ON THE NUMBER OF BOOTSTRAP REPETITIONS FOR BC_a CONFIDENCE INTERVALS

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ON THE NUMBER OF BOOTSTRAP REPETITIONS FOR BC_a CONFIDENCE INTERVALS

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This paper considers the problem of choosing the number of bootstrap repetitions B to use with the BC_a bootstrap confidence intervals introduced by Efron (1987, *Journal of the American Statistical Association* 82, 171–200). Because the simulated random variables are ancillary, we seek a choice of B that yields a confidence interval that is close to the ideal bootstrap confidence interval for which $B = \infty$. We specify a three-step method of choosing B that ensures that the lower and upper lengths of the confidence interval deviate from those of the ideal bootstrap confidence interval by at most a small percentage with high probability.

1. INTRODUCTION

In this paper, we consider the problem of choosing the number of bootstrap repetitions B for the BC_a bootstrap confidence intervals introduced by Efron (1987). We propose a three-step method for choosing B that is designed to achieve a desired level of accuracy. By accuracy, we mean closeness of the BC_a confidence interval based on B repetitions to the ideal bootstrap BC_a confidence interval for which $B = \infty$. We desire accuracy of this sort, because we do not want to be able to obtain a "different answer" from the same data merely by using different simulation draws.

More precisely, we measure accuracy in terms of the percentage deviation of the lower and upper lengths of the bootstrap confidence interval for a given value of B, from the lower and upper lengths of the ideal bootstrap confidence interval. By definition, the *lower length* of a confidence interval for a parameter θ based on a parameter estimate $\hat{\theta}$ is the distance between the lower endpoint of the confidence interval and the parameter estimate $\hat{\theta}$. The *upper length*

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is defined analogously. We want both lengths, not just the total length of the interval, to be accurate.

The accuracy obtained by a given choice of B is random, because the bootstrap simulations are random. To determine an appropriate value of B, we specify a bound on the percentage deviation, denoted *pdb*, and we require that the actual percentage deviation is less than this bound with a specified probability, $1 - \tau$, close to one. The three-step method takes *pdb* and τ as given and specifies a data-dependent method of determining a value of B, denoted B^*, such that the desired level of accuracy is achieved. For example, one might take $(pdb, \tau) = (10, .05)$. In this case, the three-step method determines a value B^* such that the percentage deviation of the upper and lower confidence interval lengths is less than 10% each with approximate probability .95.

The idea behind the three-step method is as follows. Conditional on the original sample, the BC_a confidence interval endpoints based on B repetitions are sample quantiles with random percentage points. We approximate their distributions by their asymptotic distributions as $B \to \infty$. The parameters of these asymptotic distributions are estimated in the first and second steps of the three-step method. These estimates include estimates of a density at two points. For this purpose, we use an estimator of Siddiqui (1960) with an optimal data-dependent smoothing parameter, which is a variant of that proposed by Hall and Sheather (1988). The asymptotic distributions evaluated at these estimates are used in the third step to determine how large B must be to attain the desired level of accuracy.

The three-step method is applicable whenever a BC_a confidence interval is applicable. This includes parametric, semiparametric, and nonparametric models with independent and identically distributed (i.i.d.) data, independent and nonidentically distributed (i.n.i.d.) data, and time series data (regarding the latter, see Götze and Künsch, 1996). The method is applicable when the bootstrap employed is the standard nonparametric i.i.d. bootstrap, a moving block bootstrap for time series, a parametric or semiparametric bootstrap, or a bootstrap for regression models that is based on bootstrapping residuals. Essentially, the results are applicable whenever the bootstrap samples are simulated to be i.i.d. *across* different bootstrap samples. (The simulations need not be i.i.d. within each bootstrap sample.)

We examine the small sample performance of the proposed method via simulation for two common applications in the econometrics and statistics literature. The first application is to a linear regression model. The second is to a correlation coefficient between two random variables. We find that the number of bootstrap repetitions needed to attain accurate estimates of the ideal bootstrap confidence interval is quite large. We also find that for both applications the proposed three-step method performs fairly well, although it is overly conservative. That is, the finite sample probabilities that the percentage deviations of the lower and upper lengths of the bootstrap confidence intervals are less than or equal to *pdb* are somewhat greater than their theoretical value, $1-\tau$, for most (α, pdb, τ) combinations considered.

The three-step method considered here is closely related to that specified in Andrews and Buchinsky (2000) for choosing B for bootstrap standard error estimates, percentile t confidence intervals, tests for a given significance level, p-values, and bias correction. The results of Andrews and Buchinsky (2000) are not applicable to BC_a confidence intervals, because they only apply to bootstrap sample quantiles for *fixed* percentage points. Analysis of the performance of the three-step method of Andrews and Buchinsky (2000) is given in Andrews and Buchinsky (2001).

The asymptotic approximations utilized here are equivalent to those used in Efron (1987, Sect. 9). We provide a proof of the validity of these approximations. This proof is complicated by the fact that the sample quantiles in question are from an underlying distribution that is discrete (at least for the nonparametric bootstrap) and the percentage points are random, not fixed.

Note that Hall (1986) considers the effect of B on the *unconditional* coverage probabilities of some confidence intervals (but not BC_a confidence intervals). The unconditional coverage probability is the probability with respect to the randomness in the data and the bootstrap simulations. In contrast, we consider *conditional* coverage probabilities, i.e., coverage probabilities with respect to the randomness in the data conditional on the bootstrap simulations. We do so because we do not want to be able to obtain "different answers" from the same data as a result of the use of different simulation draws.

The remainder of this paper is organized as follows. Section 2 introduces notation and defines the BC_a confidence intervals. Section 3 describes the three-step method for choosing B for these confidence intervals. Section 4 describes the asymptotic justification of the three-step method. Section 5 presents some Monte Carlo simulation results that assess the ability of the three-step method to choose B to achieve the desired accuracy in finite samples. An Appendix provides a proof of the asymptotic justification of the three-step method.

2. NOTATION AND DEFINITIONS

We begin by introducing some notation and definitions. Let $\mathbf{X} = (X_1, \ldots, X_n)'$ denote the observed data. Let $\hat{\theta} = \hat{\theta}(\mathbf{X})$ be an estimator of an unknown scalar parameter θ_0. We wish to construct an equal-tailed confidence interval for θ_0 of (approximate) confidence level $100(1-2\alpha)\%$ for some $0 < \alpha < 1$.

We assume that the normalized estimator $n^{\kappa}(\hat{\theta} - \theta_0)$ has an asymptotic normal distribution as $n \to \infty$. Let $\sigma_{\hat{\theta}}^2$ denote its asymptotic variance. We allow for $\kappa \neq \frac{1}{2}$ to cover nonparametric estimators, such as nonparametric estimators of a density or regression function at a point.

Define a bootstrap sample $\mathbf{X}^* = (X_1^*, \ldots, X_n^*)'$ and a bootstrap estimator $\hat{\theta}^* = \hat{\theta}(\mathbf{X}^*)$. Let $\hat{\theta}_{\infty}^{*(\alpha)}$ denote the α quantile of $\hat{\theta}^*$. Because the bootstrap estimator

$\hat{\theta}^*$ has a discrete distribution (at least for the nonparametric bootstrap), there typically is no constant $\hat{\theta}_\infty^{*(\alpha)}$ that satisfies the equation $P^*(\hat{\theta}^* \le \hat{\theta}_\infty^{*(\alpha)}) = \alpha$ exactly, where $P^*(\cdot)$ denotes probability with respect to the bootstrap sample $\mathbf{X}^*$ conditional on the original sample $\mathbf{X}$. Thus, to be precise, we define $\hat{\theta}_\infty^{*(\alpha)} = \inf\{k : P^*(\hat{\theta}^* \le k) \ge \alpha\}$.

The ideal bootstrap equal-tailed percentile confidence interval of approximate confidence level $100(1-2\alpha)\%$ is

$$[\hat{\theta}_\infty^{*(\alpha)}, \hat{\theta}_\infty^{*(1-\alpha)}]. \tag{1}$$

This confidence interval does not improve upon confidence intervals based on first-order asymptotics in terms of coverage probability. In consequence, Efron (1987) introduced the bias-corrected and accelerated (BC_a) confidence interval that adjusts the quantiles α and $1-\alpha$ in such a way that it exhibits higher order improvements. (For a detailed discussion of these higher order improvements, see Hall, 1988; Hall, 1992, Sect. 3.10. For an introductory discussion of BC_a confidence intervals and software to calculate them, see Efron and Tibshirani, 1993, Sect. 14.3 and Appendix.)

The ideal bootstrap BC_a confidence interval of approximate confidence level $100(1-2\alpha)\%$ is

$$CI_\infty = [\hat{\theta}_\infty^{*(\alpha_{\ell,\infty})}, \hat{\theta}_\infty^{*(\alpha_{u,\infty})}], \quad \text{where}$$

$$\alpha_{\ell,\infty} = \Phi\left(\hat{z}_{0,\infty} + \frac{\hat{z}_{0,\infty} + z^{(\alpha)}}{1 - \hat{a}(\hat{z}_{0,\infty} + z^{(\alpha)})}\right) \quad \text{and}$$

$$\alpha_{u,\infty} = \Phi\left(\hat{z}_{0,\infty} + \frac{\hat{z}_{0,\infty} + z^{(1-\alpha)}}{1 - \hat{a}(\hat{z}_{0,\infty} + z^{(1-\alpha)})}\right). \tag{2}$$

Here $\Phi(\cdot)$ is the standard normal distribution function and $z^{(\alpha)}$ is the α quantile of the standard normal distribution. The term $\hat{z}_{0,\infty}$ is the "ideal bias correction" and is defined by

$$\hat{z}_{0,\infty} = \Phi^{-1}(P^*(\hat{\theta}^* < \hat{\theta})), \tag{3}$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal distribution function.

The term $\hat{a}$ in (2) is the "acceleration constant." It can be defined in different ways. For example, in i.i.d. contexts, it can be defined to equal a jackknife estimate:

$$\hat{a} = \frac{\sum_{i=1}^{n} (\hat{\theta}_{(\cdot)} - \hat{\theta}_{(i)})^3}{6\left(\sum_{i=1}^{n} (\hat{\theta}_{(\cdot)} - \hat{\theta}_{(i)})^2\right)^{3/2}}, \tag{4}$$

where $\hat{\theta}_{(i)} = \hat{\theta}(\mathbf{X}_{(i)})$, $\mathbf{X}_{(i)}$ denotes the original sample with the ith observation deleted, and $\hat{\theta}_{(\cdot)} = \sum_{i=1}^{n} \hat{\theta}_{(i)}/n$.

Note that, when the ideal bias correction $\hat{z}_{0,\infty}$ and the ideal acceleration constant $\hat{a}$ equal zero, $\alpha_{\ell,\infty} = \Phi(z^{(\alpha)}) = \alpha$ and $\alpha_{u,\infty} = \Phi(z^{(1-\alpha)}) = 1 - \alpha$. In this case, the BC_a confidence interval reduces to the equal-tailed percentile confidence interval of (1).

Analytic calculation of the ideal bootstrap BC_a confidence interval is usually intractable. Nevertheless, one can approximate it using bootstrap simulations. Consider B bootstrap samples $\{\mathbf{X}_b^* : b = 1, \ldots, B\}$ that are independent across B, each with the same distribution as $\mathbf{X}^*$. The corresponding B bootstrap estimators are $\{\hat{\theta}_b^* = \hat{\theta}(\mathbf{X}_b^*) : b = 1, \ldots, B\}$.

Let $\{\hat{\theta}_{B,b}^* : b = 1, \ldots, B\}$ denote the ordered sample of bootstrap estimators. Define the α sample quantile of the bootstrap estimators to be $\hat{\theta}_B^{*(\alpha)} = \hat{\theta}_{B,\lfloor (B+1)\alpha \rfloor}^*$ for $\alpha \leq \frac{1}{2}$ and $\hat{\theta}_B^{*(\alpha)} = \hat{\theta}_{B,\lceil (B+1)\alpha \rceil}^*$ for $\alpha > \frac{1}{2}$, where $\lfloor a \rfloor$ denotes the largest integer less than or equal to a (i.e., the integer part of a) and $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. (If $\lfloor (B+1)\alpha \rfloor = 0$ for some $\alpha \leq \frac{1}{2}$, then let $\hat{\theta}_B^{*(\alpha)} = \hat{\theta}_{B,1}^*$. If $\lceil (B+1)\alpha \rceil = B + 1$ for some $\alpha > \frac{1}{2}$, then let $\hat{\theta}_B^{*(\alpha)} = \hat{\theta}_{B,B}^*$.)

The BC_a confidence interval of approximate confidence level $100(1 - 2\alpha)\%$ based on B bootstrap repetitions is

$$CI_B = [\hat{\theta}_B^{*(\alpha_{\ell,B})}, \hat{\theta}_B^{*(\alpha_{u,B})}], \quad \text{where}$$
$$\alpha_{\ell,B} = \Phi\left(\hat{z}_{0,B} + \frac{\hat{z}_{0,B} + z^{(\alpha)}}{1 - \hat{a}(\hat{z}_{0,B} + z^{(\alpha)})}\right) \quad \text{and}$$
$$\alpha_{u,B} = \Phi\left(\hat{z}_{0,B} + \frac{\hat{z}_{0,B} + z^{(1-\alpha)}}{1 - \hat{a}(\hat{z}_{0,B} + z^{(1-\alpha)})}\right). \tag{5}$$

The term $\hat{z}_{0,B}$ is the bias correction based on B bootstrap repetitions and is defined by

$$\hat{z}_{0,B} = \Phi^{-1}\left(\frac{1}{B}\sum_{b=1}^{B}(\hat{\theta}_b^* < \hat{\theta})\right). \tag{6}$$

We note that $\hat{z}_{0,B}$ is a random function of the bootstrap estimators $\{\hat{\theta}_b^* : b = 1, \ldots, B\}$. In consequence, $\alpha_{\ell,B}$ and $\alpha_{u,B}$ are random functions of $\{\hat{\theta}_b^* : b = 1, \ldots, B\}$. This affects the three-step method of determining B that is introduced subsequently. We also note that the acceleration constant $\hat{a}$, as defined in (4), does not depend on the bootstrap estimators. It is a function of the original sample only.

3. A THREE-STEP METHOD FOR DETERMINING THE NUMBER OF BOOTSTRAP REPETITIONS

In this section, we introduce a three-step method for determining B for the bootstrap confidence interval CI_B defined previously. Our main interest is in determining B such that CI_B is close to the ideal bootstrap confidence interval CI_∞. A secondary interest is in the unconditional coverage probability of CI_B (where "unconditional" refers to the randomness in both the data *and* the simulations).

Our primary interest is the former, because the simulated random variables are ancillary with respect to the parameter θ_0. Hence, the principle of ancillarity or conditionality (e.g., see Kiefer, 1982, and references therein) implies that we should seek a confidence interval that has a confidence level that is (approximately) $100(1-2\alpha)\%$ conditional on the simulation draws. To obtain such an interval, we need to choose B to be sufficiently large that CI_B is close to CI_∞. Otherwise, two researchers using the same data and the same statistical method could reach different conclusions due only to the use of different simulation draws.

We could measure the closeness of CI_B to CI_∞ by considering their relative lengths. However, these confidence intervals, which are based on the parameter estimate $\hat{\theta}$, are not necessarily symmetric about $\hat{\theta}$. In consequence, a more refined measure of the closeness of CI_B to CI_∞ is to consider the closeness of both their lower and upper lengths. By definition, the *lower length* of the confidence interval CI_B, denoted $L_\ell(CI_B)$, is the distance between the lower bound $\hat{\theta}_B^{*(\alpha_{\ell,B})}$ and $\hat{\theta}$. Its *upper length*, denoted $L_u(CI_B)$, is the distance from $\hat{\theta}$ to $\hat{\theta}_B^{*(\alpha_{u,B})}$. That is,

$$L_\ell(CI_B) = \hat{\theta} - \hat{\theta}_B^{*(\alpha_{\ell,B})} \quad \text{and} \quad L_u(CI_B) = \hat{\theta}_B^{*(\alpha_{u,B})} - \hat{\theta}. \tag{7}$$

The lower and upper lengths of CI_∞ are defined analogously with B replaced by ∞.

We measure the closeness of CI_B to CI_∞ by comparing the percentage deviations of the lower and upper lengths of the two intervals. The percentage deviation of the upper length of CI_B from the upper length of CI_∞ is

$$100\,\frac{|\hat{\theta}_B^{*(\alpha_{u,B})} - \hat{\theta}_\infty^{*(\alpha_{u,\infty})}|}{\hat{\theta}_\infty^{*(\alpha_{u,\infty})} - \hat{\theta}}. \tag{8}$$

The percentage deviation of the lower length of CI_B to the lower length of CI_∞ is defined analogously.

Let $1-\tau$ denote a probability close to one, such as .95. Let pdb be a bound on the percentage deviation of the lower or upper length of CI_B to the corresponding length of CI_∞. For the upper length, we want to determine $B = B(pdb, \tau)$ such that

$$P^*\left(100\,\frac{|\hat{\theta}_B^{*(\alpha_{u,B})} - \hat{\theta}_\infty^{*(\alpha_{u,\infty})}|}{\hat{\theta}_\infty^{*(\alpha_{u,\infty})} - \hat{\theta}} \le pdb\right) = 1-\tau. \tag{9}$$

For the lower length, we want to determine an analogous value of B with $\alpha_{u,B}$ and $\alpha_{u,\infty}$ replaced by $\alpha_{\ell,B}$ and $\alpha_{\ell,\infty}$, respectively.

The three-step method of determining B for CI_B is designed to obtain a specified desired level of accuracy *pdb* for both lengths, each with probability approximately equal to $1 - \tau$ (based on the asymptotic justification given subsequently).

The three-step method relies on estimators of the reciprocals of two density functions evaluated at two points, which appear in the asymptotic distributions of the sample quantiles $\hat{\theta}_B^{*(\alpha_{\ell,B})}$ and $\hat{\theta}_B^{*(\alpha_{u,B})}$. For this, we use Siddiqui's (1960) estimator (analyzed by Bloch and Gastwirth, 1968; Hall and Sheather, 1988) with plug-in estimators of the bandwidth parameters that are chosen to maximize the higher order asymptotic coverage probability of the resultant confidence interval, as calculated by Hall and Sheather (1988). To reduce the noise of the plug-in estimator, we take advantage of the fact that we know the asymptotic values of the densities, and we use them to generate our estimators of the unknown coefficients in the plug-in formulae. The density estimate makes use of the following formula, which is utilized in step 2, which follows:

$$C_\alpha = \left(\frac{1.5(z^{(1-\alpha/2)})^2 \phi^2(z^{(1-\alpha)})}{2(z^{(1-\alpha)})^2 + 1} \right)^{1/3}. \tag{10}$$

The three-step method is defined as follows.

Step 1. Compute a preliminary number of bootstrap repetitions B_1 via

$$B_1 = \lceil 10{,}000(\alpha(1-\alpha) - 2\alpha\phi(z^{(\alpha)})/\phi(0) + \phi^2(z^{(\alpha)})/\phi^2(0))(z^{(1-\tau/2)})^2 / (z^{(\alpha)}\phi(z^{(\alpha)})pdb)^2 \rceil. \tag{11}$$

Step 2. Simulate B_1 bootstrap estimators $\{\hat{\theta}_b^* : b = 1, \ldots, B_1\}$; order the bootstrap estimators, which are denoted $\{\hat{\theta}_{B_1,b}^* : b = 1, \ldots, B_1\}$; and calculate

$$\hat{z}_{0,B_1} = \Phi^{-1}\left(\frac{1}{B_1} \sum_{b=1}^{B_1} (\hat{\theta}_b^* < \hat{\theta}) \right),$$

$$\alpha_{1\ell} = \max\left\{ \Phi\left(\hat{z}_{0,B_1} + \frac{\hat{z}_{0,B_1} + z^{(\alpha)}}{1 - \hat{a}(\hat{z}_{0,B_1} + z^{(\alpha)})} \right), .01 \right\},$$

$$\alpha_{1u} = \min\left\{ \Phi\left(\hat{z}_{0,B_1} + \frac{\hat{z}_{0,B_1} + z^{(1-\alpha)}}{1 - \hat{a}(\hat{z}_{0,B_1} + z^{(1-\alpha)})} \right), .99 \right\},$$

$$\nu_{1\ell} = \lfloor (B_1 + 1)\alpha_{1\ell} \rfloor, \qquad \nu_{1u} = \lceil (B_1 + 1)\alpha_{1u} \rceil,$$

$$\hat{m}_{1\ell} = \lceil C_{\alpha_{1\ell}} B_1^{2/3} \rceil, \qquad \hat{m}_{1u} = \lceil C_{1-\alpha_{1u}} B_1^{2/3} \rceil,$$

$$\hat{\theta}^*_{B_1,\nu_{1\ell}}, \quad \hat{\theta}^*_{B_1,\nu_{1u}}, \quad \hat{\theta}^*_{B_1,\nu_{1\ell}-\hat{m}_{1\ell}}, \quad \hat{\theta}^*_{B_1,\nu_{1\ell}+\hat{m}_{1\ell}}, \quad \hat{\theta}^*_{B_1,\nu_{1u}-\hat{m}_{1u}}, \quad \hat{\theta}^*_{B_1,\nu_{1u}+\hat{m}_{1u}}. \tag{12}$$

Step 3. Take the desired number of bootstrap repetitions, B^*, to equal $B^* = \max\{B_1, B_{2\ell}, B_{2u}\}$, where

$$B_{2\ell} = \left\lceil 10{,}000(\alpha(1-\alpha) - 2\alpha\phi(z^{(\alpha)})/\phi(0) + \phi^2(z^{(\alpha)})/\phi^2(0))(z^{(1-\tau/2)})^2 \right. \\ \left. \times \left(\frac{B_1}{2\hat{m}_{1\ell}}\right)^2 (\hat{\theta}^*_{B_1,\nu_{1\ell}+\hat{m}_{1\ell}} - \hat{\theta}^*_{B_1,\nu_{1\ell}-\hat{m}_{1\ell}})^2/((\hat{\theta} - \hat{\theta}^*_{B_1,\nu_{1\ell}})pdb)^2 \right\rceil \quad \text{and}$$

$$B_{2u} = \left\lceil 10{,}000(\alpha(1-\alpha) - 2\alpha\phi(z^{(\alpha)})/\phi(0) + \phi^2(z^{(\alpha)})/\phi^2(0))(z^{(1-\tau/2)})^2 \right. \\ \left. \times \left(\frac{B_1}{2\hat{m}_{1u}}\right)^2 (\hat{\theta}^*_{B_1,\nu_{1u}+\hat{m}_{1u}} - \hat{\theta}^*_{B_1,\nu_{1u}-\hat{m}_{1u}})^2/((\hat{\theta}^*_{B_1,\nu_{1u}} - \hat{\theta})pdb)^2 \right\rceil. \tag{13}$$

Note that $z^{(\alpha)}$, $\phi(\cdot)$, and $\Phi(\cdot)$ denote the α quantile, density, and distribution function, respectively, of a standard normal distribution.

In step 2, $\alpha_{1\ell}$ and α_{1u} are truncated to be greater than or equal to .01 and less than or equal to .99, respectively. This is done to prevent potentially erratic behavior of the density estimator in step 3 if the formulae otherwise would call for estimation of the density very far in the tail. This truncation implies that the three-step method, as defined, is suitable only when $\alpha \geq .01$.

Having determined B^*, one obtains the final BC_a confidence interval by simulating $B^* - B_1$ (≥ 0) additional bootstrap estimators $\{\hat{\theta}^*_b : b = B_1 + 1, \ldots, B^*\}$, ordering the B^* bootstrap estimators, which are denoted $\{\hat{\theta}^*_{B^*,b} : b = 1, \ldots, B^*\}$, and calculating $\hat{z}_{0,B^*}$, α_{ℓ,B^*}, α_{u,B^*}, $\hat{\theta}^{*(\alpha_{\ell,B^*})}_{B^*}$, and $\hat{\theta}^{*(\alpha_{u,B^*})}_{B^*}$ using the formulae given in step 2 with B_1 replaced by B^*. The resulting BC_a confidence interval, based on B^* bootstrap repetitions, is equal to

$$CI_{B^*} = [\hat{\theta}^{*(\alpha_{\ell,B^*})}_{B^*}, \hat{\theta}^{*(\alpha_{u,B^*})}_{B^*}], \tag{14}$$

where α_{ℓ,B^*} and α_{u,B^*} are defined by (5) with B replaced by B^*.

Steps 2 and 3 could be iterated with little additional computational burden by replacing B_1 in step 2 by $\bar{B}_1 = \max\{B_1, B_{2\ell}, B_{2u}\}$, replacing $(B_{2\ell}, B_{2u})$ in step 3 by $(\bar{B}_{2\ell}, \bar{B}_{2u})$, and taking $B^* = \max\{\bar{B}_{2\ell}, \bar{B}_{2u}, \bar{B}_1\}$. In some cases, this may lead to closer finite sample and asymptotic properties of the three-step procedure.

The three-step method introduced here is based on a scalar parameter θ_0. When one is interested in separate confidence intervals for several parameters, say, M parameters, one can apply the three-step method for each of the parameters to obtain $B^*_{(1)}, B^*_{(2)}, \ldots, B^*_{(M)}$ and take B^* to equal the maximum of these values.

4. ASYMPTOTIC JUSTIFICATION OF THE THREE-STEP METHOD

We now discuss the justification of the three-step method introduced previously. The three-step method relies on the fact that $\hat{\theta}^{*(\alpha_{\ell,B})}_B$ and $\hat{\theta}^{*(\alpha_{u,B})}_B$ are sample quantiles with data-dependent percentage points based on an i.i.d. sample of random variables each with distribution given by the bootstrap distribution of $\hat{\theta}^*$. If the bootstrap distribution of $\hat{\theta}^*$ was absolutely continuous at $\hat{\theta}_B^{*(\alpha)}$, then $B^{1/2}(\hat{\theta}_B^{*(\alpha)} - \hat{\theta}_\infty^{*(\alpha)})$ would be asymptotically normally distributed as $B \to \infty$ for fixed n with asymptotic variance given by $\alpha(1-\alpha)/f^2(\hat{\theta}_\infty^{*(\alpha)})$, where $f(\cdot)$ denotes the density of $\hat{\theta}^*$. (Here and subsequently, we condition on the data, and the asymptotics are based on the randomness of the bootstrap simulations alone. We point out that it makes sense to speak of asymptotics as $B \to \infty$ for fixed n because, even though the distribution of the bootstrap sample is discrete and has a finite number n^n of atoms in the case of the nonparametric bootstrap, one can draw as many bootstrap samples B from this discrete distribution as one likes. It is not the case that $B \leq n^n$.)

But, the bootstrap distribution of $\hat{\theta}^*$ is a discrete distribution (at least for the nonparametric bootstrap, which is based on the empirical distribution). In consequence, the asymptotic distribution of $B^{1/2}(\hat{\theta}_B^{*(\alpha)} - \hat{\theta}_\infty^{*(\alpha)})$ as $B \to \infty$ for fixed n is a *pointmass at zero* for all α values except for those in a set of Lebesgue measure zero. (The latter set is the set of values that the distribution function of $\hat{\theta}^*$ takes on at its points of support.)

Although $\hat{\theta}^*$ has a discrete distribution in the case of the nonparametric bootstrap, its distribution is very nearly continuous even for small values of n. The largest probability π_n of any of its atoms is very small: $\pi_n = n!/n^n \approx (2\pi n)^{1/2}e^{-n}$ provided the original sample $\mathbf{X}$ consists of distinct vectors and distinct bootstrap samples $\mathbf{X}^*$ give rise to distinct values of $\hat{\theta}^*$ (as is typically the case; see Hall, 1992, Appendix I). This suggests that we should consider asymptotics as $n \to \infty$, and also $B \to \infty$, in order to account for the essentially continuous nature of the distribution of $\hat{\theta}^*$. If we do so, then $B^{1/2}(\hat{\theta}_B^{*(\alpha)} - \hat{\theta}_\infty^{*(\alpha)})$ has a nondegenerate asymptotic distribution with asymptotic variance that depends on the value of a density at a point, just as in the case where the distribution of $\hat{\theta}^*$ is continuous. This is what we do. It is in accord with the view of Hall (1992, p. 285) that "for many practical purposes the bootstrap distribution of a statistic may be regarded as continuous."

We note that the (potential) discreteness of $\hat{\theta}^*$ significantly increases the complexity of the asymptotic justification of the three-step method given subsequently and its proof.

The asymptotic justification of the three-step method also has to take account of the fact that the confidence interval endpoints depend on $\alpha_{\ell,B}$ and $\alpha_{u,B}$, which depend on the simulation randomness through the bootstrap bias correction $\hat{z}_{0,B} = \Phi^{-1}(\sum_{b=1}^{B}(\hat{\theta}_b^* < \hat{\theta})/B)$. The quantities $\alpha_{\ell,B}$ and $\alpha_{u,B}$ are correlated in finite samples and asymptotically with $\hat{\theta}_B^{*(\alpha)}$ for any α (see the proof of equation (18) given in the Appendix). In fact, the randomness of $\alpha_{\ell,B}$ and $\alpha_{u,B}$ is sufficiently large that it is responsible for more than half of the asymptotic variances of $B^{1/2}(\hat{\theta}_B^{*(\alpha_{\ell,B})} - \hat{\theta}_\infty^{*(\alpha)})$ and $B^{1/2}(\hat{\theta}_B^{*(\alpha_{u,B})} - \hat{\theta}_\infty^{*(1-\alpha)})$ (in the calculations carried out in Sect. 5).

We now introduce a strengthening of the assumption of asymptotic normality of the normalized estimate $n^\kappa(\hat{\theta} - \theta_0)$ that is needed for the asymptotic justification of the three-step method. We make the following assumption. For some $\xi > 0$ and all sequences of constants $\{x_n : n \geq 1\}$ for which $x_n \to \sigma_{\hat{\theta}} z^{(\alpha)}$ or $x_n \to \sigma_{\hat{\theta}} z^{(1-\alpha)}$, we have

$$P(n^{\kappa}(\hat{\theta} - \theta_0) \leq x_n) = P(\sigma_{\hat{\theta}} Z \leq x_n) + O(n^{-\xi}) \quad \text{as } n \to \infty \quad \text{and}$$

$$P^*(n^{\kappa}(\hat{\theta}^* - \hat{\theta}) \leq x_n) = P(\sigma_{\hat{\theta}} Z \leq x_n) + O(n^{-\xi}) \quad \text{as } n \to \infty, \tag{15}$$

where $Z \sim N(0,1)$. (The assumption on $n^{\kappa}(\hat{\theta}^* - \hat{\theta})$ is assumed to hold with probability one with respect to the randomness in the data, i.e., with respect to $P(\cdot)$.)

Assumption (15) holds whenever the normalized estimator $n^{\kappa}(\hat{\theta} - \theta_0)$ and the normalized bootstrap estimator $n^{\kappa}(\hat{\theta}^* - \hat{\theta})$ have one-term Edgeworth expansions. This occurs in a wide variety of contexts (e.g., see Bhattacharya and Ghosh, 1978; Hall, 1992, Sects. 2.4, 4.4, and 4.5; Hall and Horowitz, 1996). In particular, it holds in any context in which a BC_a confidence interval yields a higher order improvement in the coverage probability (see Efron, 1987; Hall, 1988). When $\kappa = \frac{1}{2}$, then (15) typically holds with $\xi = \frac{1}{2}$. When $\kappa < \frac{1}{2}$, as occurs with nonparametric estimators $\hat{\theta}$, then (15) typically holds with $\xi < \frac{1}{2}$ (see Hall, 1992, Ch. 4, and references therein).

The preceding discussion considers letting $B \to \infty$. This is not really appropriate because we want B to be determined endogenously by the three-step method. Rather, we consider asymptotics in which the accuracy measure $pdb \to 0$ and this, in turn, forces $B \to \infty$. Thus, the asymptotic justification of the three-step method of choosing B^* is in terms of the limit as *both* $pdb \to 0$ and $n \to \infty$ jointly, not sequentially.

We assume that $pdb \to 0$ sufficiently slowly that

$$pdb \times n^{\xi} \to \infty \quad \text{as } n \to \infty, \tag{16}$$

where ξ is as in (15).

We assume that

$$\hat{a} \to 0 \quad \text{as } n \to \infty \tag{17}$$

with probability one with respect to the randomness in the original data. This assumption holds for any appropriate choice of acceleration constant $\hat{a}$.

The asymptotic justification of the three-step method is that

$$P^*\left(100\,\frac{|\hat{\theta}^{*(\alpha_{j,B^*})}_{B_{2j}} - \hat{\theta}^{*(\alpha_{j,\infty})}_{\infty}|}{|\hat{\theta}^{*(\alpha_{j,\infty})}_{\infty} - \hat{\theta}|} \leq pdb\right) \to 1 - \tau \quad \text{as } pdb \to 0 \quad \text{and } n \to \infty, \quad \text{for } j = \ell, u. \tag{18}$$

As before, the probability $P^*(\cdot)$ denotes probability with respect to the simulation randomness conditional on the infinite sequence of data vectors. Under the previous assumptions, this conditional result holds with probability one with respect to the randomness in the data. The proof of (18) is given in the Appendix.

Equation (18) implies that the three-step method attains precisely the desired level of accuracy for the lower and upper lengths of the confidence interval using "small *pdb* and large *n*" asymptotics.

5. MONTE CARLO SIMULATION

5.1. Monte Carlo Design

In this section, we introduce the design of the simulation experiments. We provide simulation results for a linear regression model and a correlation coefficient. There are two purposes of the experiments. The first purpose is to illustrate the magnitudes of the values of B that are necessary to achieve different levels of accuracy. Here, *accuracy* means closeness of the BC_a confidence interval based on B repetitions to the ideal bootstrap BC_a confidence interval for which $B=\infty$. The second purpose is to see whether the three-step method yields values of B with the desired level of accuracy. More specifically, for the upper length of the confidence interval, we want to see how close $P^*(100|\hat{\theta}_B^{*(\alpha_{u,B})} - \hat{\theta}_\infty^{*(\alpha_{u,\infty})}|/(\hat{\theta}_\infty^{*(\alpha_{u,\infty})} - \hat{\theta}) \leq pdb)$ is to $1-\tau$ for values of B specified by the three-step method, for a range of values of (α, pdb, τ). We are also interested in the corresponding results for the lower length. We consider the performance of B_1, $B_{2\ell}$, B_{2u}, $B_\ell^* = \max\{B_1, B_{2\ell}\}$, $B_u^* = \max\{B_1, B_{2u}\}$, and also B^*.

Linear Regression Model. The linear regression model is

$$y_i = x_i'\beta + u_i \tag{19}$$

for $i=1,\ldots,n$, where $n=25$, $X_i = (y_i, x_i')'$ are i.i.d. over $i=1,\ldots,n$, $x_i = (1, x_{1i},\ldots,x_{5i})' \in R^6$, $(x_{1i},\ldots,x_{5i})$ are mutually independent normal random variables, x_i is independent of u_i, and u_i has a t distribution with five degrees of freedom (denoted t_5). The simulation results are invariant with respect to the means and variances of the random regressors and the value of the regression parameter β, so we need not be specific as to their values. (The results also are invariant with respect to changes in the scale of the errors.)

We estimate β by least squares (LS). We focus attention on the first slope coefficient. Thus, the parameter θ of the previous sections is β_2, the second element of β, and the estimator $\hat{\theta}$ is the LS estimator of β_2.

Correlation Coefficient. The correlation coefficient model consists of an i.i.d. sample of pairs of random variables $\{(x_i, y_i): i=1,\ldots,n\}$ with $n=25$ and correlation coefficient $\frac{1}{2}$. The random variables x_i and w_i have independent t_5 distributions, and y_i is given by

$$y_i = (1/\sqrt{3})x_i + w_i.$$

The parameter θ of the previous sections is the correlation coefficient, ρ_{xy}, between x_i and y_i. That is, $\theta = \rho_{xy} = \mathrm{Cov}(x_i, y_i)/(\mathrm{Var}(x_i)\mathrm{Var}(y_i))^{1/2}$. We estimate ρ_{xy} using the sample correlation coefficient r_{xy}:

$$\hat{\theta} = r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\left(\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2\right)^{1/2}},$$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$ and $\bar{y} = \sum_{i=1}^{n} y_i/n$.

Experimental Design. For each of the two models, we simulate 100 samples. For each of the 100 samples, we compute $\hat{\theta}$ and simulate $\hat{\theta}_\infty^{*(\alpha_\ell,\infty)}$ and $\hat{\theta}_\infty^{*(\alpha_u,\infty)}$ using 250,000 bootstrap repetitions. Here we explicitly assume that 250,000 repetitions is close enough to infinity to accurately estimate $\hat{\theta}_\infty^{*(\alpha_\ell,\infty)}$ and $\hat{\theta}_\infty^{*(\alpha_u,\infty)}$. Given $\hat{\theta}$, $\hat{\theta}_\infty^{*(\alpha_\ell,\infty)}$, and $\hat{\theta}_\infty^{*(\alpha_u,\infty)}$, we compute the lower and upper lengths of the ideal bootstrap confidence intervals for each sample.

Next, we compute 2,000 Monte Carlo repetitions for each of the 100 samples, for a total of 200,000 simulations. For a given sample, the Monte Carlo repetitions differ from each other only because of the different simulated resamples used to construct the bootstrap samples. In each Monte Carlo repetition, we compute $B_{2\ell}$, B_{2u}, and B^* for each (α, pdb, τ) combination for which $1 - 2\alpha$ is .95 or .90, pdb is 20%, 15%, or 10%, and $1 - \tau$ is .975, .95, or .90. For each sample and (α, pdb, τ) combination, we calculate the mean, median, minimum, and maximum of $B_{2\ell}$ and B_{2u} over the 2,000 Monte Carlo repetitions. In Tables 1 and 2, we report the averages of these values over the 100 samples. (For example, in column (14) of Table 1, which is headed "Med," the numbers provided are the averages of the medians of B_{2u} over the 100 samples.) For comparative purposes, we also report the value of B_1 for each (α, pdb, τ) combination. These results indicate the magnitudes of the B values needed to obtain the accuracy specified by different (pdb, τ) combinations.

In each Monte Carlo repetition, we also compute $\hat{\theta}_\infty^{*(\alpha_\ell,B)}$ for $B = B_1$, $B_{2\ell}$, B_ℓ^*, and B^* and $\hat{\theta}_\infty^{*(\alpha_u,B)}$ for $B = B_1$, B_{2u}, B_u^*, and B^*. The calculations are repeated for all of the (α, pdb, τ) combinations considered previously. For each (α, pdb, τ) combination and for each repetition, we check whether $\hat{\theta}_\infty^{*(\alpha_u,B_{2u})}$ satisfies

$$100\,\frac{|\hat{\theta}_\infty^{*(\alpha_u,B_{2u})} - \hat{\theta}_\infty^{*(\alpha_u,\infty)}|}{\hat{\theta} - \hat{\theta}_\infty^{*(\alpha_u,\infty)}} \le pdb. \quad \mathbf{(20)}$$

We compute the fraction of times this condition is satisfied out of the 2,000 Monte Carlo repetitions. Then, we compute the average of this fraction over the 100 samples. We call this fraction the *empirical level* for B_{2u} for the upper length of the BC_a confidence interval. The empirical levels for B_1, B_u^*, and B^* for the upper length also are calculated. (They are defined as before with B_1, B_u^*, and B^* in place of B_{2u}, respectively.) In addition, the empirical levels for B_1, $B_{2\ell}$, B_ℓ^*, and B^* for the lower length of the BC_a confidence interval are calculated. (They are defined analogously with ℓ in place of u.) Finally, we

TABLE 1. Simulation results for the regression model

			Empirical Levels Based on Three-Step Method					Empirical Levels with "True" Density							
			Upper				Joint	Upper		Joint		B_{2u}			
$1-2\alpha$	pdb	$1-\tau$	B_1	B_{2u}	B_u^*	B^*	B^*	B_{2ut}	B_t^*	B_t^*	B_1	Mean	Med	Min	Max
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
.950	20	.975	.850	.949	.983	.995	.990	.937	.989	.976	368	2,697	1,767	102	31,302
.950	15	.975	.868	.947	.984	.998	.996	.937	.990	.980	655	5,312	3,996	389	31,409
.950	10	.975	.883	.944	.981	.997	.993	.937	.990	.982	1,474	7,831	7,133	1,345	24,997
.950	20	.950	.799	.925	.965	.985	.972	.910	.980	.957	281	1,789	1,129	51	28,957
.950	15	.950	.818	.929	.971	.995	.991	.908	.981	.961	501	4,042	2,810	203	32,679
.950	10	.950	.834	.922	.967	.994	.989	.909	.981	.966	1,127	7,849	6,962	1,030	27,109
.950	20	.900	.727	.873	.917	.953	.914	.864	.962	.917	198	890	588	16	17,843
.950	15	.900	.740	.895	.942	.981	.965	.863	.961	.923	352	2,504	1,633	96	30,702
.950	10	.900	.761	.890	.943	.989	.979	.862	.961	.929	794	6,219	4,997	527	30,316
.900	20	.975	.902	.957	.982	.997	.995	.949	.991	.982	386	2,795	1,945	171	26,115
.900	15	.975	.912	.954	.981	.996	.992	.948	.991	.984	686	3,946	3,192	425	20,080
.900	10	.975	.920	.952	.979	.995	.990	.948	.992	.985	1,544	5,632	5,232	1,382	17,121
.900	20	.950	.856	.939	.969	.995	.990	.923	.982	.964	295	2,159	1,409	98	25,625
.900	15	.950	.868	.934	.966	.993	.986	.922	.983	.967	524	3,525	2,650	278	23,192
.900	10	.950	.878	.929	.962	.989	.980	.922	.983	.969	1,181	4,864	4,391	1,005	17,266
.900	20	.900	.779	.908	.943	.985	.973	.878	.962	.926	207	1,417	865	51	24,549
.900	15	.900	.794	.902	.939	.986	.974	.878	.962	.931	369	2,680	1,842	156	26,555
.900	10	.900	.807	.892	.930	.978	.960	.877	.962	.934	831	4,189	3,546	591	18,837

Note: The reported numbers are averages over 100 samples of the simulation results for each sample. Each sample consists of 25 observations. For each sample, 2,000 Monte Carlo repetitions are used.

calculate the *joint empirical level* for B^*, which is the fraction of times both the upper length condition (20) and the corresponding lower length condition hold with B^* in place of B_{2u} averaged over the 100 samples. We report all of the empirical levels for all of the (α, pdb, τ) combinations.

The empirical levels listed previously are subject to three types of error: (i) noisy estimates of the density and/or the upper or lower lengths of the confidence interval used in step 3 of the three-step procedure, (ii) inaccuracy of the normal approximation (even when the density and confidence interval length estimates are accurate), and (iii) simulation error. To assess the magnitude of the first type of error, we report empirical levels for the infeasible three-step procedure that uses estimates of the density and lengths of the confidence intervals in step 3 that are based on $B = 250{,}000$ rather than $B = B_1$. That is, we calculate all the quantities (except B^*) in steps 2 and 3 with B_1 replaced by 250,000. Let B_{2ut}, $B_{2\ell t}$, and B_t^* denote the analogs of B_{2u}, $B_{2\ell}$, and B^* using the "true" density and confidence interval lengths. (By definition, $B_t^* = \max\{B_1, B_{2\ell t}, B_{2ut}\}$.) We calculate the empirical levels for the upper lengths of the BC_a confidence interval that correspond to B_{2ut} and B_t^*, and also the empirical levels for the lower lengths that correspond to $B_{2\ell t}$ and B_t^*. In addition, we calculate the joint empirical level for B_t^*. We call these results the empirical levels with the "true" density.

5.2. Simulation Results

Table 1 provides the simulation results for the linear regression model. Table 1 only reports results for upper lengths because, by symmetry, the exact finite sample results for lower lengths are the same as for upper lengths in this model. Table 2 provides the results for the correlation coefficient. The first three columns of Tables 1 and 2 specify the different (α, pdb, τ) combinations that are considered in the rows of the tables. The last five columns of Table 1 and the last nine columns of Table 2 give the values of B_1 and the mean, median, minimum, and maximum values of B_{2u} (and $B_{2\ell}$ for the correlation coefficient) averaged over the 100 samples for each (α, pdb, τ) combination. The fourth to eleventh columns of Table 1 and the fourth to seventeenth columns of Table 2 give the empirical level results for the two models for each (α, pdb, τ) combination.

Linear Regression Model. Column (14) of Table 1 gives the median B_{2u} values. The median values indicate that a large number of bootstrap repetitions are required. For example, the reasonable choice of $(1 - 2\alpha, pdb, 1 - \tau) = (.90, 15, .95)$ has a median B_{2u} value of 2,650. The value does not change much when $1 - 2\alpha$ is increased to .95. This is indicative of the general insensitivity of the results to α. On the other hand, the values of B_{2u} depend greatly on the magnitudes of *pdb* and $1 - \tau$, especially *pdb*. As *pdb* decreases and $1 - \tau$ increases, the median B_{2u} values increase significantly. For example, the com-

TABLE 2. Simulation Results for the Correlation Coefficient

			Empirical Levels Based on Three-Step Method									Empirical Levels with "True" Density				
			Lower				Upper				Joint	Lower		Upper		Joint
$1-2\alpha$	*pdb*	$1-\tau$	B_1	B_{2l}	B_l^*	B^*	B_1	B_{2u}	B_u^*	B^*	B^*	B_{2lt}	B_t^*	B_{2ut}	B_t^*	B_t^*
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
.950	20	.975	.883	.959	.987	.993	.977	.926	.995	.998	.991	.928	.972	.919	.998	.970
.950	15	.975	.901	.959	.988	.996	.981	.924	.995	.999	.995	.938	.978	.919	.997	.976
.950	10	.975	.917	.956	.986	.993	.987	.925	.995	.999	.992	.944	.981	.920	.998	.979
.950	20	.950	.837	.936	.970	.979	.958	.900	.985	.995	.974	.893	.950	.886	.994	.946
.950	15	.950	.853	.941	.977	.990	.963	.899	.989	.998	.988	.908	.959	.886	.994	.954
.950	10	.950	.874	.936	.974	.988	.972	.896	.988	.998	.986	.917	.964	.887	.994	.959
.950	20	.900	.772	.884	.925	.938	.922	.851	.959	.983	.923	.837	.908	.836	.986	.897
.950	15	.900	.779	.908	.948	.967	.928	.857	.969	.991	.959	.855	.922	.834	.985	.911
.950	10	.900	.804	.906	.951	.976	.938	.852	.971	.994	.972	.869	.932	.832	.984	.919
.900	20	.975	.920	.968	.986	.992	.982	.934	.992	.999	.993	.955	.983	.933	.998	.982
.900	15	.975	.931	.966	.985	.992	.985	.935	.992	.998	.991	.959	.985	.934	.998	.984
.900	10	.975	.938	.965	.983	.994	.987	.936	.992	.998	.990	.961	.987	.934	.998	.985
.900	20	.950	.873	.952	.974	.979	.963	.907	.983	.997	.986	.929	.967	.902	.995	.963
.900	15	.950	.888	.949	.972	.980	.968	.906	.982	.996	.983	.935	.971	.902	.995	.967
.900	10	.950	.899	.946	.969	.981	.972	.906	.981	.996	.978	.938	.973	.901	.995	.969
.900	20	.900	.799	.923	.949	.971	.920	.865	.960	.990	.963	.881	.933	.849	.985	.923
.900	15	.900	.818	.921	.948	.968	.931	.860	.960	.991	.966	.892	.941	.849	.985	.930
.900	10	.900	.833	.913	.941	.965	.938	.856	.956	.988	.955	.899	.946	.850	.985	.934

				B_{2l}				B_{2u}			
$1-2\alpha$	pdb	$1-\tau$	B_1	Mean	Med	Min	Max	Mean	Med	Min	Max
(1)	(2)	(3)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)
.950	20	.975	368	1,804	1,322	92	17,725	503	403	34	3,648
.950	15	.975	655	3,901	3,033	324	25,184	1,000	839	106	6,159
.950	10	.975	1,474	6,136	5,539	1,118	22,519	1,760	1,611	382	6,627
.950	20	.950	281	1,200	857	45	14,424	358	279	18	2,925
.950	15	.950	501	2,785	2,101	192	22,046	726	597	67	4,763
.950	10	.950	1,127	6,440	5,591	835	24,500	1,728	1,515	277	7,940
.950	20	.900	198	634	464	14	6,948	216	165	8	1,738
.950	15	.900	352	1,667	1,220	82	16,811	470	376	31	3,320
.950	10	.900	794	4,772	3,840	448	26,259	1,235	1,056	148	6,947
.900	20	.975	386	2,249	1,644	165	20,640	598	482	65	4,601
.900	15	.975	686	3,568	2,879	424	19,187	1,040	890	159	6,006
.900	10	.975	1,544	5,301	4,888	1,265	16,959	1,797	1,667	493	5,826
.900	20	.950	295	1,678	1,188	97	18,756	458	358	40	4,057
.900	15	.950	524	2,998	2,311	279	21,086	807	672	109	5,286
.900	10	.950	1,181	4,577	4,069	911	17,131	1,553	1,385	344	6,555
.900	20	.900	207	1,064	735	45	14,133	311	235	22	2,979
.900	15	.900	369	2,146	1,561	150	19,907	571	459	61	4,287
.900	10	.900	831	3,913	3,271	545	18,192	1,228	1,065	207	5,942

Note: The reported numbers are averages over 100 samples of the simulation results for each sample. Each sample consists of 25 observations. For each sample, 2,000 Monte Carlo repetitions are used. The true correlation coefficient is $\rho_{xy} = .5$.

bination $(1-2\alpha, pdb, 1-\tau) = (.90, 20, .90)$ has median B_{2u} value of 865, whereas $(.90, 20, .975)$ has median B_{2u} value of 1,945, and $(.90, 10, .90)$ has median B_{2u} value of 3,546. Although the magnitudes of the B_{2u} values are large, the computation time required for the applications considered here is relatively small; always less than one minute.

The results of columns (13)–(16) of Table 1 also show that the B_{2u} values have a skewed distribution—the median is well below the mean number of bootstrap repetitions. In some cases, the required number of bootstrap repetitions is very large (see column (16)). Comparison of columns (12) and (14) shows that the median B_{2u} values are much larger than the initial B_1 values. This suggests that relying on just the first step of the three-step method, namely, B_1, is ill advised. All three steps of the three-step method are needed.

Column (4) of Table 1 reports the empirical levels based on B_1 for the regression model. These empirical levels are well below their theoretical counterparts, reported in column (3), for all $(1-2\alpha, pdb, 1-\tau)$ combinations. This corroborates the preceding supposition that reliance on the B_1 values is ill advised. The empirical levels increase significantly when B_{2u} simulations are employed (see column (5)). But the empirical levels for B_{2u} are still below the $1-\tau$ values of column (3) in most cases. The empirical levels for the B_u^* values, given in column (6), increase further. In fact, for all cases in which $1-\tau$ is .975 (.95, respectively), the empirical levels are within .009 (.021, respectively) of the exact $1-\tau$ value given in column (3). This indicates that the three-step method is performing well in terms of matching the finite sample accuracy with the desired theoretical accuracy.

The empirical levels for B^* are given in column (7). These empirical levels are higher than the empirical levels for B_u^* for the upper length. As it turns out, it is difficult to accurately estimate either the upper length of the confidence interval or the lower length. In consequence, one of the two sides of the confidence interval usually requires a relatively large number of bootstrap repetitions. As a result, the empirical levels based on B^* are quite high, well above their theoretical counterparts for some $(1-2\alpha, pdb, 1-\tau)$ combinations. The joint empirical levels for B^*, given in column (8), are somewhat lower than the upper empirical levels for B^*. But, they still tend to be conservative, i.e., greater than $1-\tau$.

The empirical levels with the "true" density are reported in columns (9)–(11). For most $(1-2\alpha, pdb, 1-\tau)$ combinations, these results do not differ very much from the results discussed previously. However, when $1-\tau$ is .900, which generates relatively small B values, there is a noticeable difference. These results indicate that estimation of the density and the confidence interval length is not a large source of inaccuracy of the three-step method unless $1-\tau$ is relatively small.

Correlation Coefficient. The results for the correlation coefficient are reported in Table 2. The general picture for the lower length results in Table 2 is very similar to that for the upper length for the regression model in Table 1 (which is the same for the upper length by symmetry). However, there is a significant difference between the two experiments. First, the B_{2u} values are much smaller than the $B_{2\ell}$ values for the correlation coefficient experiment. Second, the empirical levels for the upper length based on B_1 repetitions are quite high for the correlation coefficient experiment. These features are a consequence of the fact that the correlation coefficient is bounded between -1 and 1, the true value is $\rho_{xy} = \frac{1}{2}$, and, hence, an asymmetry occurs between the results for the lower and upper lengths. The "density" of the bootstrap distribution of $\hat{\theta}$ is much larger at the $1 - \alpha$ quantile than at the α quantile, which yields much smaller B_{2u} values than $B_{2\ell}$ values.

Table 2 indicates that even for a simple statistic, such as the correlation coefficient, the required number of bootstrap repetitions can be quite large. For example, for a 95% confidence interval estimated with $pdb = 10$ and $1 - \tau = .95$, the median number of bootstrap repetitions required is over 5,591.

The empirical levels for the lower confidence intervals are quite similar to those reported for the upper confidence interval for the regression model. The B_1 values for the upper confidence intervals are too large, however, which leads to upper empirical levels for B_1, B_u^*, and B^* that are too high. In consequence, the three-step method is conservative. It produces larger numbers of bootstrap repetitions than are required for the specified (pdb, τ) combinations.

The empirical levels with the "true" density show a similar pattern as in the regression model. However, somewhat more of the inaccuracy of the three-step method is attributable to the estimation of the density and confidence interval length in the correlation coefficient experiment.

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APPENDIX OF PROOFS

We prove (18) for $j = u$. The proof for $j = \ell$ is analogous. First we show that (18) holds with B^* replaced by the nonrandom quantity B_1. Note that $B_1 \to \infty$ as $pdb \to 0$ and B_1 does not depend on n.

Define the $1 - \alpha$ sample quantile of the *normalized* bootstrap estimates to be

$$\hat{\lambda}_{1-\alpha,B} = n^\kappa(\hat{\theta}_B^{*(1-\alpha)} - \hat{\theta}) = n^\kappa(\hat{\theta}^*_{B,\lceil (B+1)(1-\alpha)\rceil} - \hat{\theta}) \quad \text{for } \alpha < \tfrac{1}{2}. \tag{A.1}$$

Let $\hat{\lambda}_{1-\alpha,\infty}$ denote the $1 - \alpha$ quantile of $n^\kappa(\hat{\theta}^* - \hat{\theta})$. That is, $\hat{\lambda}_{1-\alpha,\infty} = n^\kappa(\hat{\theta}_\infty^{*(1-\alpha)} - \hat{\theta})$. Note that the percentage deviation of the upper length of CI_B to the upper length of CI_∞, given in (8), can be written as

$$100\,\frac{|\hat{\lambda}_{\alpha_{u,B},B} - \hat{\lambda}_{\alpha_{u,\infty},\infty}|}{\hat{\lambda}_{\alpha_{u,\infty},\infty}}. \tag{A.2}$$

We establish the asymptotic distribution of $B_1^{1/2}(\hat{\lambda}_{\alpha_{u,B_1},B_1} - \hat{\lambda}_{\alpha_{u,\infty},\infty})$ as $pdb \to 0$ and $n \to \infty$, using an argument developed for proving the asymptotic distribution of the sample median based on an i.i.d. sample of random variables that are absolutely continuous at their population median (e.g., see Lehmann, 1983, Theorem 5.3.2, p. 354). (In contrast, $\hat{\lambda}_{\alpha_{u,B_1},B_1}$ is the sample α_{u,B_1} quantile of B_1 i.i.d. observations each with the bootstrap distribution of $n^\kappa(\hat{\theta}^* - \hat{\theta})$, which depends on n and may be discrete, where α_{u,B_1} is random and data dependent.)

We have the following expression. For any $x \in R$,

$$P^*(B_1^{1/2}(\hat{\lambda}_{\alpha_{u,B_1},B_1} - \hat{\lambda}_{\alpha_{u,\infty},\infty}) \le x) = P^*(n^\kappa(\hat{\theta}^*_{B_1,\lceil (B_1+1)\alpha_{u,B_1}\rceil} - \hat{\theta}) \le \hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}). \tag{A.3}$$

Let S_B be the number of values $n^\kappa(\hat{\theta}_b^* - \hat{\theta})$'s for $b = 1,\dots,B$ that exceed $\hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}$. Here, we consider S_{B_1}. Subsequently, we consider S_{B^*}. (In both cases, the cutoff point $\hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}$ depends on B_1.) We have

$$n^\kappa(\hat{\theta}^*_{B_1,\lceil (B_1+1)\alpha_{u,B_1}\rceil} - \hat{\theta}) \le \hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2} \text{ if and only if } S_{B_1} \le B_1 - \lceil (B_1+1)\alpha_{u,B_1}\rceil. \tag{A.4}$$

The random variable S_{B_1} has a binomial distribution with parameters $(B_1, p_{B_1,n})$, where

$$p_{B_1,n} = 1 - P^*(n^\kappa(\hat{\theta}_b^* - \hat{\theta}) \leq \hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}). \quad \textbf{(A.5)}$$

The probability in (A.3) equals

$$\begin{aligned} &P^*(S_{B_1} \leq B_1 - \lceil (B_1+1)\alpha_{u,B_1} \rceil) \\ &\quad = P^*(B_1^{-1/2}(S_{B_1} - B_1 p_{B_1,n}) + B_1^{1/2}(\alpha_{u,B_1} - \alpha_{u,\infty}) \\ &\qquad \leq B_1^{1/2}(1 - p_{B_1,n} - \alpha_{u,\infty}) - B_1^{-1/2}\alpha_{u,B_1} + o(1)). \end{aligned} \quad \textbf{(A.6)}$$

We now determine the limits of the terms in the right-hand-side probability of (A.6).

Using the assumptions of (17) and (15), we have $\hat{a} = o(1)$,

$$\hat{z}_{0,\infty} = \Phi^{-1}(P^*(\hat{\theta}^* < \hat{\theta})) = \Phi^{-1}(P(Z<0) + o(1)) = o(1), \quad \text{and}$$

$$\alpha_{u,\infty} = \Phi\left(\hat{z}_{0,\infty} + \frac{\hat{z}_{0,\infty} + z^{(1-\alpha)}}{1 - \hat{a}(\hat{z}_{0,\infty} + z^{(1-\alpha)})}\right) = \Phi(z^{(1-\alpha)}) + o(1) = 1 - \alpha + o(1), \quad \textbf{(A.7)}$$

where $Z \sim N(0,1)$. These results and the assumption of (15) yield

$$\begin{aligned} \hat{\lambda}_{\alpha_{u,\infty},\infty} &= \inf\{\lambda : P^*(n^\kappa(\hat{\theta}^* - \hat{\theta}) \leq \lambda) \geq \alpha_{u,\infty}\} \\ &= \inf\{\lambda : P(\sigma_{\hat{\theta}} Z \leq \lambda) + o(1) \geq 1 - \alpha\} \\ &= \sigma_{\hat{\theta}} z^{(1-\alpha)} + o(1) \quad \text{as } n \to \infty. \end{aligned} \quad \textbf{(A.8)}$$

Next, we have

$$\begin{aligned} &B_1^{1/2}(1 - p_{B_1,n} - \alpha_{u,\infty}) \\ &\quad = B_1^{1/2}(P^*(n^\kappa(\hat{\theta}^* - \hat{\theta}) \leq \hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}) - P^*(n^\kappa(\hat{\theta}^* - \hat{\theta}) \leq \hat{\lambda}_{\alpha_{u,\infty},\infty})) \\ &\quad = B_1^{1/2}(P(\sigma_{\hat{\theta}} Z \leq \hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}) - P(\sigma_{\hat{\theta}} Z \leq \hat{\lambda}_{\alpha_{u,\infty},\infty})) + o(1) \\ &\quad = \phi(\zeta_{B_1,n}/\sigma_{\hat{\theta}})x/\sigma_{\hat{\theta}} + o(1) \\ &\quad \to \phi(z^{(1-\alpha)})x/\sigma_{\hat{\theta}} \quad \text{as } pdb \to 0 \quad \text{and} \quad n \to \infty. \end{aligned} \quad \textbf{(A.9)}$$

The first equality of (A.9) holds by the definitions of $p_{B_1,n}$ and $\hat{\lambda}_{\alpha_{u,\infty},\infty}$. The second equality holds by (15) and (16) using the fact that the latter and the definition of B_1 imply that $B_1^{1/2} = O(1/pdb) = n^\xi O(1/(pdb \times n^\xi)) = o(n^\xi)$. The third equality holds for some $\zeta_{B_1,n}$ that lies between $\hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}$ and $\hat{\lambda}_{\alpha_{u,\infty},\infty}$ by a mean value expansion. The convergence result of (A.9) holds by (A.8).

Note that (A.7) and (A.9) imply that $p_{B_1,n} \to \alpha$ as $pdb \to 0$ and $n \to \infty$.

Now, we have

$$
\begin{aligned}
B_1^{1/2}\hat{z}_{0,B_1} &= B_1^{1/2}\left(\Phi^{-1}\left(\frac{1}{B_1}\sum_{b=1}^{B_1} 1(\hat{\theta}_b^* < \hat{\theta})\right) - \Phi^{-1}\left(\frac{1}{2}\right)\right) \\
&= \left(\frac{1}{\phi(0)(1+o_p(1))}\right)\frac{1}{B_1^{1/2}}\sum_{b=1}^{B_1}\left(1(\hat{\theta}_b^* < \hat{\theta}) - \frac{1}{2}\right) \\
&= \left(\frac{1}{\phi(0)(1+o_p(1))}\right)\frac{1}{B_1^{1/2}}\sum_{b=1}^{B_1}(1(\hat{\theta}_b^* < \hat{\theta}) - P^*(\hat{\theta}_b^* < \hat{\theta})) + o_p(1), \qquad \textbf{(A.10)}
\end{aligned}
$$

where the second equality holds by a mean value expansion and the third equality holds because (15) and (16) imply that $B_1^{1/2}(P^*(\hat{\theta}_b^* < \hat{\theta}) - \frac{1}{2}) \to 0$.

Next, we have

$$
\begin{aligned}
&B_1^{1/2}(\alpha_{u,B_1} - \alpha_{u,\infty}) \\
&\quad = B_1^{1/2}\left(\Phi\left(\hat{z}_{0,B_1} + \frac{\hat{z}_{0,B_1} + z^{(1-\alpha)}}{1 - \hat{a}(\hat{z}_{0,B_1} + z^{(1-\alpha)})}\right) - \Phi\left(\hat{z}_{0,\infty} + \frac{\hat{z}_{0,\infty} + z^{(1-\alpha)}}{1 - \hat{a}(\hat{z}_{0,\infty} + z^{(1-\alpha)})}\right)\right) \\
&\quad = \phi(z^{(1-\alpha)})(1+o_p(1))B_1^{1/2} \\
&\qquad \times\left(\hat{z}_{0,B_1} + \frac{\hat{z}_{0,B_1} + z^{(1-\alpha)}}{1 - \hat{a}(\hat{z}_{0,B_1} + z^{(1-\alpha)})} - \hat{z}_{0,\infty} - \frac{\hat{z}_{0,\infty} + z^{(1-\alpha)}}{1 - \hat{a}(\hat{z}_{0,\infty} + z^{(1-\alpha)})}\right) \\
&\quad = \phi(z^{(1-\alpha)})B_1^{1/2}(2\hat{z}_{0,B_1} - 2\hat{z}_{0,\infty} - z^{(1-\alpha)}\hat{a}(\hat{z}_{0,B_1} - \hat{z}_{0,\infty}))(1+o_p(1)) \\
&\quad = 2\phi(z^{(1-\alpha)})B_1^{1/2}\hat{z}_{0,B_1}(1+o_p(1)) + o_p(1) \\
&\quad = \left(\frac{2\phi(z^{(1-\alpha)})}{\phi(0)}\right)\frac{1}{B_1^{1/2}}\sum_{b=1}^{B_1}(1(\hat{\theta}_b^* < \hat{\theta}) - P^*(\hat{\theta}_b^* < \hat{\theta}))(1+o_p(1)) + o_p(1), \qquad \textbf{(A.11)}
\end{aligned}
$$

where the second equality holds by the mean value theorem because $\hat{z}_{0,B_1} \to_p 0$, $\hat{z}_{0,\infty} \to_p 0$, and $\hat{a} \to 0$; the third equality holds because $\hat{a} \to 0$; the fourth equality holds because $B_1^{1/2}\hat{z}_{0,B_1} = O_p(1)$ by (A.10) and the Lindeberg central limit theorem, $\hat{a} \to 0$, and $B_1^{1/2}\hat{z}_{0,\infty} = B_1^{1/2}(\Phi^{-1}(P^*(n^\kappa(\hat{\theta}_b^* - \hat{\theta}) < 0)) - \Phi^{-1}(\frac{1}{2})) = o_p(1)$ by a mean value expansion, (15), and (16), and the fifth equality holds using (A.10).

Equation (A.11) gives

$$
\begin{aligned}
&B_1^{-1/2}(S_{B_1} - B_1 p_{B_1,n}) + B_1^{1/2}(\alpha_{u,B_1} - \alpha_{u,\infty}) \\
&\quad = (1+o_p(1))\frac{1}{B_1^{1/2}}\sum_{b=1}^{B_1}((n^\kappa(\hat{\theta}_b^* - \hat{\theta}) > \hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}) - p_{B_1,n} \\
&\qquad\qquad + (2\phi(z^{(1-\alpha)})/\phi(0)) \\
&\qquad\qquad \times (1(n^\kappa(\hat{\theta}_b^* - \hat{\theta}) < 0) - P^*(n^\kappa(\hat{\theta}_b^* - \hat{\theta}) < 0))) + o_p(1) \\
&\quad \xrightarrow{d} N\left(0, \alpha(1-\alpha) - 2\alpha\frac{\phi(z^{(\alpha)})}{\phi(0)} + \frac{\phi^2(z^{(\alpha)})}{\phi^2(0)}\right) \qquad \textbf{(A.12)}
\end{aligned}
$$

as $pdb \to 0$ and $n \to \infty$, where the convergence result holds by the Lindeberg central limit theorem and the fact that $p_{B_1,n} \to \alpha$ and $P^*(n^\kappa(\hat\theta_b^* - \hat\theta) < 0) \to \frac{1}{2}$.

Equations (A.3), (A.6), (A.9), and (A.12) yield

$$\begin{aligned}
&P^*(B_1^{1/2}(\hat\lambda_{\alpha_{u,B_1},B_1} - \hat\lambda_{\alpha_{u,\infty},\infty}) \le x)\\
&\quad \to \Phi\left(x\phi(z^{(\alpha)}) \Big/ \left(\sigma_{\hat\theta}\left(\alpha(1-\alpha) - 2\alpha\,\frac{\phi(z^{(\alpha)})}{\phi(0)} + \frac{\phi^2(z^{(\alpha)})}{\phi^2(0)}\right)^{1/2}\right)\right) \quad \text{and}\\
&B_1^{1/2}(\hat\lambda_{\alpha_{u,B_1},B_1} - \hat\lambda_{\alpha_{u,\infty},\infty})\\
&\quad \xrightarrow{d} N\left(0, \sigma_{\hat\theta}^2\left(\alpha(1-\alpha) - 2\alpha\,\frac{\phi(z^{(\alpha)})}{\phi(0)} + \frac{\phi^2(z^{(\alpha)})}{\phi^2(0)}\right)\Big/\phi^2(z^{(\alpha)})\right)
\end{aligned} \tag{A.13}$$

as $pdb \to 0$ and $n \to \infty$.

This result, (11), (A.2), and (A.8) imply that

$$\begin{aligned}
&P^*\left(100\,\frac{|\hat\theta_B^{*(\alpha_{u,B_1})} - \hat\theta_\infty^{*(\alpha_{u,\infty})}|}{\hat\theta_\infty^{*(\alpha_{u,\infty})} - \hat\theta} \le pdb\right)\\
&\quad = P^*\left(100\,\frac{|\hat\lambda_{\alpha_{u,B_1},B_1} - \hat\lambda_{\alpha_{u,\infty},\infty}|}{\hat\lambda_{\alpha_{u,\infty},\infty}} \le 100\left(\alpha(1-\alpha) - 2\alpha\,\frac{\phi(z^{(\alpha)})}{\phi(0)} + \frac{\phi^2(z^{(\alpha)})}{\phi^2(0)}\right)^{1/2}\right.\\
&\qquad\qquad \left. \times \left(\frac{z^{(1-\tau/2)}}{z^{(1-\alpha)}B_1^{1/2}}\right)\left(\frac{1}{\phi(z^{(1-\alpha)})}\right)(1+o(1))\right)\\
&\quad \to 1-\tau \quad \text{as } pdb \to 0 \quad \text{and} \quad n \to \infty.
\end{aligned} \tag{A.14}$$

Thus, (18) holds with B^* replaced with B_1.

Next, we show that

$$B_{2u}/B_1 \xrightarrow{p} 1 \quad \text{as } pdb \to 0 \quad \text{and} \quad n \to \infty \tag{A.15}$$

(with respect to the simulation randomness conditional on the data). By an analogous argument $B_{2\ell}/B_1 \to_p 1$ as $pdb \to 0$ and $n \to \infty$. These results imply that

$$B^*/B_1 \xrightarrow{p} 1 \quad \text{as } pdb \to 0 \quad \text{and} \quad n \to \infty. \tag{A.16}$$

Equation (A.15) follows from

$$\begin{aligned}
&n^\kappa(\hat\theta_{B_1,\nu_{1u}}^* - \hat\theta) = \hat\lambda_{\lceil (B_1+1)(1-\alpha)\rceil/(B_1+1),B_1} \xrightarrow{p} \hat\sigma_{\hat\theta} z^{(1-\alpha)} \quad \text{and}\\
&\left(\frac{B_1}{2\hat m_{1u}}\right)^2 (n^\kappa(\hat\theta^*_{B_1,\nu_{1u}+\hat m_{1u}} - \hat\theta) - n^\kappa(\hat\theta^*_{B_1,\nu_{1u}-\hat m_{1u}} - \hat\theta))^2 \xrightarrow{p} \frac{1}{\phi(\hat\sigma_{\hat\theta} z^{(1-\alpha)}/\hat\sigma_{\hat\theta})/\hat\sigma_{\hat\theta}}
\end{aligned} \tag{A.17}$$

as $pdb \to 0$ and $n \to \infty$. The former holds by the argument of (A.8) and (A.13) using the fact that $\alpha_{1u} \to 1 - \alpha$ (provided $1 - \alpha \le .99$) as $pdb \to 0$ and $n \to \infty$ because $\hat z_{0,B_1} \to 0$ by (A.10) and $\hat a \to 0$ by (17). The latter holds by an analogous argument to that given in Andrews and Buchinsky (2000, Appendix, Proofs for the Confidence Interval, Confidence Region, and Test Applications Section).

Now we use equation (A.16) and the preceding proof that (18) holds with the random quantity B^* replaced by the nonrandom quantity B_1 to establish (18) as is.

First, we have the following result. For any $x \in R$,

$$P^*(B_1^{1/2}(\hat{\lambda}_{\alpha_{u,B^*},B^*} - \hat{\lambda}_{\alpha_{u,\infty},\infty}) \leq x)$$
$$= P^*(n^\kappa(\hat{\theta}^*_{B^*,\lceil (B^*+1)\alpha_{u,B^*}\rceil} - \hat{\theta}) \leq \hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}). \quad \textbf{(A.18)}$$

(Note that we take the normalization factor to be B_1, not B^*.) Let S_{B^*} be as defined earlier. By the same argument as used in (A.4), the probability in (A.18) equals

$$P^*(S_{B^*} \leq B^* - \lceil (B^*+1)\alpha_{u,B^*}\rceil)$$
$$= P^*((B^*)^{-1/2}(S_{B^*} - B^* p_{B_1,n}) + (B^*)^{1/2}(\alpha_{u,B^*} - \alpha_{u,\infty})$$
$$\leq (B^*)^{1/2}(1 - p_{B_1,n} - \alpha_{u,\infty}) - (B^*)^{-1/2}\alpha_{u,B^*} + o(1)). \quad \textbf{(A.19)}$$

By the same argument as given in (A.7)–(A.12), we obtain

$$(B^*)^{-1/2}(S_{B^*} - B^* p_{B_1,n}) + (B^*)^{1/2}(\alpha_{u,B^*} - \alpha_{u,\infty})$$
$$= (1 + o_p(1))\,\frac{1}{(B^*)^{1/2}}\sum_{b=1}^{B^*}((n^\kappa(\hat{\theta}^*_b - \hat{\theta}) > \hat{\lambda}_{\alpha_{u,\infty},\infty} + x/B_1^{1/2}) - p_{B_1,n}$$
$$+ (2\phi(z^{(1-\alpha)})/\phi(0))(1(n^\kappa(\hat{\theta}^*_b - \hat{\theta}) < 0) - P^*(n^\kappa(\hat{\theta}^*_b - \hat{\theta}) < 0))) + o_p(1)$$
$$\xrightarrow{d} N\left(0, \alpha(1-\alpha) - 2\alpha\,\frac{\phi(z^{(\alpha)})}{\phi(0)} + \frac{\phi^2(z^{(\alpha)})}{\phi^2(0)}\right) \quad \textbf{(A.20)}$$

as $pdb \to 0$ and $n \to \infty$. The convergence result holds by the central limit theorem of Doeblin-Anscombe (e.g., see Chow and Teicher, 1978, Theorem 9.4.1, p. 317) because (i) the convergence result holds when B^* is replaced by the nonrandom quantity B_1 and (ii) $B^*/B_1 \to_p 1$ by (A.16).

Now, by the argument of (A.13) and (A.14), (18) holds as stated, which concludes the proof.

We finish by showing that the formula given in (10) for C_α, which is used to determine the bandwidth parameters $\hat{m}_{1\ell}$ and $\hat{m}_{1u}$ for the Siddiqui estimator, corresponds to that given by Hall and Sheather (1988). In our notation, Hall and Sheather's formula is

$$C_\alpha = \left(\frac{1.5(z^{(1-\alpha/2)})^2 f^4(q_{1-\alpha})}{3f'(q_{1-\alpha})^2 - f(q_{1-\alpha})f''(q_{1-\alpha})}\right)^{1/3}, \quad \textbf{(A.21)}$$

where $f(\cdot)$ denotes the density of the i.i.d. random variables upon which the sample quantile is based, $f'(\cdot)$ and $f''(\cdot)$ denote the first two derivatives of $f(\cdot)$, $q_{1-\alpha}$ denotes the population quantile, and $z^{(1-\alpha/2)}$ is as previously. In our case, we use the asymptotic analogs of $f(\cdot)$ and $q_{1-\alpha}$, namely, $\phi(\cdot/\hat{\sigma}_{\hat{\theta}})/\hat{\sigma}_{\hat{\theta}}$ and $\hat{\sigma}_{\hat{\theta}} z^{(1-\alpha)}$, respectively, in the formula. Note that $\phi'(x) = -x\phi(x)$ and $\phi''(x) = (x^2-1)\phi(x)$. Thus, $f(q_{1-\alpha}) = \phi(z^{(1-\alpha)})/\hat{\sigma}_{\hat{\theta}}$, $f'(q_{1-\alpha}) = \phi(z^{(1-\alpha)})/\hat{\sigma}^2_{\hat{\theta}}$, and $f''(q_{1-\alpha}) = ((z^{(1-\alpha)})^2 - 1)\phi(z^{(1-\alpha)})/\hat{\sigma}^3_{\hat{\theta}}$. Plugging these formulae into (A.21) gives the definition of the constant C_α in (10).