f + \epsilon(g_\alpha - f).$$

The function $\|\epsilon(g_\alpha - f)\|$ is bounded by ϵK where

$$K = \max_\tau \|g_\alpha(\tau) - f(\tau)\| \leq (n - k)\Big(\max_i \|\alpha_i\| + \max_{i,\tau} \|f_i(\tau)\|\Big).$$

Thus if we solve $H(\tau, \epsilon) = 0$ approximately for τ, that is, say $\|H(\tau, \epsilon)\| \leq \delta$ then we have solved $f(\tau) = 0$ approximately, that is, $\|f(\tau)\| \leq \epsilon K + \delta$. If we let ϵ and δ tend to zero then we have solved $f(\tau) = 0$. We turn our attention to computing an approximate zero of $H(\cdot, \epsilon)$.

The purpose of the ϵ versus zero for the interval $[\epsilon, 1]$ is simply that we are perturbing the function f by the auxiliary function g_α. With a generic choice of α we shall show that the zero set of H is well behaved; however, if ϵ were replaced by zero, this conclusion would be lost. Nevertheless, ϵ can be selected as small as one chooses, except at some point, depending upon f, one may invite numerical difficulties.

Define the set $Z = H^{-1}(0)$ of zeros of H by

$$Z = \{(\tau, \theta) \in G_k^n \times [\epsilon, 1]: H(\tau, \theta) = 0\}.$$

Define $\tilde{Z}$ to be the connected component of Z which contains $(\tau_\alpha, 1)$; we plan to follow $\tilde{Z}$ with relocalization. Recall that $H(\tau_\alpha, 1) = g_\alpha(\tau_\alpha) = 0$.

Define a route and a loop to be a (nonempty) one-dimensional manifold which is diffeomorphic to a convex subset of R^1 and to a circle in R^2, respectively. A route may have 0, 1, or 2 boundary points, whereas a compact route has exactly two boundary points. A loop is compact and has no boundary points. Every connected one-dimensional manifold is either a route or loop, and not both. Every compact one-dimensional manifold is a finite disjoint collection of loops and compact routes.

Let ∂H be the restriction of H to the boundary $G_k^n \times \{\epsilon, 1\}$ of the cylinder $G_k^n \times [\epsilon, 1]$. For almost all α in A we will show that zero is a regular value of H and ∂H. For such α, the zero set $Z = H^{-1}(0)$ of H is a neat smooth compact one-dimensional submanifold in the cylinder $G_k^n \times [\epsilon, 1]$. In particular,

(1) $\partial Z = Z \cap (G_k^n \times \{\epsilon, 1\})$ and Z is transverse to $G_k^n \times \{\epsilon, 1\}$.

(2) Z is a finite disjoint collection of loops and compact routes.

As $\partial Z = Z \cap (G_k^n \times \{\epsilon, 1\})$ we see that $(\tau_\alpha, 1)$ is a boundary point of Z and that $\tilde{Z}$ is a route. If we follow the route $\tilde{Z}$ beginning at the top of the cylinder at $(\tau_\alpha, 1)$, the route will not return to the top as Z meets the top in only one point, namely, $(\tau_\alpha, 1)$. Thus the compact route $\tilde{Z}$ leads us to the bottom of the cylinder, that is, to a point in $Z \cap (G_k^n \times \epsilon)$, that is, to a zero of $H(\cdot, \epsilon)$, thereby completing out task. We shall not directly follow the route $\tilde{Z}$ on the cylinder, but rather, we shall follow a sequence of diffeomorphic localizations of $\tilde{Z}$ in Euclidean space.

4. An atlas for the Grassmannian G_k^n. To follow the route $\tilde{Z}$ we shall need an atlas of G_k^n. Let $|\cdot|$ be the maximum norm for a vector, that is, $|x| = \max_i |x_i|$. Select r in the range $1 < r < +\infty$ and define X by

$$X = \{x \in R^{k(n-k)}: |x| < r + 1\}.$$

For the vector x in X define wrap(x) to be the $((n-k) \times k)$-matrix

$$\begin{pmatrix} x_1 & \cdots & x_k \\ x_{k+1} & \cdots & x_{k_2} \\ & \vdots & \\ x_{k(n-k)-k+1} & \cdots & x_{k(n-k)} \end{pmatrix}.$$

Let B be the set of all subsets of $\{1, \ldots, n\}$ of size k. Given β in B let ν be the complement of β in $\{1, \ldots, n\}$. We take the order of the elements in both β and ν as the natural order.

For β in B let Π_β: $R^n \to R^k$ select the k rows indexed by β; we apply Π_β to matrices also. For β in B and x in X define the $(n \times k)$-matrix $F_\beta(x)$ by

$$\Pi_\beta F_\beta(x) = I,$$

$$\Pi_\nu F_\beta(x) = \text{wrap}(x).$$

That is, the rows of $F_\beta(x)$ indexed by β form the identity matrix and the rows of $F_\beta(x)$ indexed by ν is the $((n-k) \times k)$-matrix wrap(x).

EXAMPLE. For $n = 4$, $k = 2$, $\beta = \{1, 3\}$, and $x(x_1, x_2, x_3, x_4)$ we have

$$\text{wrap}(x) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \qquad F_\beta(x) = \begin{pmatrix} 1 & 0 \\ x_1 & x_2 \\ 0 & 1 \\ x_3 & x_4 \end{pmatrix}. \quad \square$$

For β in B define $S_\beta(x)$ to be the vector space spanned by the columns of $F_\beta(x)$, that is,

$$S_\beta(x) = \operatorname{span} F_\beta(x) \subseteq R^n.$$

Observe that a vector $y \in S_\beta(x)$ is completely determined by its β coordinates, that is,

$$y = \left(F_\beta(x)\right)\Pi_\beta y.$$

Observe that $S_\beta(x) = S_\beta(y)$ implies $x = y$. The projection Π_β: $S_\beta(x) \to R^k$ is a linear bijection.

Define the multifunction Λ: $G_k^n \rightrightarrows B$ by $\Lambda(\tau) = \{\beta \in B: \tau = S_\beta(x), |x| \leq 1\}$. Given vectors spanning τ, a β in $\Lambda(\tau)$ can be effectively computed; see Eaves (1993). The multifunction is called a locator and its purpose is to "center" a chart at τ. For β in B define the open subset U_β of G_k^n by

$$U_\beta = \left\{S_\beta(x) \colon x \in X\right\}.$$

The U_β for β in B yields a finite open cover of the manifold G_k^n.

For β in B let S_β^{-1} be the inverse of S_β, that is,

$$S_\beta^{-1}(\tau) = x,$$

where $S_\beta(x) = \tau$. The collection of $\Psi = \{(S_\beta^{-1}, U_\beta)\colon \beta \in B\}$ forms a finite atlas for the manifold G_k^n. In particular, for β in B,

$$\begin{array}{l} U_\beta \subseteq G_k^n \\ {\scriptstyle S_\beta}\uparrow \\ X \subseteq R^{k(n-k)} \end{array}$$

is a diffeomorphism.

For β in B and x in X define $P_\beta(x)$ to be the $n \times n$ matrix which projects R^n to $S_\beta(x)$. That is,

$$P_\beta(x)y = \Pi_\tau(y),$$

where $\tau = S_\beta(x)$. The matrix $P_\beta(x)$ is given by

$$F_\beta(x)\left(F_\beta(x)^T F_\beta(x)\right)^{-1} F_\beta(x)^T.$$

The coordinates of $P_\beta(x)y$ relative to the frame $F_\beta(x)$ are given by

$$\begin{aligned} \Pi_\beta\left(P_\beta(x)y\right) &= \Pi_\beta P_\beta(x)y \\ &= \Pi_\beta\left(F_\beta(x)\left(\left(F_\beta(x)^T F_\beta(x)\right)^{-1} F_\beta(x)^T y\right)\right) \\ &= \Pi_\beta F_\beta(x)\left(\left(F_\beta(x)^T F_\beta(x)\right)^{-1} F_\beta(x)^T y\right) \\ &= \left(F_\beta(x)^T F_\beta(x)\right)^{-1} F_\beta(x)^T y, \end{aligned}$$

as $\Pi_\beta F_\beta(x) = I$. In particular, we see that rows $\Pi_\beta P_\beta(x)$ of $P_\beta(x)$ indexed by β are linearly independent. Also observe that

$$\nabla_x\big((\Pi_\beta P_\beta(x))y\big)$$

can be computed analytically (however, such can be avoided).

5. Localizations H_β of the homotopy H. Local coordinates are established for the homotopy H with the purpose of understanding and following a route in the zeros $Z = H^{-1}(0)$ of H. Our understanding of Z is first local then global.

Select a β in B. Such β indexes the open set $U_\beta = \{S_\beta(x) \colon x \in X\}$ of the manifold G_k^n. Define the projection operator $\Pi_\beta^{n-k} \colon R^{n(n-k)} \to R^{k(n-k)}$ by

$$\Pi_\beta^{n-k} y = \big(\Pi_\beta y_1, \Pi_\beta y_2, \ldots, \Pi_\beta y_{n-k}\big)$$

where $y = (y_1, \ldots, y_{n-k})$ and each y_i is in R^n. For each x in X observe that

$$\Pi_\beta^{n-k} \colon S_\beta(x)^{n-k} \to R^{k(n-k)}$$

is a linear bijection. Note the superscript $n-k$ indicates a cross product $n-k$ times. For all x in X and y in $S_\beta(x)^{n-k}$ we have $\Pi_\beta^{n-k} y = 0$ if and only if $y = 0$.

Let iota ι represent the identity map on $[\epsilon, 1]$, that is $\iota(\theta) = \theta$ for θ in $[\epsilon, 1]$. For β in B we shall understand and compute zeros of

$$H \colon U_\beta \times [\epsilon, 1] \to R^{n(n-k)}$$

by understanding and computing the zeros of the

$$H_\beta \colon X \times [\epsilon, 1] \to R^{k(n-k)}$$

where H_β is defined by

$$H_\beta(x, \theta) = \Pi_\beta^{n-k} H(S_\beta \times \iota)(x, \theta)$$
$$= \Pi_\beta^{n-k} H\big(S_\beta(x), \theta\big).$$

Pictorially we have the commutative diagram in Figure 2. As $S_\beta \times \iota$ is a diffeomorphism and as $\Pi_\beta^{n-k} H(\tau, \theta) = 0$ if and only if $H(\tau, \theta) = 0$, the zeros of the composite function H_β on $X \times [\epsilon, 1]$ are diffeomorphic to the zeros of H on $U_\beta \times [\epsilon, 1]$. The advantage of the function H_β over the function H is that the range and domain are $R^{k(n-k)+1}$ and $R^{k(n-k)}$ instead of U_β and $R^{n(n-k)}$. The computational task for

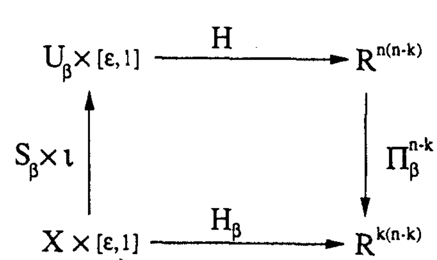

FIGURE 2. Localized homotopy H_β.

following the route of zeros of the homotopy H_β is in Euclidean space and is well formulated.

6. Generic α for regularity of zero. Our purpose in this section is to show that for almost all α in A for all β in B zero is a regular value of H_β and ∂H_β. Our vehicle for this result is the Transversality Theorem; see Guillemin and Pollack (1974, p. 68). We apply the theorem locally and then use the local results to get global results.

For β in B consider the map $H_\beta\colon X \times [\epsilon, 1] \to R^{k(n-k)}$ where for the moment we also consider α in A as an argument of the function H_β.

The derivative of $H_\beta = \Pi_\beta^{n-k} H(S_\beta \times \iota)$ with respect to α at the point (α, x, θ) is given by

$$\nabla_\alpha H_\beta(x,\theta) = \nabla_\alpha\left(\prod_\beta^{n-k} H(S_\beta \times \iota)(x,\theta)\right)$$

$$= \nabla_\alpha\left(\prod_\beta^{n-k} \theta g_\alpha(S_\beta(x))\right)$$

$$= \theta \prod_\beta^{n-k} \nabla_\alpha \begin{pmatrix} P_\beta(x)\,\alpha_1 \\ P_\beta(x)\,\alpha_\iota \\ \vdots \\ P_\beta(x)\,\alpha_{n-k} \end{pmatrix}$$

$$= \theta \begin{pmatrix} \prod_\beta P_\beta(x) & & & \\ & \prod_\beta P_\beta(x) & & \\ & & \ddots & \\ & & & \prod_\beta P_\beta(x) \end{pmatrix},$$

where θ in $[\epsilon, 1]$ is positive. We have previously shown that $\Pi_\beta P_\beta(x)$ has rank k. Thus the rank of the derivative $\nabla_\alpha H_\beta$ for all (x, θ) in $X \times [\epsilon, 1]$ is $k(n-k)$.

For β in B let Z_β be the set of zeros of H_β in $X \times [\epsilon, 1]$. Recall that $X = \{x \in R^{k(n-k)}\colon |x| < r + 1\}$, and temporarily increase r by 1. Now applying the Transversality Theorem we may conclude that there is a set $A_\beta \subseteq A$ where $A \setminus A_\beta$ has measure zero and for all α in A_β the following statement T_β is true.

T_β: If (x, θ) is in Z_β, then the rank of

$$\nabla_{(x,\theta)} H_\beta(x,\theta)$$

is $k(n-k)$. If (x,θ) is in Z_β and $\theta = \epsilon$ or $\theta = 1$, then the rank of

$$\nabla_x H_\beta(x,\theta)$$

is $k(n-k)$. □

Let A_* be the intersection of all A_β as β varies over B. As B is finite we see that A_* is generic in A, in particular, $A \setminus A_*$ has measure zero. We conclude that for all α in A_* the statement T_β is true for every β in B.

Apply the implicit function theorem at the points (x, θ) in Z_β. Upon decreasing r by 1 to its original value, it follows that Z_β is a neat smooth one-dimensional submanifold in $X \times [\epsilon, 1]$. In particular,

(1) $\partial Z_\beta = Z_\beta \cap (X \times \{\epsilon, 1\})$ and Z_β is transverse to $X \times \{\epsilon, 1\}$.

(2) Each connected component of Z_β is a route or loop with finite arc length.

(3) Only a finite number of the loops and routes meet both $|x| \leq 1$ and $|x| \geq r$.

Note, Z_β could have an infinite number of connected components. For a discussion of arc length on one-manifolds see Milnor (1969). (It seems unlikely that Z_β could have infinite arc length, but whatever the case, we do not require such conclusion herein.)

7. Global route of zeros Z. We take our local results for the zeros Z_β of H_β in $X \times [\epsilon, 1]$ and transfer them all to the domain $G_k^n \times [\epsilon, 1]$ to obtain the zero set Z of H.

For all α in A_* and β in B zero is a regular value of H_β and ∂H_β on $X \times [\epsilon, 1]$. Recall that U_β for β in B is a finite open cover of G_k^n, that $S_\beta\colon X \to U_\beta$ is a diffeomorphism, and that $H_\beta(x, \theta) = 0$, if and only if $H(S_\beta(x), \theta) = 0$ where

$$H_\beta(x, \theta) = \prod_\beta^{n-k} H(S_\beta \times \iota)(x, \theta).$$

The zero set Z of H is given by the union of the sets

$$(S_\beta \times \iota)(Z_\beta),$$

as β varies over B. Furthermore, for β and γ in B the sets

$$(S_\beta \times \iota)(Z_\beta) \quad \text{and} \quad (S_\gamma \times \iota)(Z_\gamma),$$

agree exactly on

$$(U_\beta \cap U_\gamma) \times [\epsilon, 1];$$

see Figure 3.

As zero is a regular value for the H_β and ∂H_β for all β in B it follows that zero is a regular value of H and ∂H. In particular, Z is a neat smooth compact one-dimensional submanifold in the compact manifold $G_k^n \times [\epsilon, 1]$; in particular,

(1) $\partial Z = Z \cap (G_k^n \times \{\epsilon, 1\})$ and Z is transverse to $G_k^n \times \{\epsilon, 1\}$.

(2) Z is a finite disjoint union of loops and compact routes.

THEOREM. *There is a solution τ in G_k^n to $f(\tau) = 0$.*

PROOF. Let $\tilde{Z}$ be the component of Z which contains $(\tau_\alpha, 1)$. As $Z \cap (G_k^n \times 1)$ contains the single point $(\tau_\alpha, 1)$ and $\partial Z = Z \cap G_k^n \times \{\epsilon, 1\}$ we see that $\tilde{Z}$ is a compact route. The second boundary point (τ_ϵ, θ) of $\tilde{Z}$ can only be in $G_k^n \times \epsilon$. Thus we have an approximate solution to $f(\tau) = 0$. As mentioned before G_k^n is compact, hence, if we let ϵ tend to zero we obtain an exact solution to $f(\tau) = 0$. □

8. Following the route $\tilde{Z}$ by relocalization. Using a predictor-corrector method we follow certain routes $\tilde{Z}_\beta$ in the zero sets $Z_\beta \subseteq X \times [\epsilon, 1]$ of the homotopies H_β

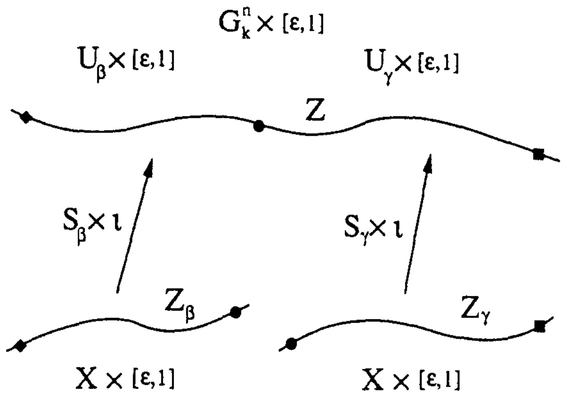

FIGURE 3. Assembling Z.

for certain β in B. Using the diffeomorphisms $S_\beta \times \iota$ these routes are transferred to $G_k^n \times [\epsilon, 1]$ to form the route of zeros $\tilde{Z}$ of the homotopy H. At the end of the route $\tilde{Z}$, at the bottom of $G_k^n \times [\epsilon, 1]$, we find an approximate solution of the system $f(\tau) = 0$.

The prediction-corrector method now has an extensive development with many safeguards and refinements. Some pertinent references include Allgower and Georg (1992), Davidenko (1953a), Kellogg, Li, and Yorke (1976), Morgan (1987), Rheinbolt (1986), Watson (1989), Watson, Billups, and Morgan (1987), and Zangwill and Garcia (1981). The papers of Scarf and Hansen (1973), Eaves (1972), Kellogg, Li, and Yorke (1976), and Smale (1976) also form part of the economic, mathematical, and computational background for this effort. It is not our intention here to develop the predictor-corrector method as such but rather to apply it to solve problems on non-Euclidean manifolds, and, at the moment, on the Grassmannian manifold G_k^n. Let us begin by giving a brief description of a typical version of a predictor-corrector iteration but couched in our framework, namely, in the domain $X \times [\epsilon, 1]$ for a function H_β. Recall $X = \{x \in R^n\colon |x| < r + 1\}$ where $r > 1$. Also we assume that α is a vector of A_*, and hence Z_β is a one-dimensional submanifold in $X \times [\epsilon, 1]$.

Predictor-corrector iteration. Assume we have a point $(x(i), \theta(i))$ on or near $\tilde{Z}_{\beta(i)} \subseteq Z_{\beta(i)} \subseteq X \times [\epsilon, 1]$. The derivative $\nabla_{(x,\theta)} H_{\beta(i)}$, or approximation thereof, is computed at $(x(i), \theta(i))$ and the system

$$\nabla_{(x,\theta)} H_{\beta(i)}(x(i), \theta(i))(\bar{x}(i), \bar{\theta}(i)) = 0 \qquad \|(\bar{x}(i), \bar{\theta}(i))\| = 1,$$

is solved or approximately solved for $(\bar{x}(i), \bar{\theta}(i))$. As the derivative is $k(n-k) \times (k(n-k)+1)$-matrix of full row rank the exact $(\bar{x}(i), \bar{\theta}(i))$ is unique up to sign.

We now take a predictor step by moving in the line $L = \{(x(i), \theta(i)) + t(\bar{x}(i), \bar{\theta}(i))\colon t \in R\}$. The predictor step moves in $L \cap (X \times [\epsilon, 1])$ in the direction $(\bar{x}(i), \bar{\theta}(i))$ or direction $-(\bar{x}(i), \bar{\theta}(i))$ so as to move away from the previously generated point. One way to accomplish this is to choose the sign of $(\bar{x}(i), \bar{\theta}(i))$ so that $\bar{x}(i-1), \bar{\theta}(i-1)) \cdot (\bar{x}(i), \bar{\theta}(i)) > 0$ where $\bar{x}(i-1), \theta(i-1))$ was the direction of movement for the previous iteration. Assuming the movement is in the direction $(\bar{x}(i), \bar{\theta}(i))$, the predictor step is to some point $(x(i), \theta(i)) + \bar{t}(\bar{x}(i), \bar{\theta}(i))$ where $\bar{t} > 0$. Basically the choice of the step size $\bar{t}$ is related to how straight $\tilde{Z}_\beta(i)$ is at $(x(i), \theta(i))$. Much has been written about particular choices for $\bar{t}$ and we will say little more about it here.

Now for the corrector step. The purpose here is to move in the orthogonal complement $L^\perp$ of L in $X \times [\epsilon, 1]$ at $(x(i), \theta(i)) + \bar{t}(\bar{x}(i), \bar{\theta}(i))$ and return to $\tilde{Z}_{\beta(i)}$;

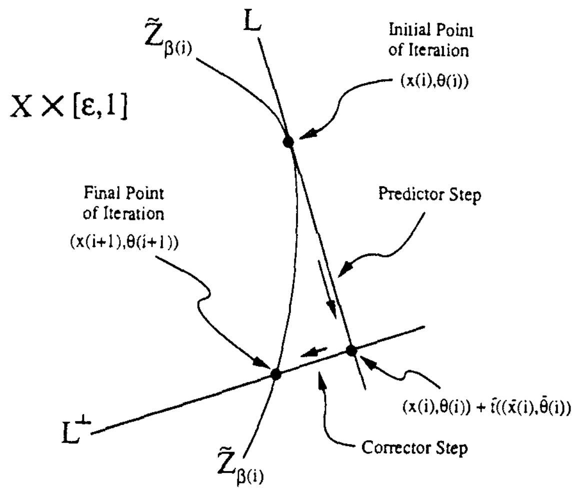

FIGURE 4. Predictor-corrector iteration following $\tilde{Z}_{\beta(i)}$.

see Figure 4. For the corrector step Newton's method, or a variation thereof, is used to find an approximation of a zero of $H_{\beta(i)}$ in the translated space $L^{\perp}$ beginning with the point $(x(i), \theta(i)) + \bar{t}(\bar{x}(i), \bar{\theta}(i))$. If the convergence here is satisfactory, $\bar{t}$ remains unaltered, and perhaps, in this case, the initial $\bar{t}$ is increased in future iterations. If the convergence Newton's method is not satisfactory, $\bar{t}$ is repeatedly reduced until convergence in the corrector step is satisfactory; and in this case, perhaps the initial $\bar{t}$ is decreased in future steps. Following satisfactory convergence in the corrector step, this iteration of the predictor-corrector method is complete. By the iterate generated by the predictor-corrector iteration, we mean that point $(x(i+1), \theta(i+1))$ where the corrector phase is terminated. Set $\beta(i+1) = \beta(i)$. Unless we have made an exit, a matter discussed shortly, the next iteration begins with a predictor step at $(x(i+1), \theta(i+1))$, etc. This completes our discussion of a typical predictor-corrector iteration. □

We have described a typical predictor-corrector iteration, however, in the present application there are a number of special considerations to be addressed.

The derivative of H_β with respect to (x, θ) is given by

$$\nabla_{(x,\theta)} H_\beta(x,\theta) = \nabla_{(x,\theta)}\left(\prod_{\beta}^{n-k}\left(\theta g_\alpha S_\beta(x) + (1-\theta) f S_\beta(x)\right)\right)$$

$$= \left(\theta \nabla_x\left(\prod_{\beta}^{n-k} g_\alpha S_\beta\right)\right)(x) + (1-\theta)\nabla_x\left(\prod_{\beta}^{n-k} f S_\beta\right)(x),$$

$$\left.\prod_{\beta}^{n-k} g_\alpha S_\beta(x) - \prod_{\beta}^{n-k} f S_\beta(x)\right).$$

We address the four summands appearing in the bracketed pair just above. The first and third terms $\nabla_x(\prod_\beta^{n-k} g_\alpha S_\beta)(x)$ and $\prod_\beta^{n-k} g_\alpha S_\beta(x)$ can be computed analytically. If we can evaluate the f_i then the fourth term

$$\prod_{\beta}^{n-k} f S_\beta(x)$$

can be evaluated. To say that the f_i are smooth implies, essentially, that the second term

$$\nabla_x\left(\prod_{\beta}^{n-k} fS_\beta\right)(x)$$

exists. To say the f_i can be differentiated analytically is essentially to say the fourth term can be differentiated analytically. We require that the derivatives $\nabla_{(x,\theta)}H_\beta$ are available, either analytically or approximately by numerical methods.

To initialize the computation procedure calculate k linearly independent vectors $p_1, \ldots, p_k$ with $p_i \cdot \alpha_j = 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n-k$. The Gramm-Schmidt procedure can be used for this purpose. Pivot on the matrix $(p_1, \ldots, p_k)$ to compute a $\beta(1)$ in $\Lambda(\tau_\alpha)$ and an $x(1)$ in X with $|x(1)| \le 1$ where $\tau_\alpha = \operatorname{span}(p_1, \ldots, p_k) = S_{\beta(1)}(x(1))$. Setting $\theta(1) = 1$ we have solved

$$H_{\beta(1)}(x(1), \theta(1)) = 0.$$

Let $\tilde{Z}_{\beta(1)}$ be the route in $Z_{\beta(1)}$ which contains the point $(x(1), \theta(1))$. Towards following the route $\tilde{z}_{\beta(1)}$, using the predictor-corrector method, we compute $(\bar{x}(1), \bar{\theta}(1))$ solving

$$\nabla_{(x,\theta)}H_{\beta(1)}(x(1), \theta(1))(\bar{x}(1), \bar{\theta}(1)) = 0 \qquad \|(\bar{x}(1), \bar{\theta}(1))\| = 1.$$

As the rank of

$$\nabla_x H_{\beta(1)}(x(1), \theta(1))$$

is $k(n-k)$ we know that $\bar{\theta}(1) \neq 0$; scale $(\bar{x}(1), \bar{\theta}(1))$ by ± 1 so that $\bar{\theta}(1)$ is negative. The predictor-connector method commences with a predictor step in the direction $(\bar{x}(1), \bar{\theta}(1))$ from $(x(1), \theta(1))$.

Advance the computation forward and assume that we are following the route $\tilde{Z}_{\beta(i)} \subseteq Z_{\beta(i)}$ in $X \times [\epsilon, 1)$ where $Z_\beta(i)$ is the zero set of the homotopy $H_{\beta(i)}$. We continue with predictor-corrector iterates until an iterate $(x(i), \theta(i))$ is generated which is about to leave $X \times [\epsilon, 1]$, that is, $\theta(i) = 1$, $\theta(i) = \epsilon$, or $|x(i)| \ge r$ with $\epsilon < \theta(i) < 1$; we refer to these three cases as a top, bottom, or side exit, respectively. See Figure 5. Let us consider each exit possibility.

TOP EXIT. If we are following $\tilde{Z}_{\beta(i)}$ closely enough, the top exit will not occur as zero is a regular value of H and as $Z \cap (G_k^n \times 1)$ is a singleton. □

BOTTOM EXIT. A bottom exit is the signal for the algorithm to terminate. That is, if $(x(i), \theta(i)) \in Z_{\beta(i)} \cap (X \times \epsilon)$ then $H_{\beta(i)}(x(i), \epsilon) = 0$ and $S_{\beta(i)}(x(i)) = \tau$ is an

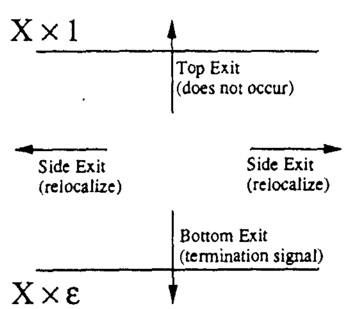

FIGURE 5. Exits from $X \times [\epsilon, 1]$.

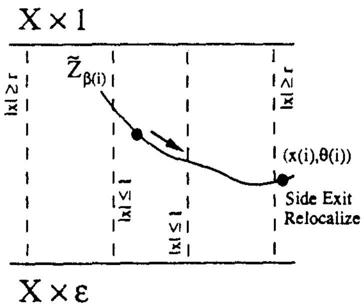

FIGURE 6. Relocalization signal.

approximate solution to $f(\tau) = 0$. The corrector step might encounter some difficulty for θ near ϵ, in which case one should try to perform the corrector step in the plane $\theta = \epsilon$ instead of the translate of $L^{\perp}$. Assuming success here one should try setting ϵ to zero and executing Newton's method, or a variant thereof, with the hope of improving the approximate solution of $f(\tau) = 0$. □

SIDE EXIT. The condition $|x(i)| \geq r$ is the signal that we have a side exit and that we should relocalize; see Figure 6. Let $(\bar{x}(i), \bar{\theta}(i))$ be the direction that the predictor iteration would use at $(x(i), \theta(i))$. Pivot on $F_{\beta(i)}(x(i))$ to compute $\beta(i+1)$ in $\Lambda(S_{\beta(i)}(x(i))$ and $x(i+1)$ with $|x(i+1)| \leq 1$ so that

$$F_{\beta(i)}(x)\Big(\prod_{\beta(i+1)} F_{\beta(i)}(x)\Big)^{-1} = F_{\beta(i+1)}(x(i+1)).$$

Set $\theta(i+1) = \theta(i)$.

We now have a solution $(x(i+1), \theta(i+1))$ to $H_{\beta(i+1)}(x(i+1), \theta(i+1)) = 0$. Define $\tilde{Z}_{\beta(i+1)}$ to be the route of $Z_{\beta(i+1)}$ which contain the point $(x(i+1), \theta(i+1))$. And now with respect to the function $H_{\beta(i+1)}$ on the domain $X \times [\epsilon, 1]$ the predictor-corrector method continues as before but now following $\tilde{Z}_{\beta(i+1)}$, and in particular, being prepared to terminate with a bottom exit and to execute side exits.

There is, however, a remaining matter to be addressed here. After the new direction $(\bar{x}(i+1), \bar{\theta}(i+1))$ is computed for the predictor step by solving

$$\nabla_{(x,\theta)} H_{\beta(i+1)}(x(i+1), \theta(i+1))(\bar{x}(i+1), \bar{\theta}(i+1)) = 0$$

$$\|(\bar{x}(i+1), \bar{\theta}(i+1))\| = 1,$$

does the movement take place in the direction $(\bar{x}(i+1), \theta(i+1))$ or $-(\bar{x}(i+1), \bar{\theta}(i+1))$? If $\bar{\theta}(i)$ is (numerically significantly) positive or negative, one merely sets the sign of $(\bar{x}(i+1), \bar{\theta}(i+1))$ so that the signs of $\bar{\theta}(i+1)$ and $\bar{\theta}(i)$ agree. If $\bar{\theta}(i)$ is (numerically essentially) zero we must examine $\bar{x}(i)$. We want to approximate the movement of

$$S_{\beta(i)}(x(i) + t\bar{x}(i)) \quad \text{with that of } S_{\beta(i+1)}(x(i+1) + t\bar{x}(i+1)),$$

in G_k^n for small t. Let $h(t)$ be the function defined by

$$F_{\beta(i+1)}(x(i+1) + h(t)) = F_{\beta(i)}(x(i) + t\bar{x}(i))\Big(\prod_{\beta(i+1)} F_{\beta(i)}(x(i) + t\bar{x}(i))\Big)^{-1},$$

that is,

$$\operatorname{wrap}(x(i+1)+h(t)) = \prod_{\nu(i+1)} F_{\beta(i)}(x(i)+t\bar{x}(i))\Big(\prod_{\beta(i+1)} F_{\beta(i)}(x(i)+t\bar{x}(i))\Big)^{-1}.$$

The sign of $(\bar{x}(i+1), \bar{\theta}(i+1))$ is chosen so that

$$\nabla h(0)\cdot \bar{x}(i+1) > 0.$$

Although $h(t)$ might be a nuisance to compute, the direction $\nabla h(0)$ is computed using the chain rule for differentiation as $\beta(i)$ is transformed to $\beta(i+1)$ one element at each step. The sign of $(\bar{x}(i+1), \bar{\theta}(i+1))$ is chosen so that the direction $\bar{x}(i+1)$ approximately equals the direction $\nabla h(0)$ as the latter corresponds to $\bar{x}(i)$ except in a different coordinate system. □

EXAMPLE. We exhibit operation of the locator Λ; see Eaves (1993). For $n = 4$, $k = 2$, $\beta(i) = \{1, 3\}$, and $r = 2$ suppose $x(i) = (x_1(i), x_2(i), x_3(i), x_4(i)) = (1, -1, 2, 1) \in X$. As $|x_3(i)| = 2 \geq r = 2$ we have a side exit. $F_{\beta(i)}(x(i))^T$ is the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 \end{pmatrix}.$$

Pivot on any element exceeding one in absolute value, namely 2 here, to get:

$$\begin{pmatrix} -1/2 & -3/2 & 1 & 0 \\ 1/2 & 1/2 & 0 & 1 \end{pmatrix}.$$

Pivot on any element exceeding one in absolute value, namely, $-3/2$ here, to get:

$$\begin{pmatrix} 1/3 & 1 & -2/3 & 0 \\ 1/3 & 0 & 1/3 & 1 \end{pmatrix}.$$

As no element exceeds one in absolute value we take this matrix as $F_{\beta(i+1)}(x(i+1))^T$ to get $\beta(i+1) = \{2, 4\} \in \Lambda(S_{\beta(i)}(x(i)))$, $x(i+1) = (1/3, 1/3, -2/3, 1/3)$, $|x(i+1)| \leq 1$, and $S_{\beta(i+1)}(x(i+1)) = S_{\beta(i)}(x(i))$. □

Our definition of the algorithm is now complete; however, we have a few remarks to be made about the algorithm. We first turn our attention to a particular point concerning finite termination.

The zero set Z_β of H_β on $X \times [\epsilon, 1]$ can have an infinite number of connected components. This situation in turn appears to admit the possibility that the algorithm would be required to change the $\beta(i)$ in B and infinite number of times in order to complete the route following task. That is, due to the burden of changing $\beta(i)$ the computation would never terminate. However, each route or loop in Z_β in $X \times [\epsilon, 1]$ has finite arc length. As there are only a finite number of routes in Z_β which meet both $|x| \leq 1$ and $|x| \geq r$ and as B is finite there is only a finite amount of arc length to be followed. For each $\beta(i)$ the algorithm begins with $|x| \leq 1$ and does not change $\beta(i)$ until there is a bottom exit, in which case the algorithm terminates, or there is a side exit, in which case $|x| \geq r$. Thus, if we have a side exit we have traveled at least $r - 1$ units of arc length. That is, for each selection of a $\beta(i)$ the algorithm travels a distance of at least $r - 1$ units, and as there is only a finite amount of arc length to be followed the $\beta(i)$ will change only a finite number of times. (One can also speak of inherent arc length on the cylinder $G_k^n \times [\epsilon, 1]$, using the metric there, and base the argument on the arc length there, but this avenue is not followed here. In more general situations such might be necessary.)

Clearly, if enough care is not exercised in the predictor-corrector step the movement could fall off of the route of interest. However, the viability of the predictor-corrector method, with suitable safeguards, has been supported both theoretically and empirically, see Allgower and Georg (1992) for a discussion and references.

Preparations are being made to implement the algorithm described herein on a computer, first for a mathematical fixed point problem as described in §1 and then extended for the economic model with incomplete assets where a price space must be adjoined.

9. Generalization and perspective. The present section offers a perspective of the foregoing development. We step back from the "fixed point" problem on the Grassmannian manifold and briefly indicate how the ideas can be used to compute zeros of equations mapping an abstract manifold into a vector bundle. See Hirsch (1976) for a definition of vector bundle.

Let M be an abstract (smooth) compact m-manifold (without boundary) with atlas Ψ and let (E, M, p, Φ) be a (smooth) vector bundle with vector bundle (total space) E of vector dimension m, base space M, projection $p\colon E \to M$, and atlas Φ. In particular, for each chart (φ, U) in Φ, the set U is open in M and

$$\varphi\colon p^{-1}(U) \to U \times R^m$$

is a diffeomorphism where $\varphi\colon p^{-1}(x) = \{x\} \times R^m$ for all x in U. The U's cover M. The set $p^{-1}(x)$ is called the fiber over x. Define the map $\varphi_U\colon p^{-1}(U) \to R^m$ by $\varphi_U(y) = z$ where $\varphi(y) = (x, z)$. If (φ', U') is a chart of Φ with x also in U', then $\varphi'_x \varphi_x^{-1}\colon R^m \to R^m$ is linear and smooth in x. For convenience and without loss of generality, we assume that for each chart (ψ, U) in Ψ there is a chart in Φ of form (φ, U).

We call a function $f\colon M \to E$ a section if $f(x)$ is in the fiber $p^{-1}(x)$ over x for all x in M. The zero section $0\colon M \to E$ is defined by $0(x) = \varphi^{-1}(x, 0)$ where (φ, U) is a chart of Φ with x in U. It might be helpful to notice that $0(M)$ is a copy of M in E. Let $f\colon M \to E$ be a (smooth) section and we consider the task of computing a zero of f, that is, of solving $f(x) = 0(x)$.

EXAMPLE. We indicate how these notions correspond to those of the previous sections, that is, of solving $f(\tau) = 0$ where $f\colon G_k^n \to R^{n(n-k)}$ and $f(\tau) \in \tau^{n-k}$ for all τ. M is the manifold G_k^n with dimension $m = k(n-k)$ and atlas $\Psi = \{(S_\beta^{-1}, U_\beta)\colon \beta \in B\}$, just as before. The vector bundle (total space) is $E = \{(\tau, y_1, \dots, y_{n-k})\colon \tau \in G_k^n,\ y_i \in \tau,\ \forall i\}$ with projection $p^{-1}(\tau) = \{\tau\} \times \tau^{n-k}$. The atlas of the vector bundle is $\{(\varphi_\beta, U_\beta)\colon \beta \in B\}$ where

$$p^{-1}(U_\beta) = \left\{(S_\beta(x), y)\colon y \in S_\beta(x)^{n-k},\ x \in X\right\}$$

and φ_β is defined by

$$\varphi_\beta(S_\beta(x), y) \equiv \left(S_\beta(x), \prod_\beta^{n-k} y\right).$$

The function $\bar{f}\colon M \to E$ is defined by $\bar{f}(\tau) = (\tau, f(\tau))$ and we want to solve $\bar{f}(\tau) = 0(\tau) = (\tau, 0)$, that is, $f(\tau) = 0$. □

Let A be an open set in R^n and g: $A \times M \to E$ be a (smooth) function; let $g_\alpha = g(\alpha, \cdot)$ and $g_x = g(\cdot, x)$. Assume that for each α in A, the function g_α is a section. As before g is an auxiliary section used to define the top of the homotopy, to initiate the computation, to assure that there is a route to follow, and to assure that the route leads to a zero of f at the bottom of the homotopy. We require two properties of g, namely, unique zero and transversality.

Unique zero of g_α. For each α in A the map g_α has a unique zero. That is, for each α in A there is a unique solution x in M of $g_\alpha(x) = 0(x)$. □

Transversality of g_x. For each chart (φ, U) of Φ the derivative $\nabla_\alpha(\varphi_U g_x)(\alpha)$: $R^n \to R^m$ of the function $\varphi_U g_x$ at each (α, x) in $A \times M$ is onto. □

$$A \overset{g_x}{\to} p^{-1}(U) \overset{\varphi_U}{\to} R^m.$$

As each g_α and f are sections we can form the section homotopy H: $M \times [\epsilon, 1] \to E$ by

$$H(x, \theta) = \varphi_x^{-1}[\theta \varphi_x g_\alpha(x) + (1 - \theta)\varphi_x f(x)]$$

where (φ, U) is a chart of Φ and x is in U. As the functions $\varphi_x' \varphi_x^{-1}$ are linear and smooth it follows that H is well-defined and smooth. Notice that $H(x, \theta)$ is in $p^{-1}(x)$ for all (x, θ) in $M \times [\epsilon, 1]$.

Again, let ι be the identity function on $[\epsilon, 1]$. For almost all α in A we see that H and ∂H is transverse to the zero section. That is, for almost all α in A we see that zero is a regular value of the maps $H_\psi \equiv \varphi_U H(\psi^{-1}, \iota)$ and ∂H_ψ for each chart (ψ, U) of Ψ and chart (φ, U) of Φ.

$$\psi^{-1}(U) \times [\epsilon, 1] \overset{(\psi^{-1}, \iota)}{\to} U \times [\epsilon, 1] \overset{H}{\to} p^{-1}(U) \overset{\varphi_U}{\longrightarrow} R^m.$$

It follows that the zero set $Z = \{(x, \theta);\ H(x, \theta) = 0(x)\}$ is a neat disjoint collection of loops and routes in $M \times [\epsilon, 1]$ and transverse to $M \times \{\epsilon, 1\}$.

As before the plan is: beginning at the point of $Z \cap (M \times 1)$ follow the corresponding route $\tilde{Z}$ of Z. The computation is carried out locally, using predictor-corrector methods, on the maps $H_\psi \equiv \varphi_U H(\psi^{-1}, z)$: $\psi^{-1}(U) \times [\epsilon, 1] \to R^m$ in Euclidean space. When a side exit occurs, relocalize. When a bottom (or top exit) occurs, terminate.

Let us again direct our attention to the matter of relocalizing. Towards organizing relocalization to assure desired convergence properties we use a locator. In R^m with norm $|\cdot|$ let $B(0, r) = \{x \in R^m\colon |x| \le r\}$ be the closed ball centered at the origin with radius r. Let Λ: $M \rightrightarrows \Psi$ be a multifunction from the manifold M to the atlas Ψ and let r be a number exceeding 1. The multifunction Λ is defined to be a locator if $\Lambda(M)$ is finite and if for each x in M and chart (ψ, U) in $\Lambda(x)$ we have $\psi(x)$ in $B(0, 1)$ and $B(0, r + 1)$ a subset of $\psi(U)$. The idea is that the charts are "centered" at x by the locator. We note that a manifold is compact if and only if it has a finite locator.

If in following the route $H_{\psi(i)}^{-1}$ in $B(0, r) \times [\epsilon, 1]$ using chart $(\psi(i), U(i))$ one makes a side exit at $(x(i), \theta(i))$, that is $|x(i)| \ge r$ and $0 < \theta(i) < 1$ in direction $(\bar{x}(i), \bar{\theta}(i))$, one relocalizes with a chart $(\psi(i + 1), U(i + 1))$ and continues at the point

$$(x(i + 1), \theta(i + 1)) = \left(\psi(i + 1)\psi(i)^{-1}(x(i)), \theta(i)\right)$$

in $B(0,1) \times [\epsilon, 1]$ in the direction

$$\big(\bar{x}(i+1), \bar{\theta}(i+1)\big) = \big(\nabla_x\big(\psi(i+1)\psi(i)^{-1}\big)(x(i))(\bar{x}(i)), \bar{\theta}(i)\big).$$

The finite convergence argument employed for the Grassmannian manifold applies here as well, and we do not repeat it. As the section g_α has a unique zero, in following $\tilde{Z}$ through the localizations one will make a bottom exit from $M \times [\epsilon, 1]$, and thereby compute an approximation solution of $f(x) = 0(x)$. Letting ϵ go to zero we obtain a zero of f.

We have given a constructive proof of the following result which is usually stated in terms of degree, intersection number, or Euler characteristic, see Hirsch (1976, Chapter 5), for example.

THEOREM. *If a section g with the unique zero and transversality properties exists, then any section has a zero.* □

To compute, that is to follow the route $\tilde{Z}$ with relocalization it is necessary, of course, to be able to evaluate the various expressions involved, for example, to select a chart in $\Lambda(x)$ for x in M. Whether or not these evaluations are feasible is a determination that cannot be made until the particular manifold, vector bundle, and section are in hand. For the Grassmannian manifold G_k^n problem treated earlier we have $m = k(n-k)$, manifold $M = G_k^n$, vector bundle $E = \{(\tau, x_1, \ldots, x_{n-k}) \colon \tau \in G_k^n, x_1 \in \tau, \ldots, x_{n-k} \in \tau\}$, section $\tau \to (\tau, f(\tau))$, auxiliary function g, and locator Λ. Whether or not the computation is feasible in the Grassmannian setting depends only upon characteristics of the function f, as all else is manageable.

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