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COWLES FOUNDATION PAPER NO. 1250

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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2008

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Econometric Theory, **23**, 2007, 1033–1082. Printed in the United States of America.
DOI: 10.1017/S0266466607070429

RANK TESTS FOR INSTRUMENTAL VARIABLES REGRESSION WITH WEAK INSTRUMENTS

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This paper considers tests in an instrumental variable (IVs) regression model with IVs that may be weak. Tests that have near-optimal asymptotic power properties with Gaussian errors for weak and strong IVs have been determined in Andrews, Moreira, and Stock (2006, *Econometrica* 74, 715–752). In this paper, we seek tests that have near-optimal asymptotic power with Gaussian errors and improved power with non-Gaussian errors relative to existing tests. Tests with such properties are obtained by introducing rank tests that are analogous to the conditional likelihood ratio test of Moreira (2003, *Econometrica* 71, 1027–1048). We also introduce a rank test that is analogous to the Lagrange multiplier test of Kleibergen (2002, *Econometrica* 70, 1781–1803) and Moreira (2001, manuscript, University of California, Berkeley).

1. INTRODUCTION

This paper is concerned with inference in the standard linear instrumental variable (IV) regression model with possibly weak IVs. We start by giving a brief account of recent developments in the literature on weak IVs to explain the contribution of this paper to the literature. It has been documented in the weak IV literature that standard methods, such as two-stage least squares–based tests and confidence intervals (CIs), perform poorly when IVs are weak, especially when endogeneity is moderate to strong. Specifically, such tests have size well in excess of their nominal level, and corresponding CIs have size well below their nominal level. See the review papers of Stock, Wright, and Yogo (2002), Dufour (2003), and Andrews and Stock (in press).

The well-known Anderson and Rubin (1949) (AR) test does not exhibit size distortions due to weak IVs. Hence, Staiger and Stock (1997) and Dufour (1997)

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 0266-4666/07 $15.00

propose basing inference on the AR test. AR-based CIs can be constructed by inverting AR tests. The AR test has good power properties when the model is just identified; see Moreira (2001) and Andrews, Moreira, and Stock (2006a) for some optimality properties for the case of Gaussian errors. However, the AR test sacrifices power when the model is overidentified. This leads to excessively long AR-based CIs.

In consequence, considerable effort has been expended recently to develop new tests that circumvent this problem. Such tests are of interest in their own right and because they can be used to construct CIs by inversion. Kleibergen (2002) and Moreira (2001) introduce a Lagrange multiplier (LM) test whose size is robust to weak IVs and whose power exceeds that of the AR test in many cases when the model is overidentified. However, this test has somewhat quirky power properties. For example, its power function can be nonmonotonic; see Andrews et al. (2006a, 2006b).

Subsequently, Moreira (2003) showed that any test can be made robust to weak IVs asymptotically by using a conditional critical value function that conditions on a statistic that is complete and sufficient under the null hypothesis. Using this method, he introduced the conditional likelihood ratio (CLR) test. Andrews et al. (2006a) investigate the power properties of the CLR test in the case of a single right-hand-side endogenous variable and show that its power is essentially on the asymptotic power envelope for two-sided invariant similar tests under the assumption of Gaussian errors. This is true under both the "weak IV asymptotics" introduced in Staiger and Stock (1997), in which the coefficient on the IVs in the first-stage regression shrinks to zero as the sample size goes to infinity, and under the standard "strong IV asymptotics." Andrews and Stock (in press) show that these optimality properties extend to the "many weak IV asymptotic scenario," in which the number of IVs increases with the sample size. Hence, the CLR test has the desirable features of having size that is robust to weak IVs and near-optimal power properties with Gaussian errors.[1]

In this paper, we aim to further improve the power properties of weak IV tests by constructing a test that has the same asymptotic behavior as the CLR test with Gaussian errors but improved power with non-Gaussian errors. To do this, we construct a *rank* analogue of the CLR test, denoted RCLR. We also construct a rank analogue of the LM test of Kleibergen (2002) and Moreira (2001), denoted RLM. As is well known from location and regression models, rank estimators and tests have more robust efficiency properties than least squares–based procedures; see Hettmansperger (1984). For example, Chernoff and Savage (1958) have shown that the asymptotic relative efficiency (ARE) of the normal scores rank test to the analogous least-squares t-test is greater than or equal to one for all symmetric error distributions with equality at the Gaussian. This holds in both location and regression models, and it also holds for estimators. This suggests that for the linear IV model rank-based tests whose size is robust to weak IVs may exhibit similarly desirable power properties under nonnormality.

Andrews and Marmer (in press) develop a rank analogue of the AR test, denoted RAR. This test has exact finite-sample size under Gaussian and non-Gaussian errors under certain circumstances. Its asymptotic power properties improve on those of the AR test and are excellent for just-identified models. However, as with the nonrank AR test, the RAR test sacrifices power in overidentified models. The RCLR and RLM tests developed here substantially improve the power properties of the RAR test in overidentified models.

We now summarize the results of the paper. The model considered is a linear IV regression model with a single structural equation with m right-hand-side endogenous variables and p exogenous variables coupled with m reduced-form equations for the right-hand-side endogenous variables. The null hypothesis is $H_0 : \beta = \beta_0$, where β is the m-dimensional coefficient on the m right-hand-side endogenous variables. The alternative hypothesis is $H_1 : \beta \neq \beta_0$.

First, we introduce rank analogues of the CLR and LM tests. This is more difficult than for the AR test because the LR and LM statistics are more complicated functions of the data than is the AR statistic. A hybrid rank/linear test statistic is required to obtain power properties of RCLR and RLM tests that are analogous to those of the CLR and LM tests under Gaussianity and superior for other distributions.

Second, we obtain the weak IV asymptotic distributions of the rank statistics under the null and fixed alternatives. These results are used to show that under Gaussian errors the normal scores (NS) RCLR and RLM tests have the same null and alternative asymptotic behavior as the nonrank versions of these tests. The same is true for the Wilcoxon scores (WS) rank and nonrank CLR and LM tests under uniform errors. Furthermore, these asymptotic distributions allow one to compare the weak IV asymptotic power of the rank to nonrank tests under different error distributions. It is shown that the same AREs for the rank versus nonrank LM and AR tests arise in the weak IV context as in the location and regression models. Hence, the Chernoff–Savage result also applies to these tests. That is, the NS-RLM test (weakly) dominates the LM test in terms of power for all symmetric error distributions, and the same is true for the NS-RAR test versus the AR test.

For the rank versus nonrank CLR tests, the weak IV asymptotic power comparison is more complicated. However, numerical calculation of the asymptotic powers of these tests shows the same pattern that is typical for rank versus nonrank procedures in other contexts. In particular, the NS-RCLR test has noticeably higher asymptotic power for thin-tailed (uniform) and thick-tailed (t_3 and difference of independent log normals (DLN)) errors than the nonrank CLR test and equal asymptotic power for Gaussian errors. The WS-RCLR test has asymptotic power that is close to that of the CLR test for Gaussian and uniform errors and substantially higher power for t_3 and DLN errors.

Third, we establish the strong IV asymptotic distributions of the rank statistics under the null and local alternatives. These results show that the RCLR and RLM tests are asymptotically equivalent under strong IV asymptotics. This is also true of the nonrank versions of these tests. The results also show that the ARE of the rank to the nonrank versions of these tests under strong IV asymptotics is the same as the standard ARE that arises in location and regression models for tests and estimators. Hence, the Chernoff–Savage result applies under strong IV asymptotics to both the NS-RCLR test and the NS-RLM test. In consequence, the NS-RCLR test (weakly) dominates the CLR test in terms of power for symmetric errors under strong IV asymptotics.

The proofs of the weak and strong IV asymptotic results make use of results and arguments given in Hájek and Sidák (1967) and Koul (1969, 1970).

Fourth, we carry out finite-sample size and power comparisons of the WS-RCLR, NS-RCLR, CLR, LM, and AR tests. For brevity, we do not report results for the RLM and RAR tests, because they are found to be inferior (both asymptotically and in finite-sample experiments) to those of the RCLR tests. We compare the tests for a variety of scenarios that differ according to the degree of endogeneity, strength of the IVs, number of IVs, and size of the sample. For each scenario we consider Gaussian, uniform, t_1, t_2, t_3, and DLN errors. The two RCLR tests perform noticeably better in terms of size than the nonrank CLR, LM, and AR tests. The finite-sample power comparisons reflect the asymptotic power comparisons discussed previously fairly closely. Specifically, the NS-RCLR test has similar power to the CLR test for Gaussian errors and higher power for non-Gaussian errors. The WS-RCLR test does not perform as well as the NS-RCLR test with uniform errors, but it performs better with thick-tailed errors.

Based on the asymptotic and finite-sample results, we recommend the NS-RCLR test over the WS-RCLR, CLR, LM, and AR tests. The WS-RCLR test also has good overall properties, but we prefer the NS-RCLR test because of its excellent power performance for both thin-tailed and thick-tailed errors.

The main drawback of the RCLR tests is that they are not robust to heteroskedasticity of the errors. That is, their size may be distorted by heteroskedasticity. This is also true of the CLR test. However, it is possible to robustify the CLR test to heteroskedasticity; see Andrews, Moreira, and Stock (2004) and Kleibergen (2005). It is not possible to robustify the RCLR tests to heteroskedasticity. Hence, there is a trade-off between power for non-Gaussian errors and robustness to heteroskedasticity for these tests. If heteroskedasticity is a possible problem, then the robustified CLR test is preferred to the NS-RCLR and WS-RCLR tests. If not, then the rank tests are preferred.

There is a vast literature on rank procedures in statistics; e.g., see Hájek and Sidák (1967), Hettmansperger (1984), Puri and Sen (1985), and Hájek, Sidák, and Sen (1999). Rank procedures have been used in both cross-section and time series econometrics. For a review, see Koenker (1996). Some more recent econometric references include Hasan and Koenker (1997), Cavanagh and Sherman (1998), Abrevaya (1999), Chen (2000, 2002), and Thompson (2004).

The remainder of this paper is organized as follows. Section 2 defines the model. Section 3 introduces the rank analogues of the CLR, LM, and AR tests.

Sections 4 and 5 provide asymptotic results for these tests under weak IV and strong IV asymptotics, respectively. These sections also give asymptotic power comparisons of rank and nonrank tests. Section 6 provides finite-sample size and power comparisons of rank and nonrank tests. An Appendix contains proofs of the results.

All limits are taken as $n \to \infty$, and $vec(\cdot)$ is the column by column vec operator.

2. MODEL

We consider the following model, which consists of a single structural equation and m reduced-form equations:

$$y_{1i} = \beta' y_{2i} + \gamma_1' X_i + u_i,$$

$$y_{2i} = \Pi' \widetilde{Z}_i + \xi_1' X_i + v_{2i}, \tag{2.1}$$

where $y_{1i} \in R$, $y_{2i} \in R^m$, $X_i \in R^p$, and $\widetilde{Z}_i \in R^k$ are observed variables; $u_i \in R$ and $v_{2i} \in R^m$ are unobserved errors; and $\beta \in R^m$, $\Pi \in R^{k\times m}$, $\gamma_1 \in R^p$, and $\xi_1 \in R^{p\times m}$ are unknown parameters.

Our interest is in testing the hypotheses

$$H_0 : \beta = \beta_0 \quad \text{and} \quad H_1 : \beta \neq \beta_0. \tag{2.2}$$

Let $\widetilde{Z}$ and X denote the $n \times k$ IV and $n \times p$ regressor matrices whose ith rows are $\widetilde{Z}_i'$ and X_i', respectively. We transform the IV matrix $\widetilde{Z}$ so that the transformed IV matrix, Z, and the regressor matrix, X, are orthogonal:

$$Z = M_X \widetilde{Z}, \qquad M_X = I_n - P_X, \qquad P_X = X(X'X)^{-1}X', \quad \text{and}$$

$$y_{2i} = \Pi' Z_i + \xi' X_i + v_{2i}, \tag{2.3}$$

where Z_i is the ith row of Z written as a column and $\xi = \xi_1 + (X'X)^{-1}X'\widetilde{Z}\Pi$. By construction, $Z'X = 0$.

Substituting the reduced-form equations for y_{2i} into the structural equation for y_{1i} yields $m+1$ reduced-form equations:

$$y_{1i} = \beta' \Pi' Z_i + \gamma' X_i + v_{1i} \quad \text{and}$$

$$y_{2i} = \Pi' Z_i + \xi' X_i + v_{2i}, \quad \text{where}$$

$$v_{1i} = u_i + \beta' v_{2i}, \tag{2.4}$$

and $\gamma = \gamma_1 + \xi\beta$. The $m+1$ reduced-form equations also can be written as

$$y_i = A\Pi' Z_i + \eta' X_i + v_i, \quad \text{where}$$

$$y_i = (y_{1i}, y_{2i}')' \in R^{m+1}, \qquad v_i = (v_{1i}, v_{2i}')' \in R^{m+1},$$

$$A = \begin{bmatrix} \beta' \\ I_m \end{bmatrix} \in R^{(m+1)\times m}, \quad \text{and} \quad \eta = [\gamma : \xi] \in R^{p\times(m+1)}. \tag{2.5}$$

Let Y and Y_2 denote the $n \times (m+1)$ and $n \times m$ matrices whose ith rows are y_i' and y_{2i}', respectively.

We make the following basic assumptions about the model. (Additional assumptions are given subsequently.)

Assumption 1.

(a) $\{(u_i, v_{2i}) : i \geq 1\}$ are independent and identically distributed (i.i.d.) random variables with mean zero.
(b) v_{2i} has nonsingular variance matrix $\Omega_{22} \in R^{m\times m}$.

Assumption 2.

(a) $\{(\tilde{Z}_i, X_i) : i \geq 1\}$ are fixed (i.e., nonrandom).
(b) The first element of X_i is 1 for all i.
(c) $n^{-1}\sum_{i=1}^{n}(\tilde{Z}_i', X_i')'(\tilde{Z}_i', X_i') \to D > 0$.
(d) $\max_{i\leq n}(\|\tilde{Z}_i\|^2 + \|X_i\|^2)/n \to 0$.

The combination of Assumptions 1 and 2(a) implies that the distribution of the errors $\{(u_i, v_{2i}) : i \geq 1\}$ does not depend on the IVs or regressors. In place of Assumption 2(a), one could treat the IVs and regressors as random. In this case, the IVs and regressors would be assumed to be independent of the errors. As is, Assumption 2(a) is consistent with random IVs and regressors provided one conditions on these variables.

Assumption 2(b) requires that the structural and reduced-form equations include an intercept. Given that $Z'X = 0$, this implies that $n^{-1}\sum_{i=1}^{n} Z_i = 0$. Assumptions 2(c) and 2(d) are standard assumptions concerning the behavior of IVs and regressors. They hold with probability one if $\{(\tilde{Z}_i, X_i) : i \geq 1\}$ is a realization of an i.i.d. sequence with positive definite (pd) variance matrix and $2+\delta$ moments finite for some $\delta > 0$; see Lemma 12 in the Appendix.

We now define the CLR test of Moreira (2003), the LM test of Kleibergen (2002) and Moreira (2001), and the AR test. The CLR test depends on an LR test statistic coupled with a "conditional" critical value defined subsequently. The LR, LM, and AR test statistics are based on the following statistics:[2]

$$S_n = (Z'Z)^{-1/2}Z'Yb_0 \cdot (b_0'\hat{\Omega}_n b_0)^{-1/2} \in R^k \quad \text{and}$$
$$T_n = (Z'Z)^{-1/2}Z'Y\hat{\Omega}_n^{-1}A_0(A_0'\hat{\Omega}_n^{-1}A_0)^{-1/2} \in R^{k\times m}, \quad \text{where}$$
$$b_0 = \begin{pmatrix} 1 \\ -\beta_0 \end{pmatrix} \in R^{m+1}, \qquad A_0 = \begin{bmatrix} \beta_0' \\ I_m \end{bmatrix} \in R^{(m+1)\times m},$$
$$\hat{\Omega}_n = (n-k-p)^{-1}Y'M_{[Z:X]}Y, \quad \text{and} \quad M_{[Z:X]} = I_n - P_Z - P_X. \tag{2.6}$$

Note that $\hat{\Omega}_n$ is an estimator of the variance matrix $\Omega = Ev_i v_i'$, which needs to be well defined and pd for S_n and T_n to be well behaved asymptotically. After proper centering, the statistics S_n and T_n have a joint multivariate normal asymptotic distribution with zero covariance under weak IV asymptotics under the null and the alternative. Hence, S_n and T_n are asymptotically independent.

The LR, LM, and AR test statistics depend on (S_n, T_n) in the following way:

$$LR_n = S_n'S_n - \lambda_{\min}([S_n : T_n]'[S_n : T_n]),$$
$$LM_n = S_n'T_n(T_n'T_n)^{-1}T_n'S_n, \quad \text{and}$$
$$AR_n = S_n'S_n/k, \tag{2.7}$$

where $\lambda_{\min}(C)$ denotes the minimum eigenvalue of the matrix C. When $m = 1$, LR_n can be written as

$$LR_n = \frac{1}{2}\left(Q_{Sn} - Q_{Tn} + \sqrt{(Q_{Sn} - Q_{Tn})^2 + 4Q_{STn}^2}\right), \quad \text{where}$$
$$Q_{Sn} = S_n'S_n, \qquad Q_{Tn} = T_n'T_n, \quad \text{and} \quad Q_{STn} = S_n'T_n; \tag{2.8}$$

see Moreira (2003) and Andrews and Stock (in press).[3]

The CLR test with asymptotic level α rejects the null hypothesis when

$$LR_n > \kappa_{LR,\alpha}(Q_{Tn}, k, m), \tag{2.9}$$

where $\kappa_{LR,\alpha}(\cdot, k, m)$ is a critical value function defined such that the CLR test has asymptotic null rejection rate α under weak IV asymptotics (under the preceding assumptions and $Eu_i^2 < \infty$). See (3.10) for the definition of $\kappa_{LR,\alpha}(\cdot, k, m)$.

The LM statistic has a chi-squared asymptotic null distribution with m degrees of freedom, denoted χ_m^2, under weak and strong IVs (under the preceding assumptions and $Eu_i^2 < \infty$). Hence, the critical value for the asymptotic level α LM test is the $1 - \alpha$ quantile of a χ_m^2 distribution.

The AR statistic times k has a chi-squared asymptotic null distribution under weak and strong IVs with k ($\geq m$) degrees of freedom (under the preceding assumptions and $Eu_i^2 < \infty$). Under the assumption of normal errors $\{v_i : i \geq 1\}$, it has an exact $F_{k,n-k-p}$ distribution. Thus, use of the $1 - \alpha$ quantile of an

$F_{k,n-k-p}$ distribution as the critical value for the level α AR test is justified asymptotically for nonnormal errors and yields an exact test for normal errors.

3. RANK CLR, LM, AND AR TESTS

In this section, we introduce rank analogues, S_n^{φ} and T_n^{φ}, of the statistics S_n and T_n, where φ is a score function defined subsequently. By design, S_n^{φ} and T_n^{φ} are asymptotically independent. Given S_n^{φ} and T_n^{φ}, we define rank statistics that are analogous to the CLR, LM, and AR statistics defined previously. We show that for normal scores, i.e., $\varphi = \varphi^{NS}$, and multivariate normal errors (u_i, v_{2i}), S_n^{φ} and T_n^{φ} are asymptotically equivalent to S_n and T_n under weak IV and strong IV asymptotics under the null and the alternative. For nonnormal errors, the rank tests have power advantages.

The statistic S_n depends on the inner product of Z and a vector of null-restricted residuals from the structural equation (2.1):

$$Z'Yb_0 = \sum_{i=1}^{n} Z_i(y_{1i} - \beta_0' y_{2i}) = \sum_{i=1}^{n} Z_i(y_{1i} - \beta_0' y_{2i} - \hat{\gamma}_{1n}' X_i), \tag{3.1}$$

where $\hat{\gamma}_{1n}$ is some estimator of γ_1 and the second equality holds because $Z'X = 0$. The rank analogue of S_n that we consider depends on the inner product of Z with the vector of ranks of $\{y_{1i} - \beta_0' y_{2i} - \hat{\gamma}_{1n}' X_i : i \leq n\}$.

Let $\hat{\gamma}_n(\beta_0)$ be some "null-restricted" estimator of γ_1. For example, one could use the least squares (LS) null-restricted estimator:

$$\hat{\gamma}_n^{LS}(\beta_0) = (X'X)^{-1} X'Y(1, -\beta_0')'. \tag{3.2}$$

Estimators other than the LS estimator could be considered, but the LS estimator is convenient because it is easy to compute. (The LS estimator satisfies the assumptions given subsequently provided the errors have finite variances.)

Let $\hat{R}_i(\beta_0)$ be the rank of $y_{1i} - \beta_0' y_{2i} - \hat{\gamma}_n(\beta_0)' X_i$ in $\{y_{1j} - \beta_0' y_{2j} - \hat{\gamma}_n(\beta_0)' X_j : j = 1, \ldots, n\}$. The ranks $\{\hat{R}_i(\beta_0) : i \leq n\}$ are referred to as *aligned* ranks.[4]

Let $\varphi : [0,1) \to R$ be a nonstochastic score function. Different score functions φ lead to different rank statistics. Of primary interest are (a) the normal (or van der Waerden) score function and (b) the Wilcoxon score function:

$$\text{(a)} \ \ \varphi^{NS}(x) = \Phi^{-1}(x) \quad \text{and} \quad \text{(b)} \ \ \varphi^{WS}(x) = x, \tag{3.3}$$

where $\Phi^{-1}(\cdot)$ is the inverse standard normal distribution function (df). Define

$$c_{\varphi} = \int_0^1 [\varphi(x) - \bar{\varphi}]^2 \, dx > 0, \quad \text{where } \bar{\varphi} = \int_0^1 \varphi(x) \, dx. \tag{3.4}$$

For normal scores, $c_{\varphi} = 1$. For Wilcoxon scores, $c_{\varphi} = \frac{1}{12}$.

Let R_φ denote the n-vector whose ith element is $\varphi(\hat{R}_i(\beta_0)/(n+1))$. The rank analogue of S_n is

$$S_n^\varphi = (Z'Z)^{-1/2} Z' R_\varphi c_\varphi^{-1/2} \in R^k. \tag{3.5}$$

The rank statistic S_n^φ replaces $Yb_0 \cdot (b_0' \hat{\Omega}_n b_0)^{-1/2}$ in S_n by $R_\varphi c_\varphi^{-1/2}$. We want the rank analogue of T_n to do the same and also to be asymptotically independent of S_n^φ. In consequence, to construct a rank analogue of T_n, it is helpful to rewrite T_n as follows:

$$T_n = (Z'Z)^{-1/2} Z' [Yb_0 \hat{\sigma}_n^{-1} : Y_2] \hat{\Omega}_{*n}^{-1} H (H' \hat{\Omega}_{*n}^{-1} H)^{-1/2}, \quad \text{where } \hat{\sigma}_n^2 = b_0' \hat{\Omega}_n b_0,$$

$$H = \begin{bmatrix} 0_m' \\ I_m \end{bmatrix} \in R^{(m+1)\times m}, \quad \hat{\Omega}_{*n} = [b_0 \hat{\sigma}_n^{-1} : H]' \hat{\Omega}_n [b_0 \hat{\sigma}_n^{-1} : H] = \begin{bmatrix} 1 & \hat{\nu}_n' \\ \hat{\nu}_n & \hat{\Omega}_{22n} \end{bmatrix},$$

$$\hat{\Omega}_{22n} = H' \hat{\Omega}_n H = (n-k-p)^{-1} Y_2' M_{[Z:X]} Y_2 \in R^{m\times m}, \quad \text{and}$$

$$\hat{\nu}_n = H' \hat{\Omega}_n b_0 \hat{\sigma}_n^{-1} \in R^m. \tag{3.6}$$

(See (A.81) in the Appendix for a proof of (3.6).) As defined, $\hat{\Omega}_{*n}$ is an estimator of the asymptotic variance matrix, Ω_*, of $n^{-1/2} \sum_{i=1}^n [b_0 \hat{\sigma}_{gn}^{-1} : H]' y_i = n^{-1/2} \sum_{i=1}^n (b_0' y_i \hat{\sigma}_{gn}^{-1}, y_{2i}')'$. The definition of $\hat{\Omega}_{*n}$ is chosen to yield asymptotic independence of S_n and T_n.

The rank analogue of T_n is[5]

$$T_n^\varphi = (Z'Z)^{-1/2} Z' [R_\varphi c_\varphi^{-1/2} : Y_2] \hat{\Omega}_{\varphi n}^{-1} H (H' \hat{\Omega}_{\varphi n}^{-1} H)^{-1/2} \in R^{k\times m}, \quad \text{where}$$

$$\hat{\Omega}_{\varphi n} = \begin{bmatrix} 1 & \hat{\nu}_{\varphi n}' \\ \hat{\nu}_{\varphi n} & \hat{\Omega}_{22n} \end{bmatrix} \quad \text{and} \quad \hat{\nu}_{\varphi n} = n^{-1} Y_2' M_{[Z:X]} R_\varphi c_\varphi^{-1/2} \in R^m. \tag{3.7}$$

Note that $\hat{\Omega}_{\varphi n}$ is an estimator of the asymptotic variance matrix of $n^{-1/2} \sum_{i=1}^n (\varphi(\hat{R}_i(\beta_0)/(n+1)) c_\varphi^{-1/2}, y_{2i}')'$. The definition of $\hat{\Omega}_{\varphi n}$ ensures that S_n^φ and T_n^φ are asymptotically independent.

We define the rank LR, LM, and AR statistics to be

$$RLR_n^\varphi = S_n^{\varphi\prime} S_n^\varphi - \lambda_{\min}([S_n^\varphi : T_n^\varphi]' [S_n^\varphi : T_n^\varphi]),$$

$$RLM_n^\varphi = S_n^{\varphi\prime} T_n^\varphi (T_n^{\varphi\prime} T_n^\varphi)^{-1} T_n^{\varphi\prime} S_n^\varphi, \quad \text{and}$$

$$RAR_n^\varphi = S_n^{\varphi\prime} S_n^\varphi / k. \tag{3.8}$$

For $m = 1$, the RLR_n^φ statistic simplifies as in (2.8) with $(S_n^\varphi, T_n^\varphi)$ in place of (S_n, T_n).

Notice that when $k = m$ (i.e., the structural equation is just identified), $k \cdot RAR_n^\varphi = S_n^{\varphi\prime} S_n^\varphi = RLM_n^\varphi = RLR_n^\varphi$.[6] That is, the rank CLR, LM, and AR tests are equivalent when $k = m$.

The rank CLR, LM, and AR tests use the same critical values as the nonrank versions of these tests. Hence, the rank LM and AR tests with asymptotic significance level α have critical values given by the $1-\alpha$ quantiles of the χ^2_m and $F_{k,n-k-p}$ distributions, respectively.

The rank CLR test rejects the null hypothesis if

$$RLR_n^{\varphi} > \kappa_{LR,\alpha}(T_n^{\varphi\prime} T_n^{\varphi}, k, m), \tag{3.9}$$

where $\kappa_{LR,\alpha}(\cdot, k, m)$ is defined as follows. For $t \in R^{k\times m}$, define $\kappa_{LR,\alpha}(t't, k, m)$ via

$$P(LR_\infty(S_0, t) > \kappa_{LR,\alpha}(t't, k, m)) = \alpha, \quad \text{where}$$

$$S_0 \sim N(0, I_k) \quad \text{and} \quad LR_\infty(s, t) = s's - \lambda_{\min}([s:t]'[s:t]) \tag{3.10}$$

for $s \in R^k$. Note that $\kappa_{LR,\alpha}(\cdot, k, m)$ depends on k (the dimension of Z_i) and m (the dimension of y_{2i}). Andrews et al. (2006b) provides detailed tables of $\kappa_{LR,\alpha}(\tau, k, m)$ for $m = 1$ and a variety of values of τ and k. Andrews, Moreira, and Stock (in press) provide a GAUSS program for computing p-values of the CLR test for $m = 1$ and arbitrary k. This program also can be used for the rank CLR test by replacing the Andrews et al. (2006a) $\widehat{LR}_n$ and $\hat{Q}_{T,n}$ statistics by RLR_n^{φ} and $T_n^{\varphi\prime} T_n^{\varphi}$, respectively.

For $m > 1$, the critical value function $\kappa_{LR,\alpha}(\cdot, k, m)$ can be simulated quite easily by simulating $S_0(r) \sim iid\ N(0, I_k)$ for $r = 1, \ldots, Reps$ and taking $\kappa_{LR,\alpha}(t't, k, m)$ to be the $1-\alpha$ sample quantile of $\{LR_\infty(S_0(r), t) : r = 1, \ldots, Reps\}$, where $Reps$ is a large integer, such as 25,000.

4. WEAK IV ASYMPTOTIC RESULTS

4.1. Weak IV Asymptotic Distributions of Rank Statistics

In this section, we establish the weak IV asymptotic distributions of the RLR_n^{φ}, RLM_n^{φ}, and RAR_n^{φ} test statistics under the null and fixed β alternatives.

We assume that the score function φ satisfies the following condition.

Assumption 3.

(a) $\varphi : [0,1) \to R$ is absolutely continuous and bounded with two derivatives that exist almost everywhere and are bounded.
(b) $0 < c_\varphi < \infty$ for c_φ defined in (3.4).

Assumption 3 holds for Wilcoxon scores. Assumption 3(b) holds for normal scores, but Assumption 3(a) does not. However, normal scores that are smoothly truncated above and below near 0 and 1 satisfy Assumptions 3(a) and 3(b). Simulation results given subsequently for untruncated normal scores indicate that truncation is not necessary in practice.

Under weak IVs, the asymptotic variance matrix, $\Omega_{\varphi g}$, of $n^{-1/2}\sum_{i=1}^{n}(\varphi(\hat{R}_i(\beta_0)/(n+1)), y_{2i}')'$ is defined by

$$\Omega_{\varphi g} = \mathrm{Var}\begin{pmatrix}\varphi(U_{gi})c_{\varphi}^{-1/2} \\ y_{2i}\end{pmatrix} = \begin{bmatrix} 1 & \nu_{\varphi g}' \\ \nu_{\varphi g} & \Omega_{22}\end{bmatrix} \in R^{(m+1)\times(m+1)}, \quad \text{where}$$

$$U_{gi} = G(u_i + (\beta-\beta_0)'v_{2i}) \in R,$$

$$\nu_{\varphi g} = \mathrm{Cov}(y_{2i}, \varphi(U_{gi})c_{\varphi}^{-1/2}) \in R^m, \tag{4.1}$$

G is the df of $u_i + (\beta-\beta_0)'v_{2i}$, and g is the density corresponding to G.[7]

Let $I(f)$ denote Fisher's information of an absolutely continuous density f. That is, $I(f) = \int[f'(x)/f(x)]^2 f(x)dx$.

The weak IV assumption is the first part of the following assumption.

Assumption 4W.

(a) $\Pi = Cn^{-1/2}$ for some matrix $C \in R^{k\times m}$.
(b) β does not depend on n.
(c) $u_i + (\beta-\beta_0)'v_{2i}$ has an absolutely continuous strictly increasing df G and an absolutely continuous and bounded density g that satisfies $I(g) < \infty$.
(d) $(u_i + (\beta-\beta_0)'v_{2i}, v_{2i})$ has an absolutely continuous bounded joint density with partial derivative with respect to its first argument that is bounded over both arguments.
(e) $\Omega_{\varphi g}$ is pd.
(f) $n^{1/2}(\hat{\gamma}_n(\beta_0) - \gamma_1 - \xi_1(\beta-\beta_0)) = O_p(1)$.

Assumption 4W(b) implies that the data-generating process satisfies the null hypothesis or a fixed β alternative. Assumptions 4W(c) and (d) require that $(u_i + (\beta-\beta_0)'v_{2i}, v_{2i})$ is absolutely continuous but otherwise are not very restrictive. Note that Assumptions 1–3 and 4W place no moment restrictions on u_i.

Assumption 4W(f) requires the null-restricted estimator $\hat{\gamma}_n(\beta_0)$ to be well behaved. It is satisfied by the LS estimator under the preceding assumptions if $Eu_i^2 < \infty$.

LEMMA 1. *Under Assumptions 1, 2, 4W(a), and 4W(b) and* $Eu_i^2 < \infty$, $\hat{\gamma}_n^{LS}(\beta_0)$ *satisfies Assumption 4W(f).*

We show that S_n^{φ} and T_n^{φ} converge in distribution to independent random quantities $S_\infty^{\varphi} \in R^k$ and $T_\infty^{\varphi} \in R^{k\times m}$, respectively, that are defined as follows. Let $D_Z \in R^{k\times k}$ be the probability limit of $n^{-1}Z'Z$:

$$D_Z = D_{11} - D_{12} D_{22}^{-1} D_{21}, \qquad D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \tag{4.2}$$

where $D_{11} \in R^{k \times k}$, $D_{12} \in R^{k \times p}$, and $D_{22} \in R^{p \times p}$.

For a score function φ and a density f, define

$$\xi(\varphi, f) = \frac{\left(\int_0^1 \varphi(x, f) \varphi(x)\, dx \right)^2}{\int_0^1 [\varphi(x) - \bar{\varphi}]^2\, dx}, \quad \text{where}$$

$$\varphi(x, f) = -\frac{f'(F^{-1}(x))}{f(F^{-1}(x))} \quad \text{for } x \in [0,1] \tag{4.3}$$

and f' denotes the derivative of f. For normal and Wilcoxon scores,

$$\xi(\varphi^{NS}, f) = \left(\int \frac{f^2(x)}{\phi(\Phi^{-1}(F(x)))}\, dx \right)^2 \quad \text{and}$$

$$\xi(\varphi^{WS}, f) = 12 \left(\int f^2(x)\, dx \right)^2, \tag{4.4}$$

respectively, where ϕ and Φ denote the standard normal density and df and $F' = f$.[8]

Let $[N_\varphi : N_2]$ be a $k \times (m+1)$ multivariate normal matrix with

$$\begin{aligned} EN_\varphi &= D_Z C \ell^{\varphi}_{g, \beta - \beta_0} \in R^k, \quad \text{where} \\ \ell^{\varphi}_{g, \beta-\beta_0} &= (\beta - \beta_0) \xi^{1/2}(\varphi, g) \in R^m, \\ EN_2 &= D_Z C \in R^{k \times m}, \quad \text{and} \end{aligned}$$

$$\operatorname{Var}(\operatorname{vec}([N_\varphi : N_2])) = \Omega_{\varphi g} \otimes D_Z, \tag{4.5}$$

where g is the density of $u_i + (\beta - \beta_0)' v_{2i}$; see Assumption 4W(c). Now, define

$$\begin{aligned} S^{\varphi}_{\infty} &= D_Z^{-1/2} N_\varphi \sim N(D_Z^{1/2} C \ell^{\varphi}_{g, \beta-\beta_0}, I_k) \in R^k, \\ T^{\varphi}_{\infty} &= D_Z^{-1/2} [N_\varphi : N_2] \Omega_{\varphi g}^{-1} H (H' \Omega_{\varphi g}^{-1} H)^{-1/2} \in R^{k \times m}, \quad \text{and} \end{aligned}$$

$$\operatorname{vec}(T^{\varphi}_{\infty}) \sim N(\operatorname{vec}(D_Z^{1/2} C [\ell^{\varphi}_{g, \beta-\beta_0} : I_m] \Omega_{\varphi g}^{-1} H (H' \Omega_{\varphi g}^{-1} H)^{-1/2}), I_{km}). \tag{4.6}$$

Under H_0, S^{φ}_{∞} has mean zero, but T^{φ}_{∞} does not. It is shown subsequently that the covariance of S^{φ}_{∞} and T^{φ}_{∞} is zero and hence these normal random variates are independent (under H_0 and H_1).

The following result holds under the null hypothesis and fixed β (i.e., nonlocal) alternative hypotheses.

THEOREM 1. *Under Assumptions 1–3 and 4W,*

(i) $(S_n^\varphi, T_n^\varphi) \to_d (S_\infty^\varphi, T_\infty^\varphi)$, *where* S_∞^φ *and* T_∞^φ *are independent,*
(ii) $RLR_n^\varphi \to_d LR_\infty^\varphi := S_\infty^{\varphi\prime} S_\infty^\varphi - \lambda_{\min}([S_\infty^\varphi : T_\infty^\varphi]'[S_\infty^\varphi : T_\infty^\varphi])$,
(iii) $RLM_n^\varphi \to_d S_\infty^{\varphi\prime} T_\infty^\varphi (T_\infty^{\varphi\prime} T_\infty^\varphi)^{-1} T_\infty^{\varphi\prime} S_\infty^\varphi$, *and*
(iv) $RAR_n^\varphi \to_d S_\infty^{\varphi\prime} S_\infty^\varphi / k$.

Remarks.

(a) Theorem 1(iv) shows that $k \cdot RAR_n^\varphi$ has an asymptotic χ_k^2 distribution under the null and a $\chi_k^2(\delta_{AR,W}^\varphi)$ distribution under fixed β alternatives, where

$$\delta_{AR,W}^\varphi = (\beta - \beta_0) C' D_Z C (\beta - \beta_0) \cdot \xi(\varphi, g). \tag{4.7}$$

This justifies using the $1 - \alpha$ quantile of the $F_{k,n-k-p}$ distribution as the critical value for the test based on RAR_n^φ because $F_{k,n-k-p} \to_d \chi_k^2/k$ as $n \to \infty$.

(b) Theorems 1(i) and (iii) imply that RLM_n^φ has an asymptotic χ_m^2 distribution under the null hypothesis (because $S_\infty^\varphi \sim N(0_k, I_k)$ under the null implies that $S_\infty^{\varphi\prime} T_\infty^\varphi (T_\infty^{\varphi\prime} T_\infty^\varphi)^{-1} T_\infty^{\varphi\prime} S_\infty^\varphi$ has a χ_m^2 distribution conditional on T_∞^φ and hence an unconditional χ_m^2 distribution also). Under the alternative, *conditional on* $P_{T_\infty^\varphi}$ $(= T_\infty^\varphi (T_\infty^{\varphi\prime} T_\infty^\varphi)^{-1} T_\infty^{\varphi\prime})$, RLM_n^φ has a noncentral chi-squared distribution, $\chi_m^2(\delta_{LM,W}^\varphi)$, with m degrees of freedom and noncentrality parameter

$$\begin{aligned}\delta_{LM,W}^\varphi = {} & (\beta - \beta_0) C' D_Z^{1/2} T_\infty^\varphi (T_\infty^{\varphi\prime} T_\infty^\varphi)^{-1} T_\infty^{\varphi\prime} D_Z^{1/2} C (\beta - \beta_0) \\ & \times \xi(\varphi, g).\end{aligned} \tag{4.8}$$

The random projection matrix $P_{T_\infty^\varphi}$ equals $P_{T_\infty^\varphi M}$, where M is any random or nonrandom nonsingular $m \times m$ matrix. In consequence, $P_{T_\infty^\varphi}$ has the same distribution as $P_{T_\infty^*}$, where $\mathrm{vec}(T_\infty^*) \sim N(\mathrm{vec}(D_Z^{1/2} C), I_{km})$. Note that the distribution of T_∞^* does not depend on φ or g. Hence, the asymptotic distribution of RLM_n^φ only depends on (φ, g) through the distribution of S_∞^φ.

(c) The statistics RLR_n^φ and RLM_n^φ and their asymptotic distributions depend on $(S_n^\varphi, T_n^\varphi)$ and $(S_\infty^\varphi, T_\infty^\varphi)$ only through $Q_n^\varphi = [S_n^\varphi : T_n^\varphi]'[S_n^\varphi : T_n^\varphi]$ and $Q_\infty^\varphi = [S_\infty^\varphi : T_\infty^\varphi]'[S_\infty^\varphi : T_\infty^\varphi]$, respectively. Given the multivariate normal distribution of $[S_\infty^\varphi : T_\infty^\varphi]$, Q_∞^φ has a noncentral Wishart distribution. It depends on unknown parameters only through

$$[ES_\infty^\varphi : ET_\infty^\varphi]'[ES_\infty^\varphi : ET_\infty^\varphi], \quad \text{where}$$

$$[ES_\infty^\varphi : ET_\infty^\varphi] = D_Z^{1/2} C [\ell_{g,\beta-\beta_0}^\varphi : [\ell_{g,\beta-\beta_0}^\varphi : I_m] \Omega_{\varphi g}^{-1} H (H' \Omega_{\varphi g}^{-1} H)^{-1/2}]. \tag{4.9}$$

The following corollary uses Theorems 1(i) and (ii) to show that the use of $\kappa_{LR,\alpha}(\tau, k, m)$ (defined in (3.10)) as the critical value function for the RLR_n^{φ} statistic yields a test with asymptotic null rejection rate α under weak IV asymptotics.

COROLLARY 1. *Under the null hypothesis, $H_0: \beta = \beta_0$, and Assumptions 1–3 and 4W,* $\lim_{n\to\infty} P(RLR_n^{\varphi} > \kappa_{LR,\alpha}(T_n^{\varphi\prime} T_n^{\varphi}, k, m)) = \alpha.$

4.2. Weak IV Asymptotic Distributions of Nonrank Statistics

To enable comparisons of the power of rank and nonrank tests, we now provide the null and nonnull weak IV asymptotic distributions of the nonrank statistics S_n and T_n under the assumption that $\Omega = Ev_i v_i'$ is well defined and pd. The results given here extend results in Andrews et al. (2006a) from $m = 1$ to $m \geq 1$. They are not covered by Moreira (2003) because Moreira (2003) only provides asymptotic results under the null hypothesis.

To make comparisons of rank and nonrank tests more transparent, we write the asymptotic distributions of the nonrank tests in a form that is analogous to that of S_∞^{φ} and T_∞^{φ}, which differs from the form given in Andrews et al. (2006a). Define

$$\begin{aligned}
\Omega_g &= \mathrm{Var}\begin{pmatrix} y_i' b_0 \sigma_g^{-1} \\ y_{2i} \end{pmatrix} = \mathrm{Var}\begin{pmatrix} (u_i + (\beta - \beta_0)' v_{2i}) \sigma_g^{-1} \\ v_{2i} \end{pmatrix} \\
&= [b_0 \sigma_g^{-1} : H]' \Omega [b_0 \sigma_g^{-1} : H] = \begin{bmatrix} 1 & \nu_g' \\ \nu_g & \Omega_{22} \end{bmatrix}, \\
\sigma_g^2 &= \mathrm{Var}(y_i' b_0) = \mathrm{Var}(u_i + (\beta - \beta_0)' v_{2i}) = b_0' \Omega b_0, \quad \text{and} \\
\nu_g &= \mathrm{Cov}(y_{2i}, (u_i + (\beta - \beta_0)' v_{2i}) \sigma_g^{-1}) = H' \Omega b_0 \sigma_g^{-1}. \qquad \textbf{(4.10)}
\end{aligned}$$

Let $[N_1 : N_2]$ be a $k \times (m+1)$ multivariate normal matrix with N_2 as before,

$$\begin{aligned}
EN_1 &= D_Z C (\beta - \beta_0) \sigma_g^{-1} \in R^k, \quad \text{and} \\
\mathrm{Var}(\mathrm{vec}([N_1 : N_2])) &= \Omega_g \otimes D_Z. \qquad \textbf{(4.11)}
\end{aligned}$$

Next, define

$$\begin{aligned}
S_\infty &= D_Z^{-1/2} N_1 \sim N(D_Z^{1/2} C \ell_{g,\beta-\beta_0}, I_k), \\
T_\infty &= D_Z^{-1/2} [N_1 : N_2] \Omega_g^{-1} H (H' \Omega_g^{-1} H)^{-1/2} \in R^{k \times m}, \\
\mathrm{vec}(T_\infty) &\sim N(\mathrm{vec}(D_Z^{1/2} C [\ell_{g,\beta-\beta_0} : I_m] \Omega_g^{-1} H (H' \Omega_g^{-1} H)^{-1/2}), I_{km}), \quad \text{and} \\
\ell_{g,\beta-\beta_0} &= (\beta - \beta_0) \sigma_g^{-1} \in R^m. \qquad \textbf{(4.12)}
\end{aligned}$$

LEMMA 2. *Under Assumptions 1–3 and 4W and* $\Omega > 0$,

(i) $(S_n, T_n) \to_d (S_\infty, T_\infty)$, *where* S_∞ *and* T_∞ *are independent,*
(ii) $LR_n \to_d S'_\infty S_\infty - \lambda_{\min}([S_\infty : T_\infty]'[S_\infty : T_\infty])$,
(iii) $LM_n \to_d S'_\infty T_\infty (T'_\infty T_\infty)^{-1} T'_\infty S_\infty$, *and*
(iv) $AR_n \to_d S'_\infty S_\infty / k$.

Remarks.

(a) Lemma 2(iv) shows that $k \cdot AR_n$ has an asymptotic χ^2_k distribution under the null and a $\chi^2_k(\delta_{AR,W})$ distribution under fixed β alternatives, where

$$\delta_{AR,W} = (\beta - \beta_0) C' D_Z C (\beta - \beta_0) \cdot \sigma_g^{-2}. \qquad \textbf{(4.13)}$$

(b) Lemma 2(i) and (iii) imply that LM_n has an asymptotic χ^2_m distribution under the null hypothesis. Under the alternative, *conditional on* P_{T_∞}, LM_n has an asymptotic noncentral chi-squared distribution, $\chi^2_m(\delta_{LM,W})$, with m degrees of freedom and noncentrality parameter

$$\delta_{LM,W} = (\beta - \beta_0) C' D_Z^{1/2} T_\infty (T'_\infty T_\infty)^{-1} T'_\infty D_Z^{1/2} C (\beta - \beta_0) \cdot \sigma_g^{-2}. \qquad \textbf{(4.14)}$$

4.3. Weak IV Power Comparisons: Rank versus Nonrank Tests

In this section, we compare the weak IV asymptotic power of the rank AR, LM, and CLR tests to that of the nonrank versions of these tests. We consider the AR and LM tests first because the comparison is simpler for these tests.

4.3.1. Anderson–Rubin and Lagrange Multiplier Tests. The RAR_n^φ and AR_n statistics have noncentral chi-squared distributions under weak IV asymptotics by Remark (a) to Theorem 1 and Remark (a) to Lemma 2(iv). Their noncentrality parameters, given in (4.7) and (4.13), respectively, differ only by the multiplicative constants $\xi(\varphi, g)$ and σ_g^{-2}. In consequence, for weak IVs, the ARE[9] of the rank AR test to the (nonrank) AR test is

$$ARE_g(RAR_n^\varphi, AR_n) = \xi(\varphi, g)\sigma_g^2 \qquad \textbf{(4.15)}$$

($ARE(T_1, T_2) > 1$ means that the T_1 test has higher power than the T_2 test). Note that the ARE in (4.15) is independent of the location and scale of g. When $k = m$, the ARE in (4.15) also applies to the rank versus nonrank CLR and LM tests because they are the same as the AR tests.

The RLM_n^φ and LM_n statistics have noncentral chi-squared distributions under weak IV asymptotics conditional on $P_{T_\infty^\varphi}$ and P_{T_∞}, respectively, by Remark (b) to Theorem 1 and Remark (b) to Lemma 2(iv). Note that the distributions of P_{T_∞} and $P_{T_\infty^\varphi}$ are equal by the argument given in Remark (b) to Theorem 1. In consequence, the ARE of the RLM_n^φ test to the LM_n test is the same as that of the rank to nonrank AR test given in (4.15).

The literature on rank tests contains extensive calculations of the ARE in (4.15) because exactly the same ARE arises when comparing a rank test with the usual t-test and F-test in a simple location model with error density g. In addition, it is the same as the ARE of a rank estimator with the sample mean in the location model. Note that the normal and Wilcoxon scores rank estimators are asymptotically efficient in the location model with normal and logistic errors, respectively.

For a density g and normal scores, $\varphi^{NS}(x) = \Phi^{-1}(x)$, the ARE is

$$ARE_g^{NS} = \xi(\varphi^{NS}, g)\sigma_g^2 = \sigma^2(g)\left(\int \frac{g^2(x)}{\phi(\Phi^{-1}(G(x)))}\,dx\right)^2, \quad \text{where}$$

$$ARE_g^{NS} = ARE_g(RAR_n^{NS}, AR_n) = ARE_g(RLM_n^{NS}, LM_n) \tag{4.16}$$

and $G(\cdot)$ of the df of g. A result due to Chernoff and Savage (1958) implies that $ARE_g^{NS} \geq 1$ for all symmetric distributions g (about some point not necessarily zero). Hence, the asymptotic power under weak IVs of the normal scores rank AR (LM) test is greater than or equal to that of the nonrank AR (LM) test for any symmetric distribution.

For a density g and Wilcoxon scores, $\varphi^{WS}(x) = x$, the ARE of the rank AR test to the nonrank AR test is

$$ARE_g^{WS} = \xi(\varphi^{WS}, g)\sigma_g^2 = 12\sigma_g^2\left(\int g^2(x)\,dx\right)^2, \quad \text{where}$$

$$ARE_g^{WS} = ARE_g(RAR_n^{WS}, AR_n) = ARE_g(RLM_n^{WS}, LM_n). \tag{4.17}$$

For the normal distribution, i.e., $g = \phi$, $ARE_\phi^{WS} = 0.955$. For the double exponential distribution g_{de}, $ARE_{g_{de}}^{WS} = 1.50$. For a contaminated normal distribution $\phi_\varepsilon(x) = (1-\varepsilon)\phi(x) + \varepsilon\phi(x/3)/3$, $ARE_{\phi_\varepsilon}^{WS} = 1.196$, 1.373, and 1.497 for $\varepsilon =$ 0.05, 0.10, and 0.15, respectively; see Hettmansperger (1984, pp. 71–72). A result due to Hodges and Lehmann states that $ARE_g^{WS} \geq 0.864$ for all symmetric distributions g (about some point not necessarily zero); see Hettmansperger (1984, Thm. 2.6.3, p. 72). Hence, the noncentrality parameter of the Wilcoxon scores rank IV test is almost as large as that of the AR test for the normal distribution, is significantly larger than that of the AR test for heavier tailed distributions, and is not much smaller for any symmetric distribution.

For any densities g_1 and g_2 symmetric about zero (with dfs G_1 and G_2), $ARE_{g_1}(RAR_n^{WS}, RAR_n^{NS}) \leq ARE_{g_2}(RAR_n^{WS}, RAR_n^{NS})$ whenever the tails of g_1 are lighter than the tails of g_2 in the sense that $G_2^{-1}(G_1(x))$ is convex for $x \geq 0$; see Theorem 2.9.5 of Hettmansperger (1984, p. 116). (The same is true with AR replaced by LM.) Thus, the comparative power of Wilcoxon scores to normal scores tests increases as the tail thickness of the distribution increases. For any symmetric density g, $ARE_g(RAR_n^{WS}, RAR_n^{NS}) \in (0, 1.91)$; see Hettmansperger (1984, Thm. 2.9.3, p. 115).

4.3.2. Conditional Likelihood Ratio Test. Next, we compare the weak IV asymptotic power of the rank CLR and (nonrank) CLR tests. Analytical comparisons are difficult because of the complicated form of the asymptotic distributions. However, the power of these tests comes primarily from the magnitude of the means of S_∞^φ and S_∞, respectively; see (4.6) and (4.12). Hence, when $u_i + (\beta - \beta_0)'v_{2i}$ has relatively heavy tails, the rank CLR test should have higher power. Furthermore, as discussed in Andrews and Stock (in press), the CLR test is a data-dependent combination of the AR and LM tests, and hence the advantage of the rank versions of the latter tests when $u_i + (\beta - \beta_0)'v_{2i}$ has relatively heavy or thin tails should carry over to that of the CLR rank test.

These conjectures are shown to hold (in the scenarios considered) by numerical comparisons of the asymptotic power of the $RCLR_n$ and CLR_n tests using the asymptotic results of Theorem 1(ii), Corollary 1, and Lemma 2(ii). Table 1 reports the weak IV asymptotic powers of the WS-RCLR, NS-RCLR, and CLR tests. For comparative purposes, asymptotic powers of the LM and AR tests also are given in Table 1.

The cases considered in Table 1 include a base case and several variations of it. The base case has $m = 1$ (i.e., β is a scalar), $\lambda = C'D_ZC = 10$ (which corresponds to moderately weak IVs), $k = 5$ (i.e., five IVs), $\rho_{uv_2} = Corr(u_i, v_{2i}) = 0.75$ (which corresponds to moderately strong endogeneity), and $\beta_0 = 0$ (without loss of generality). Two values of β are considered, namely, $\beta = 1$ and $\beta = -0.43$. These values are selected so that the CLR test has asymptotic power 0.40 with normal errors (u_i, v_{2i}). A "high endogeneity" case is the same as the base case except that $\rho_{uv_2} = Corr(u_i, v_{2i}) = 0.95$ and $\beta = 1.1$ or $\beta = -0.37$. A "weaker IV" case is the same as the base case except that $\lambda = 4.0$ and $\beta = 5.0$ or $\beta = -0.7$. A "ten IV" case is the same as the base case except that $k = 10$. In each variation of the base case, the values of β considered are chosen so that the CLR test has asymptotic power approximately equal to 0.40 with normal errors.

In all cases considered, the structural error u_i and a latent variable ε_i are taken to be independent with distribution F. We consider four distributions F, namely, standard normal, uniform $[-2\sqrt{3}, 2\sqrt{3}]$, t_3, and difference of independent log normals (DLN). The uniform distribution exhibits thin tails, whereas the t_3 and DLN distributions exhibit thick tails. The reduced-form error v_{2i} is defined to be the following function of u_i and ε_i:

$$v_{2i} = (1 - \rho_{uv_2}^2)^{1/2}\varepsilon_i + \rho_{uv_2}u_i. \tag{4.18}$$

By construction, $Corr(u_i, v_{2i}) = \rho_{uv_2}$. The distribution G, upon which the asymptotic properties of the tests depend, is the distribution of $u_i + (\beta - \beta_0)'v_{2i}$ when u_i and ε_i are independent with distribution F. When F has thin or thick tails, so does G. Details concerning the computation of the asymptotic power reported in Table 1 are given in the Appendix.

TABLE 1. Asymptotic power

Case	Distribution	WS-RCLR	NS-RCLR	CLR	LM	AR
Base case	Normal	0.37	0.38	0.39	0.39	0.21
($\beta = 1.0$)	Uniform	0.38	0.52	0.40	0.41	0.25
	t_3	0.58	0.54	0.40	0.40	0.21
	DLN	0.69	0.66	0.39	0.39	0.20
Base case	Normal	0.39	0.41	0.41	0.41	0.27
($\beta = -0.43$)	Uniform	0.37	0.50	0.41	0.41	0.26
	t_3	0.60	0.55	0.40	0.40	0.25
	DLN	0.78	0.68	0.41	0.40	0.25
High endogeneity	Normal	0.37	0.38	0.39	0.39	0.21
($\rho_{uv_2} = 0.95$, $\beta = 1.1$)	Uniform	0.39	0.59	0.39	0.40	0.22
	t_3	0.59	0.55	0.38	0.38	0.20
	DLN	0.75	0.73	0.38	0.38	0.21
High endogeneity	Normal	0.41	0.43	0.42	0.43	0.23
($\rho_{uv_2} = 0.95$, $\beta = -0.37$)	Uniform	0.40	0.61	0.42	0.42	0.24
	t_3	0.67	0.60	0.42	0.42	0.24
	DLN	0.87	0.78	0.42	0.42	0.23
Weaker IVs	Normal	0.39	0.40	0.41	0.41	0.22
($\lambda = 4$, $\beta = 5.0$)	Uniform	0.37	0.47	0.41	0.41	0.22
	t_3	0.35	0.41	0.40	0.40	0.22
	DLN	0.48	0.50	0.40	0.40	0.22
Weaker IVs	Normal	0.38	0.39	0.39	0.35	0.32
($\lambda = 4$, $\beta = -0.7$)	Uniform	0.34	0.42	0.38	0.34	0.32
	t_3	0.57	0.52	0.39	0.35	0.32
	DLN	0.73	0.63	0.40	0.35	0.33
Ten IVs	Normal	0.38	0.40	0.40	0.40	0.15
($k = 10$, $\beta = 1.0$)	Uniform	0.33	0.47	0.38	0.38	0.16
	t_3	0.59	0.56	0.41	0.41	0.16
	DLN	0.65	0.65	0.39	0.39	0.16
Ten IVs	Normal	0.35	0.37	0.37	0.37	0.19
($k = 10$, $\beta = -0.43$)	Uniform	0.33	0.44	0.36	0.36	0.19
	t_3	0.55	0.51	0.37	0.37	0.18
	DLN	0.70	0.61	0.36	0.36	0.18

Note: All cases have $\lambda = 10$, $\rho_{uv_2} = 0.75$, and $k = 5$, unless otherwise stated.

Table 1 indicates that for the normal distribution F the WS-RCLR, NS-RCLR, and CLR tests have roughly equal asymptotic power in all cases. (This is analogous to the result in Section 4.3.1 that $ARE_{\phi}^{NS} = 1$ and $ARE_{\phi}^{WS} = 0.955$.) For the (thin-tailed) uniform distribution, the NS-RCLR test has higher power than the CLR test, whereas the WS-RCLR test has lower or equal power in all cases.

(The former is analogous to the result in Section 4.3.1 that $ARE_g^{NS} \geq 1$ for all symmetric distributions g. The latter is analogous to the result in Section 4.3.1 that $ARE_g\ (RAR_n^{WS}, RAR_n^{NS}) \leq ARE_\phi(RAR_n^{WS}, RAR_n^{NS})$ for any distribution g that has thinner tails than ϕ.) For the (thick-tailed) t_3 and DLN distributions, the WS-RCLR and NS-RCLR tests have noticeably higher power than the CLR test except in one case (viz., the "weaker IV" case with positive β and t_3 distribution). In the base case, for the t_3 distribution, the rank CLR tests' powers are 33% higher or more than the nonrank CLR test. In the base case, for the DLN distribution, the rank tests' powers are more than 50% higher. (This is analogous to the results in Section 4.3.1 that $ARE_g^{NS} \geq 1$ for all symmetric distributions g and $ARE_g\ (RAR_n^{WS}, RAR_n^{NS}) \geq ARE_\phi(RAR_n^{WS}, RAR_n^{NS})$ for any distribution g that has thicker tails than ϕ.)

Table 1 shows that the NS-RCLR and WS-RCLR tests cannot be rank ordered in an overall sense because the NS-RCLR test has noticeably higher power for the uniform distribution, but lower power for the t_3 and DLN distributions in most cases. Table 1 also shows that the AR test has lower asymptotic power than the other tests considered (because $k = 5 > m = 1$ or $k = 10 > m = 1$). Also, the LM test has comparable asymptotic power to the CLR test in the scenarios considered except the "weaker IV" case with negative β, in which case it has lower power.

We conclude from Table 1 that the WS-RCLR and NS-RCLR tests have weak IV asymptotic power advantages over the CLR test. For the NS-RCLR test, this is true both for thin- and thick-tailed distributions. Furthermore, there is little or no cost asymptotically for using the WS-RCLR or NS-RCLR test in place of the CLR test for the normal distribution. Because it is shown in Andrews et al. (2006a) that the CLR test is nearly asymptotically universally most powerful in the class of invariant similar tests under normality, the results suggest that the NS-RCLR test also inherits this property.

4.3.3. Asymptotic Equivalence. We now provide a result that establishes when the rank and nonrank versions of the CLR, LM, and AR tests are asymptotically equivalent. We show that for a given score function $\varphi(x)$ there is a distribution G of $u_i + (\beta - \beta_0)'v_{2i}$ (and vice versa) such that the rank and nonrank versions of these tests are asymptotically equivalent under weak IV asymptotics.

LEMMA 3. *Let $L(\cdot)$ be some df with finite variance. Suppose $(u_i + (\beta - \beta_0)'v_{2i})\kappa \sim L(\cdot)$ for some $\kappa > 0$ and $\varphi(x) = L^{-1}(x)$; then*

(i) $\varphi(U_{gi})c_\varphi^{-1/2} = (u_i + (\beta - \beta_0)'v_{2i})\sigma_g^{-1}$,

(ii) $\Omega_{\varphi g} = \Omega_g$,

(iii) $\int_0^1 \varphi(x, g)\varphi(x)\, dx \cdot c_\varphi^{-1/2} = \sigma_g^{-1}$, *and*

(iv) $N_\varphi \sim N_1$, $S_\infty^\varphi \sim S_\infty$, *and* $T_\infty^\varphi \sim T_\infty$, *where* $\sim$ *denotes "has the same distribution as."*

Remark. Lemmas 2 and 3 and Theorem 1 imply that if $u_i + (\beta - \beta_0)'v_{2i}$ has a normal distribution, then the normal score function leads to asymptotic equivalence between the rank and nonrank versions of the CLR, LM, and AR tests. Likewise, if $u_i + (\beta - \beta_0)'v_{2i}$ has a uniform $[-a, a]$ distribution for some $a > 0$, then the Wilcoxon score function leads to asymptotic equivalence between the rank and nonrank versions of these statistics.

5. STRONG IV ASYMPTOTIC RESULTS

5.1. Strong IV Asymptotic Distributions of Rank Statistics

In this section, we provide the asymptotic distributions of the RLR_n^φ, RLM_n^φ, and RAR_n^φ test statistics under standard strong IV asymptotics under the null hypothesis and local alternatives.

In place of Assumption 4W, we use the following assumption. The first part of this assumption is the local alternative assumption.

Assumption 4S.

(a) $\beta = \beta_0 + Bn^{-1/2}$ for some vector $B \in R^m$.
(b) Π does not depend on n and is full column rank m.
(c) $v_{2i} = \varepsilon_i + \rho u_i$ for $i \geq 1$, where ε_i is a random m-vector and $\rho \in R^m$ is a vector of constants.
(d) $\{\varepsilon_i : i \geq 1\}$ are i.i.d. and independent of $\{u_i : i \geq 1\}$, and $E\|\varepsilon_i\|^{2+\delta} < \infty$ for some $\delta > 0$.
(e) u_i has an absolutely continuous strictly increasing df F and an absolutely continuous and bounded density f that satisfies $I(f) < \infty$.
(f) (u_i, v_{2i}) has an absolutely continuous bounded joint density with partial derivative with respect to its first argument that is bounded over both arguments.
(g) $\Omega_{\varphi f}$ is pd.
(h) $\sum_{i=1}^{\infty} \|\tilde{Z}_i\|^2/i^2 < \infty$ and $\sum_{i=1}^{\infty} \|X_i\|^2/i^2 < \infty$.
(i) $n^{1/2}(\hat{\gamma}_n(\beta_0) - \gamma_1) = O_p(1)$.

Assumption 4S(c) allows for dependence between the structural error u_i and the reduced-form error v_{2i}, but it must be of a special form. The special form is needed to make the asymptotic results for the rank statistic S_n^φ tractable. Assumption 4S(h) is not very restrictive.[10] Assumption 4S(i) holds for the null-restricted LS estimator under Assumptions 1, 2, and 4S(a)–(c).[11] The combination of Assumptions 1 and 4S(c) implies that $Eu_i^2 < \infty$.

Under strong IV asymptotics, S_n^φ has a nondegenerate asymptotic distribution given by that of $S_{f\infty}^\varphi$, and $n^{-1/2}T_n^\varphi$ converges in probability to a constant $\alpha_T^\varphi \neq 0$, where

$$S^{\varphi}_{f\infty} \sim N(\alpha^{\varphi}_S, I_k), \qquad \alpha^{\varphi}_S = D_Z^{1/2}\Pi\ell^{\varphi}_{f,B} \in R^k,$$

$$\alpha^{\varphi}_T = D_Z^{1/2}\Pi(H'\Omega^{-1}_{\varphi f}H)^{1/2} \in R^{k\times m},$$

$$\Omega_{\varphi f} = \mathrm{Var}\begin{pmatrix} \varphi(F(u_i))c_{\varphi}^{-1/2} \\ y_{2i}\end{pmatrix} = \begin{bmatrix} 1 & \nu'_{\varphi f} \\ \nu_{\varphi f} & \Omega_{22}\end{bmatrix} \in R^{(m+1)\times(m+1)}, \quad \text{and}$$

$$\nu_{\varphi f} = \mathrm{Cov}(y_{2i}, \varphi(F(u_i))c_{\varphi}^{-1/2}) \in R^m. \tag{5.1}$$

Note that $S^{\varphi}_{f\infty}$ differs from S^{φ}_{∞} only in that $\ell^{\varphi}_{f,B}$ replaces $\ell^{\varphi}_{g,\beta-\beta_0}$ (both of which are defined by the expression for $\ell^{\varphi}_{g,\beta-\beta_0}$ in (4.5)) in its mean.

The main result of this section is the following theorem.

THEOREM 2. *Under Assumptions 1–3 and 4S,*

(i) $(S^{\varphi}_n, n^{-1/2}T^{\varphi}_n) \to_d (S^{\varphi}_{f\infty}, \alpha^{\varphi}_T)$,
(ii) $RLR^{\varphi}_n \to_d S^{\varphi\prime}_{f\infty}\alpha^{\varphi}_T(\alpha^{\varphi\prime}_T\alpha^{\varphi}_T)^{-1}\alpha^{\varphi\prime}_T S^{\varphi}_{f\infty} \sim \chi^2_m(\delta^{\varphi}_{LM,S})$, *where* $\delta^{\varphi}_{LM,S} = \alpha^{\varphi\prime}_S\alpha^{\varphi}_T(\alpha^{\varphi\prime}_T\alpha^{\varphi}_T)^{-1}\alpha^{\varphi\prime}_T\alpha^{\varphi}_S$,
(iii) $RLM^{\varphi}_n \to_d S^{\varphi\prime}_{f\infty}\alpha^{\varphi}_T(\alpha^{\varphi\prime}_T\alpha^{\varphi}_T)^{-1}\alpha^{\varphi\prime}_{T\infty}S^{\varphi}_{f\infty} \sim \chi^2_m(\delta^{\varphi}_{LM,S})$, *and*
(iv) $RAR^{\varphi}_n \to_d S^{\varphi\prime}_{f\infty}S^{\varphi}_{f\infty}/k \sim \chi^2_k(\delta^{\varphi}_{AR,S})/k$, *where* $\delta^{\varphi}_{AR,S} = \alpha^{\varphi\prime}_S\alpha^{\varphi}_S$.

Remarks.

(a) Theorem 2(ii) and (iii) show that under strong IV asymptotics the RLR and RLM test statistics are asymptotically equivalent under the null and local alternatives for any values of k and m. (As noted previously, when $k = m$, the RLR and RLM test statistics are the same, and so the tests are trivially asymptotically equivalent.)
(b) Theorem 2(ii)–(iv) shows that the RAR test statistic has a different asymptotic distribution from that of the RLR and RLM statistics when $k > m$. When $k = m$, $k\cdot RAR^{\varphi}_n = RLM^{\varphi}_n = RLR^{\varphi}_n$, and so the three rank statistics are asymptotically equivalent.

5.2. Strong IV Asymptotic Distributions of Nonrank Statistics

For comparative purposes, we now provide the strong IV asymptotic distributions under the null hypothesis and local alternatives of the nonrank LR_n, LM_n, and AR_n test statistics. The results for LR_n with $m > 1$ are new. (Andrews et al., 2006a, provides the same results for $m = 1$.) Let

$$S_{f\infty} \sim N(\alpha_S, I_k), \qquad \alpha_S = D_Z^{1/2}\Pi B\sigma_f^{-1} \in R^k,$$

$$\alpha_T = D_Z^{1/2}\Pi(H'\Omega_f^{-1}H)^{1/2} \in R^{k\times m}, \quad \text{and}$$

$$\Omega_f = \mathrm{Var}((u_i\sigma_f^{-1}, v_{2i})') = \begin{bmatrix} 1 & \nu'_f \\ \nu_f & \Omega_{22}\end{bmatrix}, \qquad \nu_f = \mathrm{Cov}(y_{2i}, u_i\sigma_f^{-1}). \tag{5.2}$$

LEMMA 4. *Under Assumptions 1–3 and 4S and* $\Omega > 0$*,*

(i) $(S_n, n^{-1/2}T_n) \to_d (S_{f\infty}, \alpha_T)$,
(ii) $LR_n \to_d S'_{f\infty}\alpha_T(\alpha_T'\alpha_T)^{-1}\alpha_T' S_{f\infty} \sim \chi^2_m(\delta_{LM,S})$, *where* $\delta_{LM,S} = \alpha_S'\alpha_T(\alpha_T'\alpha_T)^{-1}\alpha_T'\alpha_S$,
(iii) $LM_n \to_d S'_{f\infty}\alpha_T(\alpha_T'\alpha_T)^{-1}\alpha_T' S_{f\infty} \sim \chi^2_m(\delta_{LM,S})$, *and*
(iv) $AR_n \to_d S'_{f\infty}S_{f\infty}/k \sim \chi^2_k(\delta_{AR,S})/k$, *where* $\delta_{AR,S} = \alpha_S'\alpha_S$.

5.3. Strong IV Power Comparisons: Rank versus Nonrank Tests

Theorem 2 and Lemma 4 allow calculation of the ARE of the rank and nonrank tests with strong IVs. The calculation is analogous to that given in Section 4.3.1 for weak IVs but with three differences. The first difference is that α_T^φ and α_T are fixed in the strong IV case, whereas T_∞^φ and T_∞ are random in the weak IV case. This does not affect the ARE calculations. The second difference is that the asymptotic distributions depend on the density f of u_i rather than the density g of $u_i + (\beta - \beta_0)'v_{2i}$. This occurs because β converges to β_0 under strong IV local alternatives and hence $(\beta - \beta_0)'v_{2i} \to 0$ as $n \to \infty$. The third difference is that under strong IVs the asymptotic distributions of RLR_n^φ and RLM_n^φ are the same and, analogously, those of LR_n and LM_n are the same.

Combining the results of Section 4.3.1 with these differences, we find that under strong IVs the ARE of the rank to nonrank AR tests is the same as for the rank to nonrank LM and CLR tests and is equal to the usual ARE for rank to nonrank procedures based on the density f. That is,

$$ARE_f(RAR_n^\varphi, AR_n) = ARE_f(RLM_n^\varphi, LM_n) = ARE_f(RLR_n^\varphi, LR_n) \\ = \xi(\varphi^{NS}, f)\sigma_f^2, \tag{5.3}$$

where $\xi(\varphi^{NS}, f)\sigma_f^2$ is given in (4.16) and (4.17) for normal and Wilcoxon scores, respectively, with f in place of g.[12]

In sum, all of the statements in Section 4.3.1 concerning (4.15) apply to the ARE of the rank to nonrank versions of the AR, LM, and CLR tests under strong IVs but with f in place of g.

5.4. Asymptotic Equivalence

The next result establishes when the rank and nonrank versions of the CLR, LM, and AR tests are asymptotically equivalent under strong IV asymptotics.

LEMMA 5. *Let* $L(\cdot)$ *be some df with finite variance. Suppose* $u_i\kappa \sim L(\cdot)$ *for some* $\kappa > 0$ *and* $\varphi(x) = L^{-1}(x)$*; then*

(i) $\varphi(F(u_i))c_\varphi^{-1/2} = u_i\sigma_f^{-1}$,
(ii) $\Omega_{\varphi f} = \Omega_f$,

(iii) $\int_0^1 \varphi(x,f)\varphi(x)\,dx \cdot c_\varphi^{-1/2} = \sigma_f^{-1}$,
(iv) $S_{f\infty}^{\varphi} \sim S_{f\infty}$, *and* $\alpha_T^{\varphi} = \alpha_T$.

Remarks.

(a) Lemmas 4 and 5 and Theorem 2 imply that if u_i has a normal distribution, then the normal score function leads to asymptotic equivalence between the rank and nonrank versions of the CLR, LM, and AR tests. Likewise, if u_i has a uniform $[-a,a]$ distribution for some $a > 0$, then the Wilcoxon score function leads to asymptotic equivalence between these statistics.

(b) For the case of normal errors, the (nonrank) CLR and LM tests are asymptotically efficient under strong IV asymptotics; see Andrews et al. (2006a). This combined with Remark 1 implies that the normal scores rank CLR and LM tests also are asymptotically efficient under normal errors and strong IV asymptotics. When $k > m$, the rank AR statistic has a different asymptotic distribution from that of the rank LR and LM statistics (see Remark (a) to Theorem 2), and hence it is not asymptotically efficient.

6. FINITE-SAMPLE RESULTS

In this section, we report simulation results concerning the finite-sample size of some of the rank and nonrank tests discussed previously. We also provide power comparisons of size-corrected versions of these tests.

We consider the Wilcoxon scores rank CLR test, denoted $RCLR_n^{WS}$, and the (untruncated) normal scores CLR rank test, denoted $RCLR_n^{NS}$. For comparative purposes, we also consider the CLR, LM, and AR tests. We do not report results for the rank LM and rank AR tests both for brevity and for the following reasons. First, when the model is overidentified, the AR test has distinctly lower power than the CLR test (see Andrews et al., 2006a, 2006b), and simulations show that the same is true for the rank versions of these tests. Second, the LM test has quirky power properties in parts of the parameter space (see, e.g., Andrews et al., 2006a, 2006b), and simulations show that the rank LM test inherits these properties.

6.1. Experimental Design

We take the model to be as in (2.1) with y_{2i} and β being scalars ($m = 1$) and v_{2i} defined as in (4.18), where $\rho_{uv_2} \in [-1,1]$. Let $\tilde{Z}_i = (\tilde{Z}_{i1}, \ldots, \tilde{Z}_{ik})'$ and $X_i = (1, X_{i2}, \ldots, X_{ip})'$. We take $\tilde{Z}_{ij}, X_{is}, u_i, \varepsilon_i$ to be i.i.d. with distribution F for all $j = 1, \ldots, k$, $s = 2, \ldots, p$, and $i = 1, \ldots, n$.[13]

The test statistics considered are invariant with respect to γ_1, ξ_1, and the location and scale of F. Hence, without loss of generality we take γ_1 and ξ_1 to be zero, and we take F to have mean zero (if its mean is well defined), center of symmetry zero (if it is symmetric), and variance one (if its variance is well defined).

The parameter vector $\pi \in R^k$ determines the strength of the IVs. It is taken to be proportional to a k-vector of ones:

$$\pi = \frac{\rho_{IV}}{k^{1/2}(1-\rho_{IV}^2)^{1/2}}(1,\ldots,1)' \quad \text{for some } \rho_{IV} \in [-1,1], \tag{6.1}$$

where ρ_{IV} is the correlation between the reduced-form regression function, $Z_i'\pi$, and the endogenous variable y_{2i} (when F has a finite variance). The parameter ρ_{IV} can be related to a parameter λ that directly measures the strength of the IVs (and is closely related to the so-called concentration parameter):

$$\lambda = \frac{n\rho_{IV}^2}{1-\rho_{IV}^2} = n\pi' E\widetilde{Z}_i\widetilde{Z}_i'\pi \approx \pi'\widetilde{Z}'\widetilde{Z}\pi, \tag{6.2}$$

where the first equality defines λ, the second equality holds provided $\widetilde{Z}_i$ has a finite variance, and $a_n \approx b_n$ means $a_n/b_n \to_p 1$ as $n \to \infty$.

The hypotheses of interest are $H_0: \beta = \beta_0$ and $H_1: \beta \neq \beta_0$. Without loss of generality, we take $\beta_0 = 0$.[14]

For both the size and power results, we first consider a base case with moderately weak IVs $\lambda = 10$ (equivalently, $\rho_{IV} = 0.302$ when $n = 100$), moderately strong endogeneity $\rho_{uv_2} = 0.75$, sample size $n = 100$, number of IVs $k = 5$, no exogenous variables beyond a constant $p = 1$, and distribution F equal to the normal, uniform, t_1, t_2, t_3, or difference of DLNs. The uniform distribution exhibits thin tails, and the t-distributions exhibit heavy tails (e.g., t_1 is the Cauchy distribution) as does the DLN distribution. For the power results, both positive and negative true β values are considered. The β values are selected so that the level 0.05 CLR test has power around 0.4 for the given choice of λ, ρ_{uv_2}, n, k, and p when F is normal.

We also consider a number of variations of the base case to illustrate the effect of changes in the level of endogeneity: $\rho_{uv_2} = 0, 0.95$; strength of IVs: $\lambda = 4, 20$; number of IVs: $k = 1, 10$; and sample size: $n = 50, 200$. In each variation of the base case, only one of these parameters is different from the base case. In the base case, we find that when F is normal the power of the normal scores rank CLR test is slightly higher than that of the nonrank CLR test, but the opposite is true for negative β. (These differences disappear asymptotically under weak and strong IV asymptotics.) In consequence, to maintain fair comparisons and for brevity, in each variation of the base case we report average power for two β values—one positive and one negative—each of which is chosen so that the CLR test has power approximately equal to 0.4 when F is normal.[15]

For the power results, the tests are all size-corrected (where the size-correcting critical values are based on the same distribution F and the same parameters λ,

ρ_{uv_2}, n, k, and p as for the corresponding power results but with $\beta = 0$). The size-correcting critical values are obtained via simulation with 100,000 simulation repetitions. The number of simulation repetitions is 20,000 for the size results and 5,000 for the power results.

Note that the size results for the AR test are invariant to ρ_{uv_2} and λ.

6.2. Size Results

Table 2 presents the size results. The two rank CLR tests perform noticeably better in terms of size than the nonrank CLR, LM, and AR tests. Nine different cases are considered with six different distributions for each case. Over the 54 trials, the range of null rejection rates for each test is WS-RCLR: [0.027, 0.052]; NS-RCLR: [0.033, 0.051]; CLR: [0.047, 0.091]; LM: [0.042, 0.070]; and AR: [0.049, 0.127]. For the two rank tests, the majority of rejection rates are in the desired [0.040, 0.050] range, which corresponds to no overrejection and sufficiently small underrejection as to minimize the power loss. In particular, 44/54 for WS-RCLR and 38/54 for NS-RCLR are in this range. In contrast, for the nonrank tests a small number of rejection rates are in this desired range: 1/54 for CLR, 3/54 for LM, and 11/54 for AR. Not surprisingly, the largest overrejections for the nonrank tests occur for the thickest tailed distributions. If one widens the range to [0.04, 0.06], which includes overrejection by a small amount, then the RCLR tests still outperform the CLR and AR tests, but the LM test performs best of all. The numbers of cases in this range are 44/54 for WS-RCLR, 39/54 for NS-RCLR, 33/54 for CLR, 49/54 for LM, and 29/54 for AR.

6.3. Power Comparisons

Table 3 presents the power results. The general pattern of finite-sample power in Table 3 reflects that of asymptotic power given in Table 1. In particular, the NS-RCLR and CLR tests have comparable power for the normal distribution, the NS-RCLR test has higher power than the CLR test for the uniform distribution in many cases and much higher power for the thick-tailed distributions. This occurs in the base case and in the variations of the base case. For example, in the base case with two β values the (average) power of the NS-RCLR test for the t_2 distribution is 0.67 compared to 0.46 for the CLR test. The WS-RCLR and NS-RCLR tests have similar power with the NS-RCLR test having slightly higher power for the normal distribution, noticeably higher power for the uniform distribution, and slightly worse power for the thick-tailed distributions. The LM test has similar power to the CLR test, but with lower power in the weaker IVs case with normal distribution and slightly higher power for the heavy-tailed distributions. The AR test has significantly lower power than the other tests except in the case with $k = 1$.

In sum, the NS-RCLR test has power that essentially dominates that of the (nonrank) CLR, LM, and AR tests. Its power is comparable to that of the CLR

Table 2. Finite-sample null rejection rates of nominal level 0.05 tests

Case	Distribution	WS-RCLR	NS-RCLR	CLR	LM	AR
Base case	Normal	0.050	0.043	0.056	0.054	0.049
	Uniform	0.049	0.041	0.054	0.052	0.053
	t_1	0.032	0.045	0.073	0.610	0.108
	t_2	0.044	0.042	0.065	0.058	0.077
	t_3	0.046	0.039	0.060	0.056	0.058
	DLN	0.044	0.038	0.062	0.055	0.071
No endogeneity	Normal	0.048	0.039	0.058	0.055	0.049
($\rho_{uv_2} = 0$)	Uniform	0.048	0.040	0.059	0.053	0.053
	t_1	0.030	0.039	0.077	0.058	0.108
	t_2	0.042	0.039	0.072	0.058	0.077
	t_3	0.046	0.037	0.063	0.057	0.058
	DLN	0.043	0.038	0.069	0.055	0.071
High endogeneity	Normal	0.050	0.045	0.054	0.053	0.049
($\rho_{uv_2} = 0.95$)	Uniform	0.050	0.045	0.052	0.051	0.053
	t_1	0.033	0.047	0.064	0.056	0.108
	t_2	0.046	0.045	0.059	0.057	0.077
	t_3	0.047	0.042	0.056	0.055	0.058
	DLN	0.043	0.042	0.057	0.055	0.071
Weaker IVs	Normal	0.049	0.041	0.058	0.055	0.049
($\lambda = 4$)	Uniform	0.049	0.043	0.058	0.053	0.053
	t_1	0.031	0.045	0.078	0.058	0.108
	t_2	0.043	0.041	0.073	0.058	0.077
	t_3	0.046	0.041	0.064	0.056	0.058
	DLN	0.043	0.039	0.070	0.055	0.071
Stronger IVs	Normal	0.049	0.043	0.054	0.054	0.049
($\lambda = 20$)	Uniform	0.049	0.041	0.054	0.052	0.053
	t_1	0.032	0.045	0.068	0.056	0.108
	t_2	0.045	0.041	0.063	0.058	0.077
	t_3	0.047	0.041	0.057	0.055	0.058
	DLN	0.043	0.039	0.070	0.055	0.071
One IV	Normal	0.048	0.041	0.053	0.053	0.050
($k = 1$)	Uniform	0.052	0.045	0.055	0.055	0.053
	t_1	0.031	0.041	0.047	0.047	0.046
	t_2	0.041	0.041	0.055	0.054	0.053
	t_3	0.046	0.041	0.054	0.054	0.052
	DLN	0.045	0.047	0.057	0.057	0.054
Ten IVs	Normal	0.052	0.048	0.053	0.052	0.050
($k = 10, n = 200$)	Uniform	0.050	0.046	0.053	0.053	0.051
	t_1	0.032	0.051	0.058	0.044	0.127
	t_2	0.049	0.048	0.058	0.050	0.090
	t_3	0.049	0.045	0.054	0.052	0.062
	DLN	0.045	0.041	0.057	0.057	0.054

(*continued*)

TABLE 2. Continued

Case	Distribution	WS-RCLR	NS-RCLR	CLR	LM	AR
Smaller sample size	Normal	0.045	0.036	0.061	0.056	0.048
(n = 50)	Uniform	0.050	0.039	0.065	0.058	0.050
	t_1	0.027	0.035	0.091	0.070	0.127
	t_2	0.039	0.033	0.078	0.064	0.078
	t_3	0.044	0.034	0.071	0.061	0.063
	DLN	0.044	0.037	0.076	0.064	0.074
Larger sample size	Normal	0.048	0.046	0.052	0.052	0.049
(n = 200)	Uniform	0.049	0.044	0.052	0.052	0.053
	t_1	0.032	0.049	0.052	0.042	0.090
	t_2	0.044	0.045	0.058	0.051	0.076
	t_3	0.048	0.045	0.056	0.053	0.056
	DLN	0.050	0.046	0.056	0.054	0.067

Note: All cases have $\beta = \beta_0 = 0$, $\lambda = 10$ (equivalently, $\rho_{IV} = 0.302$ for $n = 100$), $\rho_{uv_2} = 0.75$, $n = 100$, $k = 5$, and $p = 1$ (an intercept), unless otherwise stated.

TABLE 3. Finite-sample power of size-corrected level 0.05 tests

Case	Distribution	WS-RCLR	NS-RCLR	CLR	LM	AR
Base case	Normal	0.42	0.46	0.40	0.40	0.26
(β = 1.35)	Uniform	0.41	0.48	0.40	0.40	0.25
	t_1	0.92	0.95	0.56	0.61	0.40
	t_2	0.66	0.66	0.46	0.49	0.26
	t_3	0.53	0.53	0.42	0.43	0.26
	DLN	0.61	0.60	0.43	0.45	0.23
Base case	Normal	0.35	0.35	0.39	0.38	0.25
($\beta = -0.44$)	Uniform	0.32	0.38	0.39	0.39	0.25
	t_1	0.94	0.95	0.55	0.61	0.40
	t_2	0.72	0.68	0.46	0.49	0.27
	t_3	0.53	0.50	0.41	0.43	0.25
	DLN	0.65	0.59	0.42	0.44	0.22
Base case	Normal	0.38	0.40	0.39	0.39	0.26
(β = 1.35 and $\beta = -0.44$)	Uniform	0.37	0.43	0.40	0.40	0.25
	t_1	0.93	0.95	0.55	0.61	0.40
	t_2	0.69	0.67	0.46	0.49	0.27
	t_3	0.53	0.51	0.42	0.43	0.26
	DLN	0.63	0.60	0.42	0.44	0.23

(*continued*)

TABLE 3. Continued

Case	Distribution	WS-RCLR	NS-RCLR	CLR	LM	AR
No endogeneity	Normal	0.44	0.47	0.41	0.37	0.34
($\rho_{uv_2} = 0$,	Uniform	0.43	0.47	0.42	0.37	0.34
$\beta = 0.975$ and $\beta = -1.05$)	t_1	0.94	0.97	0.55	0.61	0.42
	t_2	0.74	0.72	0.46	0.47	0.33
	t_3	0.57	0.56	0.43	0.40	0.34
	DLN	0.67	0.64	0.41	0.42	0.30
High endogeneity	Normal	0.38	0.39	0.41	0.41	0.22
($\rho_{uv_2} = 0.95$,	Uniform	0.39	0.46	0.41	0.41	0.22
$\beta = 0.95$ and $\beta = -1.25$)	t_1	0.93	0.96	0.61	0.64	0.41
	t_2	0.70	0.66	0.49	0.50	0.25
	t_3	0.53	0.51	0.42	0.42	0.22
	DLN	0.67	0.62	0.44	0.44	0.19
Weaker IVs	Normal	0.33	0.33	0.36	0.31	0.30
($\lambda = 4$,	Uniform	0.31	0.33	0.36	0.32	0.29
$\beta = 25$ and $\beta = -0.725$)	t_1	0.91	0.94	0.52	0.59	0.40
	t_2	0.61	0.58	0.41	0.43	0.28
	t_3	0.43	0.42	0.37	0.35	0.29
	DLN	0.55	0.49	0.36	0.37	0.25
Stronger IVs	Normal	0.40	0.42	0.40	0.40	0.23
($\lambda = 20$,	Uniform	0.38	0.47	0.41	0.41	0.23
$\beta = 0.62$ and $\beta = -0.325$)	t_1	0.94	0.96	0.58	0.63	0.40
	t_2	0.75	0.72	0.48	0.50	0.25
	t_3	0.57	0.54	0.42	0.43	0.23
	DLN	0.59	0.54	0.38	0.40	0.22
One IV	Normal	0.37	0.38	0.39	0.39	0.39
($k = 1$,	Uniform	0.34	0.42	0.39	0.39	0.39
$\beta = 1.05$ and $\beta = -0.41$)	t_1	0.88	0.89	0.44	0.44	0.44
	t_2	0.67	0.64	0.42	0.42	0.42
	t_3	0.54	0.51	0.42	0.42	0.42
	DLN	0.66	0.61	0.40	0.40	0.40
Ten IVs	Normal	0.38	0.39	0.41	0.41	0.24
($k = 10$,	Uniform	0.38	0.43	0.45	0.45	0.27
$\beta = 1.9$ and $\beta = -0.49$)	t_1	0.94	0.90	0.60	0.67	0.43
	t_2	0.72	0.69	0.50	0.54	0.28
	t_3	0.54	0.52	0.46	0.46	0.28
	DLN	0.64	0.60	0.46	0.48	0.24

Note: All cases have $\beta_0 = 0$, $\lambda = 10$ (equivalently, $\rho_{IV} = 0.302$ for $n = 100$), $\rho_{uv_2} = 0.75$, $n = 100$, $k = 5$, and $p = 1$ (an intercept), unless otherwise stated.

and LM tests for the normal distribution and higher for the other distributions, especially the thick-tailed ones. The power of the WS-RCLR test is similar to that of the NS-RCLR test.

NOTES

1. We note that the CLR test reduces to the AR test when the model is just-identified, and so the optimality properties mentioned in the text are consistent with those mentioned previously for the AR test.

2. The statistics S_n and T_n are denoted $\bar{S}$ and $\bar{T}$, respectively, in Moreira (2003).

3. The statistic LR_n is the likelihood ratio statistic for the case of normal errors v_i with known covariance matrix Ω and with $\hat{\Omega}_n$ plugged in for Ω. One can also consider the likelihood ratio statistic for the case of normal errors and unknown covariance matrix; see Moreira (2003).

4. If there are any ties in the ranks, then we determine a unique ranking by randomization. For example, if $y_{1i} - \beta' y_{2i} - \hat{\gamma}_n(\beta)'X_i = y_{1j} - \beta' y_{2j} - \hat{\gamma}_n(\beta)'X_j$ for some $i \neq j$ and these observations are the ℓth largest in the sample, then $\hat{R}_i(\beta) = \ell$ with probability 0.5, $\hat{R}_i(\beta) = \ell + 1$ with probability 0.5, $\hat{R}_j(\beta) = \ell + 1$ if $\hat{R}_i(\beta) = \ell$, and $\hat{R}_j(\beta) = \ell$ if $\hat{R}_i(\beta) = \ell + 1$. Ties only occur with positive probability if the distribution of $y_{1i} - \beta' y_{2i} - \hat{\gamma}_n(\beta)'X_i$ is not absolutely continuous. In consequence, in practice ties rarely occur.

The matrix programming languages GAUSS and Matlab have very fast built-in procedures for finding the ranks of a given vector. The GAUSS procedure is called *rankindx.*

5. The definition of T_n^{φ} uses the ranks R_{φ} of $\{y_{1j} - \beta_0' y_{2j} - \hat{\gamma}_n(\beta_0)'X_j : j = 1, \dots, n\}$ but is linear in Y_2 (or equivalently, in $Y_2 - P_X Y_2$ because $Z'P_X = 0$). One might think that it is more natural to replace Y_2 in the definition of T_n^{φ} by the ranks of $Y_2 - P_X Y_2$. We do not do this for the following reason. For power purposes one wants the Y_2 term in the definition of T_n^{φ} to be (asymptotically) linear in its mean $Z\Pi$. If one replaces Y_2 by the ranks of $Y_2 - P_X Y_2$, then (asymptotic) linearity does not hold under strong IV asymptotics, defined in Section 5, because $Z\Pi$ is not an $n^{-1/2}$ perturbation from the zero vector; see Lemma 6 in the Appendix. Hence, one does not obtain the desired power properties under strong IV asymptotics. Under weak IV asymptotics, defined in Section 4, (asymptotic) linearity holds because $Z\Pi = ZCn^{-1/2}$ for some matrix C and the latter is an $n^{-1/2}$ perturbation from the zero vector. Hence, power problems with this alternative definition of T_n^{φ} only arise under strong IV asymptotics.

6. The second equality holds because T_n^{φ} is a square invertible matrix when $k = m$. The last equality holds because $[S_n : T_n]'[S_n : T_n]$ is positive semidefinite and singular, which implies that $\lambda_{\min}([S_n : T_n]'[S_n : T_n]) = 0$. Singularity holds because $[S_n : T_n]'$ is an $(m + 1) \times m$ matrix and $[S_n : T_n]'[S_n : T_n]$ is $(m + 1) \times (m + 1)$ when $k = m$.

7. $\mathrm{Var}(\varphi(U_{gi})) = c_{\varphi}$ because U_{gi} has a $U[0,1]$ distribution.

8. The expressions for $\xi(\varphi, f)$ for normal and Wilcoxon scores are established by change of variables and integration by parts.

9. The ARE of one test to another is usually defined, roughly speaking, to be the limit of the ratio of the sample sizes of the second test to the first required for the two tests to have the same power; see Lehmann (1986, p. 321). In standard scenarios—in which the two tests have noncentral chi-square asymptotic distributions—the ARE reduces to the ratio of the (asymptotic) noncentrality parameter of the first test to the second. In Section 4.3.1, which involves nonstandard weak IV asymptotics—in which the power of a test does not necessarily increase with the sample size—we adopt the ratio of the (asymptotic) noncentrality parameters to be the definition of the ARE. That is, by definition, the ARE of one test to another is the ratio of the noncentrality parameter of the asymptotic distribution of the first test to that of the second test provided this ratio is nuisance parameter free and the two tests have noncentral chi-square asymptotic distributions or mixed noncentral chi-square asymptotic distributions (and the ratio of the noncentrality parameters is the same for all values of the mixing variable).

10. A sufficient condition for Assumption 4S(h) is the same condition with 2 replaced by $1 + \delta$, and the latter holds with probability one for sequences $\{(\widetilde{Z}_i, X_i) : i \geq 1\}$ that are realizations of i.i.d. random vectors with finite $1 + \delta$ moments; see Lemma 12 in the Appendix.

11. The proof is the same as for Lemma 1 except that in place of (A.69) we have $n^{1/2}(\xi - \xi_1)(\beta - \beta_0) = O(1)$ because $\beta - \beta_0 = O(n^{-1/2})$ by Assumption 4S(a) and $\xi - \xi_1 = O(1)$ by Assumptions 2(c) and 4S(b).

12. The AREs discussed in Section 5.3 can be defined by the usual method involving the limit of ratios of sample sizes or in terms of the ratio of noncentrality parameters—see note 9 regarding these definitions. Under strong IV asymptotics, the two definitions are equivalent for the tests considered here.

13. Thus, we consider a model with random exogenous variables and IVs. The tests considered have the correct size asymptotically both conditionally and unconditionally on the exogenous variables and IVs.

14. There is no loss of generality in taking $\beta_0 = 0$ because the structural equation $y_{1i} = y_{2i}\beta + \gamma_1' X_i + u_i$ and hypothesis $H_0 : \beta = \beta_0$ can be transformed into $\tilde{y}_{1i} = y_{2i}\tilde{\beta} + \gamma_1' X_i + u_i$ and $H_0 : \tilde{\beta} = 0$, where $\tilde{y}_{1i} = y_{1i} - y_{2i}\beta_0$ and $\tilde{\beta} = \beta - \beta_0$.

15. The reported power of the CLR test for the case where λ or n is small is less than 0.4 because the CLR test has power less than 0.4 for all values of β.

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APPENDIX: Proofs

The proofs of Lemmas 1–5 and Corollary 1 are given at the end of the Appendix, as is the description of the numerical calculation of asymptotic power under weak IVs.

The proofs of Theorems 1(i) and 2(i) rely on the following lemmas. Lemma 6 follows from results of Koul (1970) and Hájek and Sidák (1967).

LEMMA 6. *Let* $\Psi_n(t) = n^{-1}\sum_{i=1}^{n}(c_i - \bar{c}_n)\varphi(r_i(t)/(n+1))$, *where*

(i) $r_i(t)$ *is the rank of* $Q_i - d_i' t$ *among* $\{Q_j - d_j' t : 1 \le j \le n\}$ *for a constant vector* $t \in R^{\delta_d}$,

(ii) $\{Q_i : i \ge 1\}$ *is a sequence of i.i.d. random variables with absolutely continuous strictly increasing df H and absolutely continuous and bounded density h that satisfies* $I(h) < \infty$,

(iii) $\{c_i : i \le n, n \ge 1\}$ *and* $\{d_i : i \le n, n \ge 1\}$ *are triangular arrays of nonrandom* δ_c*-vectors and* δ_d*-vectors, respectively (with dependence of* c_i *and* d_i *on n sup-*

pressed for brevity), that satisfy $\lim_{n\to\infty} \max_{1\le i\le n} \|c_i - \bar{c}_n\|^2 / \sum_{i=1}^n \|c_i - \bar{c}_n\|^2 = 0$ *and* $\lim_{n\to\infty} n^{-1} \sum_{i=1}^n \|c_i - \bar{c}_n\|^2 < \infty$ *and likewise with* $c_i - \bar{c}_n$ *replaced by* $d_i - \bar{d}_n$*, where* $\bar{c}_n = n^{-1}\sum_{i=1}^n c_i$ *and* $\bar{d}_n = n^{-1}\sum_{i=1}^n d_i$*, and*

(iv) the score function φ *satisfies Assumption 3.*

Then,

(a) for all $\varepsilon > 0$ *and* $b < \infty$*,*

$$\lim_{n\to\infty} P\left(\sup_{\|t\|\le b} n^{1/2}|\Psi_n(tn^{-1/2}) - \Psi_n(0) - n^{-1/2}\dot{A}_n(0)t| > \varepsilon\right) = 0,$$

where

$$\dot{A}_n(0) = -n^{-1}\sum_{i=1}^n (c_i - \bar{c}_n)(d_i - \bar{d}_n)' \int_0^1 \varphi(x,h)\varphi(x)\,dx,$$

(b) for any sequence of random δ_d*-vectors* $\{\hat{\tau}_n : n \ge 1\}$ *for which* $n^{1/2}\hat{\tau}_n = O_p(1)$*,*

$$n^{1/2}\Psi_n(\hat{\tau}_n) = n^{1/2}\Psi_n(0) + \dot{A}_n(0)n^{1/2}\hat{\tau}_n + o_p(1),$$

(c) $n^{1/2}\Psi_n(0) = n^{-1/2}\sum_{i=1}^n (c_i - \bar{c}_n)\varphi(H(Q_i)) + o_p(1)$.

Remarks.

(a) Lemma 6(a) is an extension of Theorem 2.1 and Lemma 2.3 of Koul (1970) from scalar constants c_i and d_i to vectors. As Koul (1970, p. 1280) notes, his proof of these results goes through for this extension with virtually no changes. Lemma 6(b) follows from part (a). Lemma 6(c) follows from the proofs of the Hájek and Sidák (1967) Theorem V.1.5a (p. 160), Theorem VI.1.6a (p. 163), and Lemma VI.1.6a (p. 164), which show that in the scalar c_i case $E(n^{-1/2}\sum_{i=1}^n (c_i - \bar{c}_n)\varphi(H(Q_i)) - n^{-1/2}\sum_{i=1}^n (c_i - \bar{c}_n)a_n^{\varphi}(i))^2 = o(1)$ and $E(n^{-1/2}\sum_{i=1}^n (c_i - \bar{c}_n)a_n^{\varphi}(i) - n^{1/2}\Psi_n(0))^2 = o(1)$, respectively, where $a_n^{\varphi}(i) = E(\varphi(H(Q_1))|r_1(0) = i)$.

(b) The expression for $\dot{A}_n(0)$ on p. 1277 of Koul (1970) is correct, but the expression for $\dot{A}_n(0)$ given on p. 1278 (which is of the form given previously) contains a typo—a minus sign is missing. Also, the proof of Theorem 2.1 of Koul (1970) contains a typo that could be confusing to the reader. The term $\varphi(q_n)$ that appears at the end of the expression on the first two lines of the first equation on p. 1276 should be $\varphi'(q_n)$ in both places.

(c) We do not require φ to satisfy the second condition of (i) on p. 1274 of Koul (1970) because this is a normalization condition that implies that $\varphi(\frac{1}{2}) = 0$ which is not needed for his Theorem 2.1 or Lemma 2.3. It is needed for his $n^{1/2}S_n(0)$ to have an asymptotic normal distribution. We do not require it for $n^{1/2}\Psi_n(0)$ to have an asymptotic normal distribution because we consider demeaned constant vectors $c_i - \bar{c}_n$, which yields $n^{1/2}\Psi_n(0)$ invariant to additive constants in φ, whereas Koul (1970) does not.

The next lemma is used to establish the probability limit of $\hat{\nu}_{\varphi n}$.

LEMMA 7. *Suppose*

(i) $\{(Q_{1i}, Q_{2i}) : i \geq 1\}$ *is an i.i.d. sequence of random* $(m+1)$*-vectors with* $Q_{1i} \in R$,
(ii) (Q_{1i}, Q_{2i}) *has an absolutely continuous and bounded joint df* H_{Q_1,Q_2} *that satisfies* $\sup_{(q_1,q_2)} |\partial H_{Q_1,Q_2}(q_1,q_2)/\partial q_1| < \infty$,
(iii) $E\|Q_{2i}\| < \infty$,
(iv) $r_i(t)$ *is the rank of* $Q_{1i} - d_i' t$ *among* $\{Q_{1j} - d_j' t : j \leq n\}$*, where* $t \in R^{\delta_d}$,
(v) $\{d_i : i \leq n, n \geq 1\}$ *is a triangular array of nonrandom* δ_d*-vectors that satisfies* $\overline{\lim}_{n\to\infty} n^{-1} \sum_{i=1}^n \|d_i\| < \infty$*, and*
(vi) the score function φ *satisfies Assumption 3.*

Then,

(a) for all $b < \infty$,

$$\sup_{t:\|t\|\leq b} \left| n^{-1} \sum_{i=1}^n \varphi\left(\frac{r_i(tn^{-1/2})}{n+1}\right) Q_{2i} - n^{-1} \sum_{i=1}^n \varphi\left(\frac{r_i(0)}{n+1}\right) Q_{2i} \right| = o_p(1),$$

(b) for any sequence of random δ_d*-vectors* $\{\hat{\tau}_n : n \geq 1\}$ *for which* $n^{1/2}\hat{\tau}_n = O_p(1)$,

$$n^{-1} \sum_{i=1}^n \varphi\left(\frac{r_i(\hat{\tau}_n)}{n+1}\right) Q_{2i} = n^{-1} \sum_{i=1}^n \varphi\left(\frac{r_i(0)}{n+1}\right) Q_{2i} + o_p(1),$$

(c) $n^{-1} \sum_{i=1}^n \varphi(r_i(0)/(n+1)) Q_{2i} = E\varphi(H_{Q_1}(Q_{1i})) Q_{2i} + o_p(1)$*, where* H_{Q_1} *is the df of* Q_{1i}.

Remark. Lemma 7(a) follows from arguments similar to those used to prove Lemma 2.2 in Koul (1970), which was originally proved, under different assumptions, as Theorem 3.1 in Koul (1969). The result established in Lemma 7(a) is different from the results established in Koul (1969, 1970), but the idea of the argument is essentially the same. The results in Koul (1969, 1970) are for a linear regression model with deterministic regressors. Hence, using our notation, the results in Koul (1969, 1970) are restricted to the case where $\{Q_{2i} : i \leq n\}$ are nonrandom real numbers and $\{(Q_{1i}, Q_{2i}) : i \leq n\}$ and $\{d_i : i \leq n\}$ satisfy the relation imposed by a linear regression equation. Hence, the conditions in Lemma 7(a) generalize those in Lemma 2.2 of Koul (1970). On the other hand, Lemma 2.2 of Koul (1970) establishes that the left-hand side in Lemma 7(a) is $o_p(n^{-1/2})$, which is a stronger result than that given in Lemma 7(a).

Let Φ be the n-vector with ith element given by $\varphi(U_{gi}) = \varphi(G(u_i + (\beta - \beta_0)' v_{2i}))$.

LEMMA 8. *Under Assumptions 1–3 and 4W,*

(i) $n^{-1/2} Z' R_\varphi = n^{-1/2} Z'(\Phi + ZC\ell^\varphi_{g,\beta-\beta_0} c_\varphi^{1/2} n^{-1/2}) + o_p(1)$,
(ii) $S_n^\varphi = (Z'Z)^{-1/2} Z'(\Phi c_\varphi^{-1/2} + ZC\ell^\varphi_{g,\beta-\beta_0} n^{-1/2}) + o_p(1)$,
(iii) $n^{-1} Z'Z \to D_Z > 0$*, and*
(iv) $n^{-1/2} Z'[(\Phi c_\varphi^{-1/2} + ZC\ell^\varphi_{g,\beta-\beta_0} n^{-1/2}) : Y_2] \to_d [N_\varphi : N_2]$.

LEMMA 9. *Under Assumptions 1–3 and 4W,*

(i) $\hat{\nu}_{\varphi n} \to_p \nu_{\varphi g}$ *and*
(ii) $\hat{\Omega}_{22n} \to_p \Omega_{22}$.

LEMMA 10. *Under Assumptions 1–3 and 4S,*

(i) $n^{-1/2}Z'R_\varphi = n^{-1/2}Z'(\Phi + Z\Pi\ell_{f,B}^{\varphi}c_\varphi^{1/2}n^{-1/2}) + o_p(1),$
(ii) $S_n^\varphi = (Z'Z)^{-1/2}Z'(\Phi c_\varphi^{-1/2} + Z\Pi\ell_{f,B}^{\varphi}n^{-1/2}) + o_p(1)$, *and*
(iii) $n^{-1}Z'[\Phi : Y_2] \to_p D_Z[0_k : \Pi]$.

LEMMA 11. *Under Assumptions 1–3 and 4S,*

(i) $\hat{\nu}_{\varphi n} \to_p \nu_{\varphi f}$ *and*
(ii) $\hat{\Omega}_{22n} \to_p \Omega_{22}$.

The following lemma gives sufficient conditions for an i.i.d. sequence to satisfy Assumptions 2(d) and 4S(h) a.s.

LEMMA 12. *Suppose* $\{\psi_i : i \geq 1\}$ *is an i.i.d. sequence of nonnegative random variables with* $E\psi_i^{1+\delta} < \infty$ *for some* $\delta > 0$. *Then,*

(i) $\sum_{i=1}^{\infty} \psi_i^{1+\delta}/i^{1+\delta} < \infty$ *a.s. and*
(ii) $\max_{i \leq n} \psi_i / n \to 0$ *a.s.*

The last lemma is a Glivenko–Cantelli theorem for triangular arrays of random variables, which is used in the proof of Lemma 7. It is proved by verifying the conditions in Pollard (1990, Thm. 8.3).

LEMMA 13. *Suppose*

(i) $\{(Q_{1i}, Q_{2i}) : i \geq 1\}$ *is an i.i.d. sequence of random* $(m+1)$*-vectors with* $Q_{1i} \in R$ *and*
(ii) $\{d_i : i \geq 1\}$ *is any sequence of nonrandom* δ_d*-vectors.*

Then, for any $b < \infty$,

$$\sup_{(q_1,q_2)\in R^{m+1}} \; \sup_{t \in R^{\delta_d} : \|t\| \leq b} \left| n^{-1} \sum_{i=1}^{n} [h_{ni}(q_1,q_2,t) - Eh_{ni}(q_1,q_2,t)] \right| \to 0 \text{ a.s.},$$

where

$$h_{ni}(q_1,q_2,t) = 1(Q_{1i} \leq q_1 + d_i' t n^{-1/2}, Q_{2i} \leq q_2).$$

The proofs of Lemmas 7–13 are given after the proofs of Theorems 1 and 2.

Proof of Theorem 1. Lemma 9 and Assumption 4W(e) imply that

$$\hat{\Omega}_{\varphi n} \to_p \Omega_{\varphi g} \quad \text{and} \quad \hat{\Omega}_{\varphi n}^{-1}H(H'\hat{\Omega}_{\varphi n}^{-1}H)^{-1/2} \to_p \Omega_{\varphi g}^{-1}H(H'\Omega_{\varphi g}^{-1}H)^{-1/2}. \tag{A.1}$$

This, Lemma 8, the continuous mapping theorem, and the definitions of $(S_n^\varphi, T_n^\varphi)$ and $(S_\infty^\varphi, T_\infty^\varphi)$ combine to establish part (i).

Independence of S_∞^φ and T_∞^φ is implied by zero covariance between the normal variates N_φ and $[N_\varphi : N_2]\Omega_{\varphi g}^{-1}H$. The latter holds by the following argument. Let $N_{\varphi,j}$, $N_{2,\ell}$, and $D_{Z,j\ell}$ denote the jth element of N_φ, the ℓth row of N_2, and the (j,ℓ) element of D_Z, respectively. Let e_1 denote an $m+1$-vector of ones. The covariance between $N_{\varphi,j}$ and the ℓth row of $[N_\varphi : N_2]\Omega_{\varphi g}^{-1}H$ for $j, \ell = 1, \ldots, k$ is

$$\mathrm{Cov}(N_{\varphi,j},[N_{\varphi,\ell}:N_{2,\ell}]\Omega_{\varphi g}^{-1}H)$$

$$= Ee_1'\begin{pmatrix} N_{\varphi,j}-EN_{\varphi,j} \\ N_{2,j}'-EN_{2,j}' \end{pmatrix}[N_{\varphi,\ell}:N_{2,\ell}]\Omega_{\varphi g}^{-1}H = D_{Z,j\ell}\cdot e_1'\Omega_{\varphi g}\Omega_{\varphi g}^{-1}H = 0. \tag{A.2}$$

Parts (ii)–(iv) of the theorem follow immediately from part (i) and the continuous mapping theorem. ■

Proof of Theorem 2. The result $S_n^\varphi \to_d S_{f\infty}^\varphi$ of part (i) follows from Lemma 10(ii), Lemma 8(iii) (which does not rely on Assumption 4W), and the Lindeberg CLT applied to $n^{-1/2}Z'\Phi c_\varphi^{-1/2}$. The CLT applies by the same argument as given in the proof of Lemma 8(iv) subsequently.

The result $n^{-1/2}T_n^\varphi \to_d \alpha_T^\varphi$ (or $n^{-1/2}T_n^\varphi \to_p \alpha_T^\varphi$) is established as follows:

$$\begin{aligned}
n^{-1/2}T_n^\varphi &= n^{-1/2}(Z'Z)^{-1/2}Z'[R_\varphi c_\varphi^{-1/2}:Y_2]\hat{\Omega}_{\varphi n}^{-1}H(H'\hat{\Omega}_{\varphi n}^{-1}H)^{-1/2} \\
&= (n^{-1}Z'Z)^{-1/2}[n^{-1}Z'R_\varphi c_\varphi^{-1/2}:n^{-1}Z'Y_2]\Omega_{\varphi f}^{-1}H(H'\Omega_{\varphi f}^{-1}H)^{-1/2}+o_p(1) \\
&= D_Z^{-1/2}[n^{-1}Z'(\Phi c_\varphi^{-1/2}+Z\Pi\ell_{f,B}^\varphi n^{-1/2}):n^{-1}Z'Y_2] \\
&\quad \times \Omega_{\varphi f}^{-1}H(H'\Omega_{\varphi f}^{-1}H)^{-1/2}+o_p(1) \\
&= D_Z^{1/2}[0_k:\Pi]\Omega_{\varphi f}^{-1}H(H'\Omega_{\varphi f}^{-1}H)^{-1/2}+o_p(1) \\
&= D_Z^{1/2}\Pi(H'\Omega_{\varphi f}^{-1}H)^{1/2}+o_p(1),
\end{aligned} \tag{A.3}$$

where the second equality holds because Lemma 11 and Assumption 4S(g) imply that $\hat{\Omega}_{\varphi n}^{-1} \to_p \Omega_{\varphi f}^{-1}$, the third equality holds by Lemma 8(iii) and Lemma 10(ii), the fourth equality holds by Lemma 10(iii), and the fifth equality holds because $[0_k:\Pi] = \Pi[0_m:I_m] = \Pi H'$. The convergence of $(S_n^\varphi, n^{-1/2}T_n^\varphi)$ holds jointly because α_T^φ is a constant.

Parts (iii) and (iv) follow immediately from part (i) using the continuous mapping theorem noting that $\alpha_T^{\varphi\prime}\alpha_T^\varphi$ is pd by Assumptions 2(c), 4S(b), and 4S(g).

We now prove part (ii). Given the definition of RLR_n^φ in (3.8) and the result of Theorem 2(iii), it suffices to show that

$$\lambda_{\min}([S_n^\varphi:T_n^\varphi]'[S_n^\varphi:T_n^\varphi]) = S^{\perp\prime}S^\perp + o_p(1), \quad \text{where}$$

$$S^\perp = S_n^\varphi - T_n^\varphi(T_n^{\varphi\prime}T_n^\varphi)^{-1}T_n^{\varphi\prime}S_n^\varphi. \tag{A.4}$$

For notational simplicity, let $[S:T]$ denote $[S_n^\varphi:T_n^\varphi]$ and let $T_j \in R^{m+1}$ denote the jth column of T for $j = 1,\ldots,m$. We rotate $[S:T]$ by an orthogonal matrix $B \in R^{(m+1)\times(m+1)}$ whose first column, b_1, is designed to be such that $[S:T]b_1 = d_1S^\perp$, where d_1 is a positive scalar that equals $1+o_p(1)$. Then, we have

$$\lambda_{\min}([S:T]'[S:T]) = \lambda_{\min}(B'[S:T]'[S:T]B), \tag{A.5}$$

and the $(1,1)$ element of the matrix on the right-hand side equals $\lambda_1^2 d_1^2 S^{\perp\prime}S^\perp$.

Let b_j denote the jth column of B and let b_{ij} denote the (i,j)th element of B. Define

$$b_1 = d_1 \begin{pmatrix} 1 \\ -(T'T)^{-1}T'S \end{pmatrix} \in R^{m+1}, \tag{A.6}$$

where d_1 is a constant such that $b_1' b_1 = 1$. Next, we define the orthogonal vectors $\{b_j : j = 2, \ldots, m+1\}$ via the Gramm–Schmidt procedure applied to the vectors $b_1, e_2, \ldots, e_{m+1}$, where e_j is the jth elementary vector (whose jth element is one and whose other elements are zero). We have

$$\begin{aligned} b_2 &= d_2(e_2 - (e_2' b_1)b_1) = d_2(e_2 - b_{12} b_1), \\ b_3 &= d_3(e_3 - (e_3' b_2)b_2 - (e_3' b_1)b_1) = d_3(e_3 - b_{23} b_2 - b_{13} b_1), \end{aligned} \tag{A.7}$$

and so on, where d_j is the constant that yields $\|b_j\| = 1$ for $j = 1, \ldots, m$.

The constants $\{d_j : j = 1, \ldots, m+1\}$ satisfy

$$\begin{aligned} d_1 &= (1 + n^{-1}(n^{-1/2}S'T)(n^{-1}T'T)^{-2}(n^{-1/2}T'S))^{-1/2} = 1 + o_p(1), \\ d_2 &= (1 - b_{12}^2)^{-1/2} = 1 + o_p(1), \\ d_3 &= (1 - b_{23}^2 - b_{13}^2)^{-1/2} = 1 + o_p(1), \end{aligned} \tag{A.8}$$

and so on, using Theorem 2(i) and the fact that

$$\begin{aligned} b_{1j} &= n^{-1/2}[-d_1(n^{-1}T'T)^{-1}n^{-1/2}T'S]_j = O_p(n^{-1/2}) \quad \text{for } j = 2, \ldots, m, \\ b_{2j} &= d_2(-b_{12} b_{1j}) = O_p(n^{-1}) \quad \text{for } j = 3, \ldots, m, \\ b_{3j} &= d_3(-b_{23} b_{2j} - b_{13} b_{1j}) = O_p(n^{-1}) \quad \text{for } j = 4, \ldots, m, \end{aligned} \tag{A.9}$$

and so on.

Let $\lambda = (\lambda_1, \ldots, \lambda_{m+1})' = (\lambda_1, \tilde{\lambda}_2')' \in R^{m+1}$ be such that $\|\lambda\| = 1$. Then, we have

$$\begin{aligned} &\lambda_{\min}(B'[S:T]'[S:T]B) = \inf_{\lambda \in R^{m+1} : \|\lambda\| = 1} J(\lambda), \quad \text{where} \\ &J(\lambda) := \|[S:T]B\lambda\|^2 = \lambda_1^2 d_1^2 S^{\perp\prime} S^{\perp} + 2\lambda_1 d_1 S^{\perp\prime}[S:T][b_2 \ldots b_{m+1}]\tilde{\lambda}_2 + J_3(\lambda), \\ &J_3(\lambda) := \tilde{\lambda}_2'[b_2 \ldots b_{m+1}]'[S:T]'[S:T][b_2 \ldots b_{m+1}]\tilde{\lambda}_2. \end{aligned} \tag{A.10}$$

The cross-product summand of $J(\lambda)$ in (A.10) equals

$$2\lambda_1 d_1 [S^{\perp\prime} S : 0_{1\times m}][b_2 \ldots b_{m+1}]\tilde{\lambda}_2 = O_p(\|\tilde{\lambda}_2\|), \tag{A.11}$$

using $S^{\perp\prime}T = 0$, $(S^{\perp\prime}S)^2 \le (S^{\perp\prime}S^{\perp})S'S \le (S'S)^2 = O_p(1)$, $|b_{ij}| \le 1$, and $d_1 = 1 + o_p(1)$. For the third summand $J_3(\lambda)$ of $J(\lambda)$, we have

$$\begin{aligned} &[S:T][b_2 \ldots b_{m+1}] \\ &\quad = [d_2(T_1 - b_{12}S^{\perp}) : d_3(T_2 - b_{23} d_2(T_1 - b_{12}S^{\perp}) - b_{13}S^{\perp}) : \ldots]. \end{aligned} \tag{A.12}$$

Combining this with (A.8), (A.9), $S^{\perp\prime}T = 0$, $S^{\perp} = O_p(1)$, and $n^{-1/2}T \to_p \alpha_T^{\varphi}$ (by part (i) of the theorem), we obtain

$$0 \le J_3(\lambda) = n\tilde{\lambda}_2'(\alpha_T^{\varphi\prime}\alpha_T^{\varphi} + o_p(1))\tilde{\lambda}_2, \tag{A.13}$$

where $\alpha_T^{\varphi\prime}\alpha_T^{\varphi}$ is pd by Assumptions 2(c), 4S(b), and 4S(g).

Let $\lambda^* = (\lambda_1^*, \ldots, \lambda_{m+1}^*)' = (\lambda_1^*, \tilde{\lambda}_2^{*\prime})' \in R^{m+1}$ be an $m + 1$-vector that minimizes $J(\lambda)$ over $\lambda \in R^{m+1}$ such that $\|\lambda\| = 1$. If $\|\tilde{\lambda}_2^*\| = o_p(n^{-1})$, then

$$J(\lambda^*) = S^{\perp\prime}S^{\perp} + o_p(1) \tag{A.14}$$

by (A.10)–(A.13) and $S^{\perp\prime}S^{\perp} = O_p(1)$ by part (i) of the theorem.

On the other hand, suppose that $\|\tilde{\lambda}_2^*\| = o_p(1)$ and $\|\tilde{\lambda}_2^*\| \neq o_p(n^{-1})$; then $|\lambda_1^*| = 1 + o_p(1)$,

$$J(\lambda^*) = S^{\perp\prime}S^{\perp} + o_p(1) + J_3(\lambda^*), \quad \text{and}$$

$$0 \le J_3(\lambda^*) = n\tilde{\lambda}_2^{*\prime}(\alpha_T^{\varphi\prime}\alpha_T^{\varphi} + o_p(1))\tilde{\lambda}_2^* \neq o_p(1). \tag{A.15}$$

This contradicts the assumption that λ^* minimizes $J(\lambda)$ over λ such that $\|\lambda\| = 1$ because a different choice of λ, namely, λ such that $\|\tilde{\lambda}_2\| = o_p(n^{-1})$, yields a smaller value $J(\lambda)$ as indicated in (A.14).

Next, suppose that $\|\tilde{\lambda}_2^*\| \neq o_p(1)$. Then,

$$J(\lambda^*) = O_p(1) + J_3(\lambda^*),$$

$$0 \le J_3(\lambda^*) \neq o_p(n), \quad \text{and} \quad J(\lambda^*) \neq O_p(1) \tag{A.16}$$

by (A.10)–(A.13). In particular, for some $\varepsilon > 0$ and some (infinite) subsequence $\{\ell_n\}$ of $\{n\}$, $P(J_3(\lambda^*) > \ell_n\varepsilon) > \varepsilon$ when the sample size is ℓ_n for all $n \ge 1$. Again this is a contradiction, because a different choice of λ, namely, λ such that $\|\tilde{\lambda}_2\| = o_p(n^{-1})$, yields a smaller value $J(\lambda)$, namely, one that is $O_p(1)$ as indicated in (A.14). We conclude that $\|\tilde{\lambda}_2^*\|$ must satisfy $\|\tilde{\lambda}_2^*\| = o_p(n^{-1})$, and hence (A.14), (A.4), (A.5), and (A.10) combine to establish the result of part (ii). ■

Proof of Lemma 7. Because $E\|Q_{2i}\| < \infty$, given any $\varepsilon > 0$, there exists a constant $c_\varepsilon < \infty$ such that

$$E\|Q_{2i}\|1(\|Q_{2i}\| > c_\varepsilon) < \varepsilon. \tag{A.17}$$

Hence, using the boundedness of φ, say, by C, and Markov's inequality, we have for any $\eta > 0$ and $\varepsilon > 0$,

$$P\left(\sup_{t:\|t\|\le b}\left|n^{-1}\sum_{i=1}^{n}\varphi\left(\frac{r_i(tn^{-1/2})}{n+1}\right)Q_{2i}1(\|Q_{2i}\| > c_\varepsilon)\right| > \eta\right)$$
$$\le \frac{C}{\eta}E\|Q_{2i}\|1(\|Q_{2i}\| > c_\varepsilon) < \frac{C\varepsilon}{\eta}. \tag{A.18}$$

Therefore, without loss of generality, we can assume that Q_{2i} is bounded.

Define

$$L_{1n}(q_1,t) = n^{-1}\sum_{i=1}^{n} 1(Q_{1i} - d_i' t \le q_1) \quad \text{and}$$

$$L_{12n}(q_1,q_2,t) = n^{-1}\sum_{i=1}^{n} 1(Q_{1t} - d_i' t \le q_1, Q_{2i} \le q_2). \tag{A.19}$$

Note that

$$EL_{1n}(q_1,t) = n^{-1}\sum_{i=1}^{n} H_{Q_1}(q_1 + d_i' t) \quad \text{and}$$

$$EL_{12n}(q_1,q_2,t) = n^{-1}\sum_{i=1}^{n} H_{Q_1,Q_2}(q_1 + d_i' t, q_2). \tag{A.20}$$

Now, we have

$$\begin{aligned}
& n^{-1}\sum_{i=1}^{n}\varphi\left(\frac{r_i(t)}{n+1}\right)Q_{2i} \\
&\quad = n^{-1}\sum_{i=1}^{n}\varphi\left(\frac{1}{n+1}\sum_{j=1}^{n}1(Q_{1j} - d_j' t \le Q_{1i} - d_i' t)\right)Q_{2i} \\
&\quad = n^{-1}\sum_{i=1}^{n}\varphi\left(\frac{nL_{1n}(Q_{1i} - d_i' t, t)}{n+1}\right)Q_{2i} \\
&\quad = \iint \varphi\left(\frac{nL_{1n}(q_1,t)}{n+1}\right) q_2\, dL_{12n}(q_1,q_2,t) \\
&\quad = \iint \left[\varphi\left(\frac{nL_{1n}(q_1,t)}{n+1}\right) - \varphi\left(\frac{nEL_{1n}(q_1,t)}{n+1}\right)\right] q_2\, dL_{12n}(q_1,q_2,t) \\
&\qquad + \iint \varphi\left(\frac{nEL_{1n}(q_1,t)}{n+1}\right) q_2\, dL_{12n}(q_1,q_2,t).
\end{aligned} \tag{A.21}$$

Therefore, using the triangle inequality,

$$\begin{aligned}
& \sup_{t:\|t\|\le b}\left| n^{-1}\sum_{i=1}^{n}\varphi\left(\frac{r_i(tn^{-1/2})}{n+1}\right)Q_{2i} - n^{-1}\sum_{i=1}^{n}\varphi\left(\frac{r_i(0)}{n+1}\right)Q_{2i}\right| \\
&\quad \le A_{1n}(b) + A_{1n}(0) + A_{2n},
\end{aligned} \tag{A.22}$$

where, for $b \geq 0$,

$$A_{1n}(b) = \sup_{t:\|t\|\leq b} \Bigg| \iint \left[\varphi\left(\frac{nL_{1n}(q_1, tn^{-1/2})}{n+1} \right) - \varphi\left(\frac{nEL_{1n}(q_1, tn^{-1/2})}{n+1} \right) \right] \times q_2\, dL_{12n}(q_1, q_2, tn^{-1/2}) \Bigg| \quad \text{and}$$

$$A_{2n} = \sup_{t:\|t\|\leq b} \Bigg| \iint \varphi\left[\frac{nEL_{1n}(q_1, tn^{-1/2})}{n+1} \right] q_2\, dL_{12n}(q_1, q_2, tn^{-1/2}) - \iint \varphi\left[\frac{nEL_{1n}(q_1, 0)}{n+1} \right] q_2\, dL_{12n}(q_1, q_2, 0) \Bigg|. \tag{A.23}$$

Now, by Lemma 13,

$$\sup_{q_1 \in R} \sup_{t:\|t\|\leq b} |L_{1n}(q_1, tn^{-1/2}) - EL_{1n}(q_1, tn^{-1/2})| = \sup_{q_1 \in R} \sup_{t:\|t\|\leq b} \left| n^{-1} \sum_{i=1}^{n} [1(Q_{1i} \leq q_1 + d_i' tn^{-1/2}) - H_{Q_1}(q_1 + d_i' tn^{-1/2})] \right| \to_p 0. \tag{A.24}$$

This implies that $A_{1n}(b) \to_p 0$ and $A_{1n}(0) \to_p 0$, because φ is absolutely continuous, Q_{2i} is bounded, and $0 \leq A_{1n}(0) \leq A_{1n}(b)$.

Using the triangle inequality again, we have $A_{2n} \leq B_{1n} + B_{2n}$, where

$$B_{1n} = \sup_{t:\|t\|\leq b} \Bigg| \iint \varphi\left[\frac{nEL_{1n}(q_1, tn^{-1/2})}{n+1} \right] q_2\, dL_{12n}(q_1, q_2, tn^{-1/2}) - \iint \varphi\left[\frac{nEL_{1n}(q_1, 0)}{n+1} \right] q_2\, dL_{12n}(q_1, q_2, tn^{-1/2}) \Bigg| \quad \text{and} \tag{A.25}$$

$$B_{2n} = \sup_{t:\|t\|\leq b} \Bigg| \iint \varphi\left[\frac{nEL_{1n}(q_1, 0)}{n+1} \right] q_2\, d\{L_{12n}(q_1, q_2, tn^{-1/2}) - L_{12n}(q_1, q_2, 0)\} \Bigg|.$$

To bound B_{1n} and B_{2n}, we write

$$\begin{aligned} &\sup_{(q_1,q_2)\in R^{m+1}} \sup_{t:\|t\|\leq b} |L_{12n}(q_1, q_2, tn^{-1/2}) - L_{12n}(q_1, q_2, 0)| \\ &\quad \leq \sup_{(q_1,q_2)\in R^{m+1}} \sup_{t:\|t\|\leq b} |L_{12n}(q_1, q_2, tn^{-1/2}) - EL_{12n}(q_1, q_2, tn^{-1/2})| \\ &\qquad + \sup_{(q_1,q_2)\in R^{m+1}} \sup_{t:\|t\|\leq b} |EL_{12n}(q_1, q_2, tn^{-1/2}) - EL_{12n}(q_1, q_2, 0)| \\ &\qquad + \sup_{(q_1,q_2)\in R^{m+1}} \sup_{t:\|t\|\leq b} |EL_{12n}(q_1, q_2, 0) - L_{12n}(q_1, q_2, 0)|. \end{aligned} \tag{A.26}$$

The first and last terms on the right-hand side converge to zero a.s. by Lemma 13. The second term on the right-hand side converges to zero because it equals

$$\sup_{(q_1,q_2)} \sup_{t:\|t\|\le b} \left| n^{-1} \sum_{i=1}^{n} H_{Q_1,Q_2}(q_1 + d_i' t n^{-1/2}, q_2) - H_{Q_1,Q_2}(q_1,q_2) \right|$$

$$= \sup_{(q_1,q_2)} \sup_{t:\|t\|\le b} \left| n^{-1} \sum_{i=1}^{n} \frac{\partial H_{Q_1,Q_2}(q_1 + d_i' t^* n^{-1/2}, q_2)}{\partial q_1} d_i' t n^{-1/2} \right| = o(1), \quad \textbf{(A.27)}$$

where t^* lies between 0 and t, the first equality holds by a mean-value expansion around $t = 0$, and the second equality holds because $\partial H_{Q_1,Q_2}/\partial q_1$ is bounded (Assumption 4W(d)) and $\overline{\lim}_{n\to\infty} \sum_{i=1}^{n} \|d_i\| < \infty$. Therefore, using the boundedness of φ and Q_{2i}, we have $B_{2n} \to_p 0$.

Equation (A.27) and a mean-value expansion yield $B_{1n} \to_p 0$ because φ has a bounded first derivative by Assumption 3(a). In consequence, $A_{2n} \to_p 0$, which completes the proof of part (a).

Part (b) of the lemma follows from part (a) using a standard argument.

To prove part (c), as in part (a), we can assume that Q_{2i} is bounded without loss of generality. We have

$$n^{-1} \sum_{i=1}^{n} \varphi\left(\frac{r_i(0)}{n+1}\right) Q_{2i} = \iint \varphi\left(\frac{nL_{1n}(q_1,0)}{n+1}\right) q_2 \, dL_{12n}(q_1,q_2,0)$$

$$= \iint \varphi\left(\frac{nEL_{1n}(q_1,0)}{n+1}\right) q_2 \, dL_{12n}(q_1,q_2,0) + o_p(1)$$

$$= \iint \varphi\left(\frac{nH_{Q_1}(q_1)}{n+1}\right) q_2 \, dH_{Q_1,Q_2}(q_1,q_2) + o_p(1)$$

$$= E\varphi(H_{Q_1}(Q_{1i}))Q_{2i} + o_p(1), \quad \textbf{(A.28)}$$

where the first equality holds by (A.21) with $t = 0$, the second equality holds because $A_{1n}(0) \to_p 0$, the third equality holds by (A.20), and the fourth equality holds because $n/(n+1) \to 1$, Q_{2i} is bounded, and φ has a bounded derivative. ■

Proof of Lemma 8. We prove part (i) first. Using (2.1) and Assumption 4W(a),

$$y_{1i} - \beta_0' y_{2i} - \hat{\gamma}_n(\beta_0)' X_i$$

$$= (\beta - \beta_0)' y_{2i} - (\hat{\gamma}_n(\beta_0) - \gamma_1)' X_i + u_i$$

$$= (\beta - \beta_0)' C' \widetilde{Z}_i n^{-1/2} - (\hat{\gamma}_n(\beta_0) - \gamma_1 - \xi_1(\beta - \beta_0))' X_i + u_i + (\beta - \beta_0)' v_{2i}.$$

(A.29)

In consequence, we apply Lemma 6 with

$$\Psi_n(\hat{\tau}_n) = n^{-1}Z'R_\varphi, \qquad Q_i = u_i + (\beta - \beta_0)'v_{2i}, \qquad c_i = Z_i, \qquad d_i = (\widetilde{Z}_i', X_i')',$$

$$\hat{\tau}_n = \begin{pmatrix} -C(\beta - \beta_0)n^{-1/2} \\ \hat{\gamma}_n(\beta_0) - \gamma_1 - \xi_1(\beta - \beta_0) \end{pmatrix}, \quad \text{and } h = g. \tag{A.30}$$

Note that $\bar{c}_n = \bar{Z}_n = 0$ because X_i contains an intercept by Assumption 2(b) and $Z'X = 0$ by construction. The required conditions of Lemma 6 on d_i are satisfied by Assumption 2. The assumptions on Q_i are satisfied by Assumptions 1(a) and 4W(b) and (c). The condition $n^{1/2}\hat{\tau}_n = O_p(1)$ holds by Assumption 4W(f).

We now verify the conditions of Lemma 6 on $c_i = Z_i$. By construction, $Z_i = \widetilde{Z}_i - \widetilde{Z}'X(X'X)^{-1}X_i$, where $\widetilde{Z}'X(X'X)^{-1} \to_p D_{12}D_{22}^{-1}$ by Assumption 2(c). In consequence, by standard arguments using Assumptions 2(c) and (d), we obtain $\lim_{n\to\infty} n^{-1}\sum_{i=1}^n \|Z_i\|^2 < \infty$ and $\lim_{n\to\infty} \max_{1\le i\le n} \|Z_i\|^2/n = 0$. Hence, all of the conditions of Lemma 6 hold.

Now, using (A.30) and $\bar{Z}_n = 0$, $\dot{A}_n(0)n^{1/2}\hat{\tau}_n(\int_0^1 \varphi(x,g)\varphi(x)\,dx)^{-1}$ equals

$$n^{-1}\sum_{i=1}^n Z_i\widetilde{Z}_i'C(\beta - \beta_0) - n^{-1}\sum_{i=1}^n Z_iX_i'n^{1/2}(\hat{\gamma}_n(\beta_0) - \gamma_1 - \xi_1(\beta - \beta_0)). \tag{A.31}$$

The second summand is zero because $Z'X = 0$. The first summand equals $n^{-1}Z'ZC(\beta - \beta_0)$ because $Z'\widetilde{Z} = Z'M_X\widetilde{Z} = Z'Z$. Hence, by Lemma 6(b), we have

$$n^{-1/2}Z'R_\varphi = n^{1/2}\Psi_n(0) + n^{-1}Z'ZC(\beta - \beta_0)\int_0^1 \varphi(x,g)\varphi(x)\,dx + o_p(1). \tag{A.32}$$

(By definition, $n^{1/2}\Psi_n(0) = n^{-1/2}Z'R_\varphi^0$, where R_φ^0 is the n-vector whose ith element is $\varphi(R_i/(n+1))$ and R_i is the rank of $u_i + (\beta - \beta_0)'v_{2i}$ in $\{u_j + (\beta - \beta_0)'v_{2j} : j \le n\}$.)

Finally, Lemma 6(c) implies that

$$n^{1/2}\Psi_n(0) = n^{-1/2}Z'\Phi + o_p(1). \tag{A.33}$$

Combining (A.32), (A.33), and the definition of $\ell^\varphi_{g,\beta-\beta_0}$ in (4.5) establishes Lemma 8(i).

Lemma 8(ii) follows from part (i) and Assumption 2(c).

Lemma 8(iii) follows from Assumption 2(c) and $Z'Z = \widetilde{Z}'\widetilde{Z} - \widetilde{Z}'X(X'X)^{-1}X'\widetilde{Z}$. Positive definiteness of D_Z follows from that of D.

Lemma 8(iv) follows from the Lindeberg CLT for triangular arrays applied to $n^{-1/2}Z'[\Phi c_\varphi^{-1/2} : Y_2 - EY_2]$ plus the facts that

$$n^{-1/2}Z'(ZC\ell^\varphi_{g,\beta-\beta_0}n^{-1/2}) = D_ZC\ell^\varphi_{g,\beta-\beta_0} + o(1),$$

$$n^{-1/2}Z'EY_2 = Z'ZCn^{-1} = D_ZC + o(1), \quad \text{and}$$

$$\mathrm{Var}(n^{-1/2}\mu_1'Z'[\Phi c_\varphi^{-1/2} : Y_2 - EY_2]\mu_2) \to \mu_2'\Omega_{\varphi g}\mu_2 \cdot \mu_1'D_Z\mu_1, \tag{A.34}$$

for arbitrary fixed nonzero vectors $\mu_1 \in R^k$ and $\mu_2 \in R^{m+1}$. Note that $EZ'\Phi = 0$ because $\bar{Z}_n = 0$ and $E\varphi(U_{gi})$ does not depend on i.

The Lindeberg condition is verified for $n^{-1/2}\mu_1' Z'[\Phi c_\varphi^{-1/2} : Y_2 - EY_2]\mu_2$ (for μ_1 and μ_2 as before), as follows. Let $\zeta_i = (\varphi^2(U_{gi})c_\varphi^{-1/2}, v_{2i}')\mu_2 \in R$. For any $\varepsilon > 0$,

$$
\begin{aligned}
&n^{-1}\sum_{i=1}^{n}(\mu_1' Z_i)^2 E\zeta_i^2 1((\mu_1' Z_i)^2\zeta_i^2 > n\varepsilon) \\
&\quad \le n^{-1}\sum_{j=1}^{n}(\mu_1' Z_j)^2 \cdot E\zeta_i^2 1\left(\max_{j\le n}(\mu_1' Z_j)^2\zeta_i^2 > n\varepsilon\right) \to 0, \qquad \text{(A.35)}
\end{aligned}
$$

where the inequality uses $(\mu' Z_i)^2 \le \max_{j\le n}(\mu' Z_j)^2$ in the indicator function and the convergence to zero holds by Assumption 2, $E\|v_{2i}\|^2 < \infty$ (by Assumption 1(b)), $E\varphi^2(U_{gi}) < \infty$ (by Assumption 3), and the dominated convergence theorem. ■

Proof of Lemma 9. We prove part (i) first. Let V_2 be the $n \times m$ matrix whose ith row is v_{2i}'. Using $Z'X = 0$, we have

$$
\begin{aligned}
\hat{\nu}_{\varphi n} &= n^{-1}V_2' R_\varphi c_\varphi^{-1/2} - n^{-1}V_2' Z(n^{-1}Z'Z)^{-1}n^{-1}Z'R_\varphi c_\varphi^{-1/2} \\
&\quad - n^{-1}V_2' X(n^{-1}X'X)^{-1}n^{-1}X'R_\varphi c_\varphi^{-1/2}. \qquad \text{(A.36)}
\end{aligned}
$$

We have $n^{-1}V_2'Z \to_p 0$ and $n^{-1}V_2'X \to_p 0$ because they have mean zero and variance $O(n^{-1})$ by Assumptions 1 and 2(a) and (c). Assumption 2(c) implies that $(n^{-1}Z'Z)^{-1}$ and $(n^{-1}X'X)^{-1}$ are $O_p(1)$. Lemma 8(i) and (iv) implies that $n^{-1/2}Z'R_\varphi = O_p(1)$. These results combine to show that the second term on the right-hand side of (A.36) is $o_p(1)$. Next, we have

$$
n^{-1}\|X'R_\varphi\| = n^{-1}\left\|\sum_{i=1}^{n} X_i\varphi\left(\frac{\hat{R}_i(\beta_0)}{n+1}\right)\right\| \le Cn^{-1}\sum_{i=1}^{n}\|X_i\| = O(1) \qquad \text{(A.37)}
$$

for some constant $C < \infty$, using the triangle inequality, the boundedness of φ, and Assumption 2(c). This result and the others given previously imply that the third term on the right-hand side of (A.36) is $o_p(1)$. Hence, $\hat{\nu}_{\varphi n} = n^{-1}V_2'R_\varphi c_\varphi^{-1/2} + o_p(1)$.

We apply Lemma 7 with $Q_{1i} = u_i + (\beta - \beta_0)'v_{2i}$, $Q_{2i} = v_{2i}$, and d_i and $\hat{\tau}_n$ as in (A.30) to get

$$
\begin{aligned}
n^{-1}V_2'R_\varphi c_\varphi^{-1/2} &= E[\varphi(G(u_i + (\beta - \beta_0)'v_{2i}))v_{2i}]c_\varphi^{-1/2} + o_p(1) \\
&= \mathrm{Cov}[\varphi(G(u_i + (\beta - \beta_0)'v_{2i})), y_{2i}]c_\varphi^{-1/2} + o_p(1) = \nu_{\varphi g} + o_p(1).
\end{aligned}
\qquad \text{(A.38)}
$$

The conditions of Lemma 7 on $\hat{\tau}_n$, d_i, and (Q_{1i}, Q_{2i}) hold by Assumptions 4W(f), 2(c), and 4W(d), respectively.

Next, we prove part (ii). For simplicity, we replace $n - k - p$ by n in the definition of $\hat{\Omega}_{22n}$. We have

$$
\begin{aligned}
\hat{\Omega}_{22n} &= n^{-1}Y_2(I_n - P_Z - P_X)'Y_2 \\
&= n^{-1}(V_2'V_2 - V_2'P_Z V_2 - V_2'P_X V_2) \to_p \Omega_{22}, \qquad \text{(A.39)}
\end{aligned}
$$

where V_2 denotes the $n \times m$ matrix whose ith row is v_{2i}', $n^{-1}V_2'V_2 \to_p \Omega_{22}$ by Kolmogorov's law of large numbers for i.i.d. random variables, and $n^{-1}Z'V_2 \to_p 0$ and $n^{-1}X'V_2 \to_p 0$ because they have mean zero and variances that are $O(n^{-1})$ by Assumptions 1 and 2(a) and (c). ■

Proof of Lemma 10. We prove part (i) first. It suffices to show that $(S_n^\varphi, n^{-1/2}T_n^\varphi) \to_d (S_{f\infty}^\varphi, \alpha_T^\varphi)$ conditional on an $\{\varepsilon_i : i \geq 1\}$ sequence that satisfies certain properties, and that $\{\varepsilon_i : i \geq 1\}$ sequences satisfy these properties with probability one. Because conditional probabilities are bounded by zero and one, this implies that $(S_n^\varphi, n^{-1/2}T_n^\varphi) \to_d (S_{f\infty}^\varphi, \alpha_T^\varphi)$ unconditionally by the bounded convergence theorem. The desired properties are

$$\lim_{n\to\infty} \max_{1\leq i\leq n} \|\varepsilon_i - \bar{\varepsilon}_n\|^2 \Big/ \sum_{i=1}^n \|\varepsilon_i - \bar{\varepsilon}_n\|^2 = 0, \tag{A.40}$$

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^n \|\varepsilon_i - \bar{\varepsilon}_n\|^2 < \infty, \tag{A.41}$$

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^n \widetilde{Z}_i \varepsilon_i' = 0, \quad \text{and} \tag{A.42}$$

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^n X_i \varepsilon_i' = 0. \tag{A.43}$$

Conditions (A.40) and (A.41) hold a.s. by Assumption 4S(d), Lemma 12(ii), and Kolmogorov's strong law of large numbers. Conditions (A.42) and (A.43) hold a.s. by Assumptions 4S(d) and (h) and the strong law of large numbers of Theorem 5.2.1 of Chow and Teicher (1978, p. 121) applied with $\alpha_n = 2$. Consequently, sequences $\{\varepsilon_i : i \geq 1\}$ that satisfy (A.40)–(A.43) occur with probability one.

Using (2.1) and Assumptions 4S(a)–(c), we have

$$\begin{aligned} & y_{1i} - \beta_0' y_{2i} - \hat{\gamma}_n(\beta_0)' X_i \\ &\quad = B'\Pi'\widetilde{Z}_i n^{-1/2} - (\hat{\gamma}_n(\beta_0) - \gamma_1 - \xi_1 B n^{-1/2})' X_i + B'\varepsilon_i n^{-1/2} + (1 + \rho' B n^{-1/2}) u_i. \end{aligned} \tag{A.44}$$

Let $\zeta_n = (1 + \rho' B n^{-1/2})^{-1}$. Because $\zeta_n > 0$ for n sufficiently large, $\{\hat{R}_i(\beta_0) : i \leq n\}$ are equal to the ranks of the i.i.d. random variables $\{u_i : i \leq n\}$ plus the terms

$$\{\zeta_n B'\Pi'\widetilde{Z}_i n^{-1/2} - \zeta_n(\hat{\gamma}_n(\beta_0) - \gamma_1 - \xi_1 B n^{-1/2})' X_i + \zeta_n B'\varepsilon_i n^{-1/2} : i \leq n\}. \tag{A.45}$$

Hence, we apply Lemma 6, conditional on an $\{\varepsilon_i : i \geq 1\}$ sequence that satisfies (A.40)–(A.43), with

$$\Psi_n(\hat{\tau}_n) = n^{-1} Z' R_\varphi, \qquad Q_i = u_i, \qquad c_i = Z_i,$$

$$d_i = \begin{pmatrix} \widetilde{Z}_i \\ X_i \\ \varepsilon_i \end{pmatrix}, \qquad \hat{\tau}_n = \begin{pmatrix} -\zeta_n \Pi B n^{-1/2} \\ \zeta_n(\hat{\gamma}_n(\beta_0) - \gamma_1 - \xi_1 B n^{-1/2}) \\ -\zeta_n B n^{-1/2} \end{pmatrix}, \quad \text{and} \quad h = f. \tag{A.46}$$

The assumptions of Lemma 6 on Q_i are satisfied by Assumptions 1 and 4S(e). The required conditions for c_i are verified by the same argument as in the proof of Theorem 1. The assumptions on d_i are satisfied by Assumption 2, (A.40), and (A.41). The assumptions on $\hat{\tau}_n$ are satisfied by Assumption 4S(i) because $\zeta_n \to 1$.

Using the definitions of c_i, d_i, and $\hat{\tau}_n$, $\dot{A}_n(0)n^{1/2}\hat{\tau}_n(\int_0^1 \varphi(x,f)\varphi(x)\,dx)^{-1}$ equals

$$\zeta_n n^{-1}\sum_{i=1}^n Z_i \widetilde{Z}_i' \Pi B - \zeta_n n^{-1}\sum_{i=1}^n Z_i X_i' n^{1/2}(\hat{\gamma}_n(\beta_0) - \gamma_1 - \xi_1 B n^{-1/2})$$
$$+ \zeta_n n^{-1}\sum_{i=1}^n Z_i \varepsilon_i' B. \tag{A.47}$$

The first term in (A.47) equals $Z'Z\Pi B + o(1)$ because $\zeta_n \to 1$. The second term is zero because $Z'X = 0$. The third term equals

$$\zeta_n n^{-1}\sum_{i=1}^n \widetilde{Z}_i \varepsilon_i' B - \zeta_n(n^{-1}\widetilde{Z}'X)(n^{-1}X'X)^{-1}n^{-1}\sum_{i=1}^n X_i\varepsilon_i' B = o(1), \tag{A.48}$$

using (A.42), (A.43), and Assumption 2(c). Hence, by Lemma 6(b) and (c), we have

$$n^{-1/2}Z'R_\varphi = n^{-1/2}Z'\Phi + Z'Z\Pi B\int_0^1 \varphi(x,f)\varphi(x)\,dx + o_p(1), \tag{A.49}$$

which establishes part (i).

Lemma 10(ii) follows from part (i) and Assumption 2(c).

To establish Lemma 10(iii), we have

$$n^{-1}Z'Y_2 = n^{-1}Z'(Z\Pi + X\xi + V_2) = n^{-1}Z'Z\Pi + n^{-1}Z'V_2 \to_p D_Z\Pi, \tag{A.50}$$

where V_2 denotes the $n \times m$ matrix whose ith row is v_{2i}' and using $n^{-1}Z'V_2 \to_p 0$ because its mean is zero and its variance is $O(n^{-1})$ by Assumptions 1 and 2(a) and (c).

In addition, we have

$$n^{-1}Z'\Phi = n^{-1}Z'(\Phi - E\Phi) \to_p 0, \tag{A.51}$$

where the equality holds because $E\Phi$ is proportional to 1_n and $Z'1_n = 0$ and the convergence to 0 holds by the strong law of large numbers referenced in the previous paragraph. ■

Proof of Lemma 11. We prove part (i) first. By the same argument as in the proof of Lemma 9(i), but with Lemma 8 replaced by Lemma 10, we have $\hat{\nu}_{\varphi n} = n^{-1}V_2' R_\varphi c_\varphi^{-1/2} + o_p(1)$. As in the proof of Lemma 10(i), it suffices to show the result conditional on an $\{\varepsilon_i : i \geq 1\}$ sequence that satisfies certain properties and that $\{\varepsilon_i : i \geq 1\}$ sequences satisfy these properties with probability one. In the present case, we need the property

$$\overline{\lim_{n\to\infty}}\, n^{-1}\sum_{i=1}^n \|\varepsilon_i\| < \infty. \tag{A.52}$$

Condition (A.52) holds a.s. by Kolmogorov's strong law of large numbers using Assumption 4S(d).

Lemma 7 applied conditional on a sequence $\{\varepsilon_i : i \geq 1\}$ that satisfies (A.52), with $(Q_{1i}, Q_{2i}) = (u_i, v_{2i})$ and d_i and $\hat{\tau}_n$ as in (A.46), gives

$$n^{-1} V_2' R_\varphi c_\varphi^{-1/2} = E[\varphi(F(u_i)) v_{2i}] c_\varphi^{-1/2} + o_p(1)$$
$$= \mathrm{Cov}[\varphi(F(u_i)), y_{2i}] c_\varphi^{-1/2} + o_p(1) = \nu_{\varphi f} + o_p(1). \quad \textbf{(A.53)}$$

The conditions of Lemma 7 on $\hat{\tau}_n$, d_i, and (Q_{1i}, Q_{2i}) hold by Assumption 4S(i), condition (A.52) and Assumptions 2(c) and (d), and Assumption 4S(f), respectively.

The proof of part (ii) is the same as for Lemma 9(ii). ∎

Proof of Lemma 12. Part (i) holds because $E\sum_{i=1}^\infty \psi_i^{1+\delta}/i^{1+\delta} = E\psi_1^{1+\delta} \sum_{i=1}^\infty i^{-(1+\delta)} < \infty$ implies that $\sum_{i=1}^\infty \psi_i^{1+\delta}/i^{1+\delta} < \infty$ a.s. Part (ii) holds because the result of part (i) and Kronecker's lemma (see, e.g., Chow and Teicher, 1978, p. 111) imply that $n^{-1-\delta}\sum_{i=1}^n \psi_i^{1+\delta} \to 0$ a.s. Hence, $n^{-1-\delta}\max_{i\leq n}\psi_i^{1+\delta} \leq n^{-1-\delta}\sum_{i=1}^n \psi_i^{1+\delta} \to 0$ a.s. In turn, this gives $n^{-1}\max_{i\leq n}\psi_i \to 0$ a.s. ∎

Proof of Lemma 13. We prove the lemma by verifying the conditions of Theorem 8.3 in Pollard (1990). To match the notation in Pollard (1990), view the sequence $\{(Q_{1i}, Q_{2i}) : i \geq 1\}$ as depending on $\omega \in \Omega$, where the probability space is $\{\Omega, \mathcal{F}, \mathbb{P}\}$, and let $(Q_{1i}, Q_{2i})(\omega)$ denote the ith element of this sequence. Also, view the sequence of independent processes

$$\{h_{ni}(q_1, q_2, t) : (q_1, q_2, t) \in \mathcal{T} \subset R^{m+1+\delta_d}\} \quad \textbf{(A.54)}$$

for $i \geq 1$ as a sequence of independent processes indexed by $\tau \in \mathcal{T}$:

$$\{h_{ni}(\omega; \tau) : \tau \in \mathcal{T}\}, \quad \text{where } \tau = (q_1, q_2, t) \quad \text{and}$$
$$h_{ni}(\omega; \tau) = 1((Q_{1i}, Q_{2i})(\omega) \leq (q_1 + d_i' t n^{-1/2}, q_2)). \quad \textbf{(A.55)}$$

Each of the processes $h_{ni}(\omega; \tau)$ has envelope $H_{ni}(\omega) = 1\ \forall \omega \in \Omega$, and these envelope functions satisfy

$$\sum_{i=1}^\infty \frac{EH_{ni}}{i^2} = \sum_{i=1}^\infty \frac{1}{i^2} < \infty, \quad \textbf{(A.56)}$$

which is the first condition of Theorem 8.3 of Pollard (1990).

Now, we verify that the processes $\{h_{ni}(\omega; \tau) : \tau \in \mathcal{T}\}$ and the envelope functions $\{H_{ni}(\omega) = 1\ \forall \omega \in \Omega\}$ for $i \geq 1$ satisfy the second condition of Theorem 8.3 of Pollard (1990). For each ω, define the sets

$$\mathcal{H}_{\omega n} = \{(h_{n1}(\omega; \tau), \ldots, h_{nn}(\omega; \tau)) \in R^n : \tau \in \mathcal{T}\} \quad \text{and}$$
$$\alpha \odot \mathcal{H}_{\omega n} = \{(\alpha_1 h_{n1}(\omega; \tau), \ldots, \alpha_n h_{nn}(\omega; \tau)) \in R^n : \tau \in \mathcal{T}\} \quad \textbf{(A.57)}$$

for some $\alpha \in R^n$.

Denote the largest number κ for which there exist points in a subset of a metric space T with $d(t_i,t_j) > \varepsilon$, for $i \neq j$, by $D(\varepsilon,T)$. The number $D(\varepsilon,T)$ is called the *packing number*. Denote the ℓ_1 distance in R^n by $|\alpha|_1 = \sum_{i=1}^n |\alpha_i|$.

By Definition 7.9 of Pollard (1990), $\{h_{ni}(\omega;\tau) : \tau \in \mathcal{T}\}$ for $i \geq 1$ is *manageable* with respect to the envelopes $\{H_{ni}(\omega) = 1 \ \forall \omega \in \Omega\}$ for $i \geq 1$ if there exists a function $\lambda(\varepsilon)$ such that

1. $\int_0^1 \sqrt{\lambda(\varepsilon)}d\varepsilon < \infty$,
2. $D(\varepsilon|\alpha|_1, \alpha \odot \mathcal{H}_{\omega n}) \leq \lambda(\varepsilon)$ for $0 < \varepsilon \leq 1$, all $\omega \in \Omega$, all vectors of nonnegative weights α, and all n. (Because $H_{ni}(\omega) = 1 \ \forall \omega$ we have that $|\alpha \odot \mathbf{H}|_1 = |\alpha|_1$, where $\mathbf{H} = (H_{ni}(\omega), \ldots, H_{nn}(\omega))$.)

The second condition of Theorem 8.3 of Pollard (1990) is that $\{h_{ni}(\omega;\tau) : \tau \in \mathcal{T}\}$ for $i \geq 1$ is *manageable* with respect to the envelopes $\{H_{ni}(\omega) = 1 \ \forall \omega \in \Omega\}$ for $i \geq 1$.

For any ω, the class $\mathcal{H}_{\omega n}$ belongs to a larger class of functions $\mathcal{H}$ defined by

$$\mathcal{H} = \{h | h(q_1,q_2) = 1((q_1,q_2) \in C) \quad \text{for } C \text{ of the type } (-\infty, c_1] \times (-\infty, c_3]^m\}. \quad \textbf{(A.58)}$$

The collection of all cells $(-\infty,c_1] \times (-\infty,c_2]^m$ has *VC*-index equal to $(m+1)+1$, which implies that the class of indicator functions $\mathcal{H}$ has *VC*-index equal to $(m+1)+1$ also (see van der Vaart and Wellner, 1996). From Theorem 2.6.7 in van der Vaart and Wellner (1996), it follows that there exist constants A_1 and W such that

$$N(\varepsilon/2, \mathcal{H}) \leq A_1(\varepsilon/2)^{-W} \quad \text{for } 0 < \varepsilon \leq 2, \quad \textbf{(A.59)}$$

where $N(\varepsilon/2,\mathcal{H})$ is the smallest number of closed balls with radius $\varepsilon/2$ that covers $\mathcal{H}$. The number $N(\varepsilon/2,\mathcal{H})$ is called the *covering number* of $\mathcal{H}$. Because $D(\varepsilon,\mathcal{H}) \leq N(\varepsilon/2,\mathcal{H})$ and $\mathcal{H}_{\omega n} \subset \mathcal{H}$ for every n and ω, it follows that there exist constants A_2 and W,

$$D(\varepsilon, \mathcal{H}_{\omega n}) \leq A_2 \varepsilon^{-W} \quad \text{for } 0 < \varepsilon \leq 1. \quad \textbf{(A.60)}$$

Now, using an argument similar to the one used in the proof of Theorem 4.8 of Pollard (1990), we can show that for all n and ω, there exist constants A_3 and W such that

$$D(\varepsilon|\alpha|_1, \alpha \odot \mathcal{H}_{\omega n}) \leq A_3 \varepsilon^{-W} \quad \text{for } 0 < \varepsilon \leq 1. \quad \textbf{(A.61)}$$

Take $\mathcal{H}_{\omega n}^*$ to be the set of rescaled coordinates

$$h_{ni}^* = \frac{\alpha_i h_{ni}}{2\sum_{i=1}^n \alpha_i} \quad \text{for } h_n \in \mathcal{H}_{\omega n}. \quad \textbf{(A.62)}$$

Let h_1^* and h_2^* in $\mathcal{H}_{\omega n}^*$ be rescaled coordinates of h_1 and h_2 in $\mathcal{H}_{\omega n}$. Then,

$$|h_1^* - h_2^*|_1 \leq \sum_{i=1}^n \left| \frac{\alpha_i}{2\sum_{i=1}^n \alpha_i} \right| |h_{1i} - h_{2i}| \leq |h_1 - h_2|_1. \quad \textbf{(A.63)}$$

Hence,

$$D(\varepsilon, \mathcal{H}^*_{\omega n}) \leq A_2 \varepsilon^{-W} \quad \text{for } 0 < \varepsilon \leq 1. \tag{A.64}$$

Now, we have

$$|h_1^* - h_2^*|_1 < \varepsilon/2 \Leftrightarrow \sum_{i=1}^n \left| \frac{\alpha_i}{2\sum_{i=1}^n \alpha_i} (h_{1i} - h_{2i}) \right| < \varepsilon/2$$

$$\Leftrightarrow |\alpha \odot h_1 - \alpha \odot h_2|_1 < \varepsilon \left| \sum_{i=1}^n \alpha_i \right| < \varepsilon |\alpha|_1. \tag{A.65}$$

Therefore, (A.61) holds with $A_3 = 2^W A_2$. This establishes that $\{h_{ni}(\omega;\tau) : \tau \in \mathcal{T}\}$ is *manageable* with respect to the envelopes $\{H_{ni}(\omega) = 1 \ \forall \omega \in \Omega\}$. Theorem 8.3 of Pollard (1990) then gives

$$n^{-1} \sup_{\tau \in \mathcal{T}} \left| \sum_{i=1}^n (h_{ni}(\omega;\tau) - Eh_{ni}(\omega;\tau)) \right| \to 0 \quad \text{a.s.,} \tag{A.66}$$

which gives the result of the lemma. ■

Proof of Lemma 1. By the definition of $\hat{\gamma}_n^{LS}(\beta_0)$, we have

$$\hat{\gamma}_n^{LS}(\beta_0) = (n^{-1}X'X)^{-1} n^{-1} \sum_{i=1}^n X_i((\beta - \beta_0)' y_{2i} + \gamma_1' X_i + u_i)$$

$$= \gamma_1 + \xi(\beta - \beta_0) + (n^{-1}X'X)^{-1} n^{-1} \sum_{i=1}^n X_i(u_i + (\beta - \beta_0)' v_{2i}), \tag{A.67}$$

using $y_{2i} = \Pi' Z_i + \xi' X_i + v_{2i}$ and $X'Z = 0$. Hence, we obtain

$$n^{1/2}(\hat{\gamma}_n^{LS}(\beta_0) - \gamma_1 - \xi_1(\beta - \beta_0)) = (n^{-1}X'X)^{-1} n^{-1/2} \sum_{i=1}^n X_i(u_i + (\beta - \beta_0)' v_{2i})$$
$$+ n^{1/2}(\xi - \xi_1)(\beta - \beta_0). \tag{A.68}$$

Assumption 2(c) implies that $(n^{-1}X'X)^{-1} = D_{11}^{-1} + o(1)$. The second multiplicand of the first term on the right-hand side of (A.68) is asymptotically normal by the Lindeberg central limit theorem using Assumptions 1 and 2 and $Eu_i^2 < \infty$. The Lindeberg condition is verified by an argument analogous to that in (A.35), where $E(u_i + (\beta - \beta_0)' v_{2i})^2 < \infty$ by Assumption 1(b) and $Eu_i^2 < \infty$. Thus, the first term on the right-hand side of (A.68) is $O(1)$. Next, we have

$$\xi - \xi_1 = (n^{-1}X'X)^{-1} n^{-1} X' \tilde{Z} \Pi = (n^{-1}X'X)^{-1} n^{-1} X' \tilde{Z} C n^{-1/2} = O(n^{-1/2}), \tag{A.69}$$

where the first equality holds by the definition of ξ stated following (2.3), the second equality holds by Assumption 4W(a), and the last equality holds by Assumption 2(c). Assumption 4W(b) states that $\beta - \beta_0$ is a constant. Hence, $n^{1/2}(\xi - \xi_1)(\beta - \beta_0) = O(1)$, which completes the proof of the lemma. ■

Proof of Corollary 1. We have

$$
\begin{aligned}
P(RLR_n^{\varphi} > \kappa_{LR,\alpha}(T_n^{\varphi\prime}T_n^{\varphi}, k, m)) &= P(LR_\infty(S_n^{\varphi}, T_n^{\varphi}) > \kappa_{LR,\alpha}(T_n^{\varphi\prime}T_n^{\varphi}, k, m)) \\
&\to P(LR_\infty(S_\infty^{\varphi}, T_\infty^{\varphi}) > \kappa_{LR,\alpha}(T_\infty^{\varphi\prime}T_\infty^{\varphi}, k, m)) \\
&= \int P(LR_\infty(S_\infty^{\varphi}, t) > \kappa_{LR,\alpha}(t't, k, m))\, dF_{T_\infty^{\varphi}}(t) = \alpha,
\end{aligned} \tag{A.70}
$$

where $F_{T_\infty^{\varphi}}(\cdot)$ is the df of T_∞^{φ}, the convergence holds by Theorem 1(i) and the continuous mapping theorem, the second equality holds by the independence of S_∞^{φ} and T_∞^{φ}, and the last equality holds by the definition of $\kappa_{LR,\alpha}(t't, k, m)$ in (3.10) and the fact that $S_\infty^{\varphi} \sim N(0, I_k)$ under the null by (4.6). ■

Proof of Lemma 2. The proof is very similar to that of Theorem 1 with $Yb_0\hat{\sigma}_n^{-1}$ in place of Φc_φ^{-1}. First, by the same proof as for Lemma 9(ii) but with (Y_2, V_2) replaced by (Y, V), we get $\hat{\Omega}_n \to_p \Omega$, where V is the $n \times (m+1)$ matrix with ith row equal to v_i'. This implies $\hat{\sigma}_n^2 \to_p b_0'\Omega b_0 = \sigma_g^2$ and $\hat{\Omega}_{*n} \to_p \Omega_g$. Next, we need the following analogues of Lemma 8(i), (ii), and (iv):

$$
n^{-1/2}Z'Yb_0 = n^{-1/2}Z'(Yb_0 - EYb_0 + ZC(\beta - \beta_0)n^{-1/2}), \tag{A.71}
$$

$$
S_n = (Z'Z)^{-1/2}Z'(Yb_0\sigma_g^{-1} - EYb_0\sigma_g^{-1} + ZC(\beta - \beta_0)\sigma_g^{-1}n^{-1/2}) + o_p(1), \tag{A.72}
$$

$$
n^{-1/2}Z'[(Vb_0\sigma_g^{-1} + ZC(\beta - \beta_0)\sigma_g^{-1}n^{-1/2}) : Y_2] \to_d [N_1 : N_2], \tag{A.73}
$$

where (A.71) holds by (2.1), (2.3), Assumption 1(a), and $Z'X = 0$, (A.72) holds by (A.71) and $\hat{\sigma}_n^2 \to_p \sigma_g^2$, and (A.73) holds by the same proof as that of Lemma 8(iv) (given earlier) except with $\Phi c_\varphi^{-1/2}$, $\ell_{g,\beta-\beta_0}^{\varphi}$, and $\varphi(U_{gi})c_\varphi^{-1/2}$ replaced by $Vb_0\sigma_g^{-1}$, $(\beta - \beta_0)\sigma_g^{-1}$, and $v_{2i}'b_0\sigma_g^{-1}$, respectively, and with $E(v_{2i}'b_0)^2 < \infty$ by the assumption that Ω is well defined. Given these analogues of Lemma 8(i), (ii), and (iv), the rest of the proof of Lemma 2 is the same as that of Theorem 1. ■

Proof of Lemma 3. We establish part (i) first. Let σ_L^2 denote the variance of the df L. Because $\varphi(x) = L^{-1}(x)$, $c_\varphi = \sigma_L^2$ by change of variables. Also, we have

$$
\begin{aligned}
&\sigma_L^2 = \mathrm{Var}((u_i + (\beta - \beta_0)'v_{2i})\kappa) = \sigma_g^2\kappa^2, \quad \kappa = \sigma_L/\sigma_g, \quad \text{and} \\
&G(x) = L(\kappa x) = L(\sigma_L x/\sigma_g).
\end{aligned} \tag{A.74}
$$

Combining these results gives the result of part (i):

$$
\begin{aligned}
\varphi(U_{gi})c_\varphi^{-1/2} &= \varphi(G[(u_i + (\beta - \beta_0)'v_{2i})])c_\varphi^{-1/2} \\
&= L^{-1}(L[\sigma_L(u_i + (\beta - \beta_0)'v_{2i})/\sigma_g])\sigma_L^{-1} \\
&= (u_i + (\beta - \beta_0)'v_{2i})/\sigma_g.
\end{aligned} \tag{A.75}
$$

Part (i) implies that $\nu_{\varphi g} = \nu_g$, and so $\Omega_{\varphi g} = \Omega_g$ and part (ii) holds.

For part (iii), we have

$$\int_0^1 \varphi(x,g)\varphi(x)\,dx = -\int_0^1 \frac{g'(G^{-1}(x))}{g(G^{-1}(x))} L^{-1}(x)\,dx = -\int_{-\infty}^{\infty} g'(y)L^{-1}(G(y))\,dy$$
$$= -\int_{-\infty}^{\infty} g'(y)L^{-1}(L(\kappa y))\,dy = -\kappa\int_{-\infty}^{\infty} g'(y)y dy = \kappa = \sigma_L/\sigma_g, \tag{A.76}$$

where the second equality holds by change of variables with $y = G^{-1}(x)$, the third and last equalities hold by (A.74), and the fourth equality holds by integration by parts. Combining (A.76) with $c_\varphi = \sigma_L^2$ establishes part (iii).

Part (iv) follows from parts (ii) and (iii) and the definitions of N_φ, N_1, S_∞^φ, S_∞, T_∞^φ, and T_∞. ∎

Proof of Lemma 4. The proof is like that of Theorem 2 with $Yb_0\hat\sigma_n^{-1}$ in place of $R_\varphi c_\varphi^{-1}$. By essentially the same proof as for Lemma 9(ii) but with (Y_2, V_2) replaced by (Y, V), we get $\hat\Omega_n - Ev_iv_i' \to_p 0$. Under Assumption 4S(a), we have

$$v_{1i} = u_i + (\beta_0 + Bn^{-1/2})'v_{2i},$$
$$Ev_iv_i' \to E\begin{pmatrix} u_i + \beta_0'v_{2i} \\ v_{2i}\end{pmatrix}\begin{pmatrix} u_i + \beta_0'v_{2i} \\ v_{2i}\end{pmatrix}',$$
$$b_0'E\begin{pmatrix} u_i + \beta_0'v_{2i} \\ v_{2i}\end{pmatrix}\begin{pmatrix} u_i + \beta_0'v_{2i} \\ v_{2i}\end{pmatrix}'b_0 = Eu_i^2 = \sigma_f^2, \quad \text{and}$$
$$H'E\begin{pmatrix} u_i + \beta_0'v_{2i} \\ v_{2i}\end{pmatrix}\begin{pmatrix} u_i + \beta_0'v_{2i} \\ v_{2i}\end{pmatrix}'b_0 = Ev_{2i}u_i. \tag{A.77}$$

In consequence,

$$\hat\sigma_n^2 = b_0'\hat\Omega_n b_0 \to_p \sigma_f^2, \qquad \hat\nu_n = H'\hat\Omega_n b_0\hat\sigma_n^{-1} \to_p Ev_{2i}u_i\sigma_f^{-1} = \nu_f, \quad \text{and} \quad \hat\Omega_{*n} \to_p \Omega_f. \tag{A.78}$$

We need the following analogue of Lemma 10(ii):

$$S_n = (Z'Z)^{-1/2}Z'(Yb_0\sigma_f^{-1} - EYb_0\sigma_f^{-1} + ZB\sigma_f^{-1}n^{-1/2}) + o_p(1), \tag{A.79}$$

which holds by (2.1), (2.3), Assumption 1(a), $Z'X = 0$, and $\hat\sigma_n^2 \to_p \sigma_f^2$. Next, an analogue of (A.3) holds with $\hat\Omega_{\varphi n}$, $\Omega_{\varphi f}$, and $n^{-1}Z'R_\varphi$ replaced by $\hat\Omega_{*n}$, Ω_f, and $n^{-1}Z'Yb_0 = n^{-1}Z'(Vb_0 + Z\Pi Bn^{-1/2})$, respectively, using the result that $\hat\Omega_{*n} \to_p \Omega_f$ and the fact that $n^{-1}Z'Vb_0 \to_p 0$ because its mean is zero and its variance is $O(n^{-1})$. Given these analogues of Lemma 10(ii) and (A.3), the rest of the proof of Lemma 4 is the same as that of Theorem 2. ∎

Proof of Lemma 5. The proofs of parts (i)–(iii) are analogous to those of parts (i)–(iii) of Lemma 3. Part (iv) follows from parts (ii) and (iii) and the definitions of $S_{f\infty}^\varphi$, $S_{f\infty}$, α_T^φ, and α_T. ∎

Proof of an Alternative Expression for T_n. We now provide a proof of (3.6), which gives an alternative expression for T_n from its definition in (2.6). Let $M = [b_0\hat\sigma_{gn}^{-1} : H] \in R^{(m+1)\times(m+1)}$. Straightforward calculations yield

$$YM = [Yb_0 \hat{\sigma}_{gn}^{-1} : Y_2], \qquad M'A_0 = H, \qquad \hat{\Omega}_{*n} = M'\hat{\Omega}_n M, \quad \text{and}$$

$$A_0'\hat{\Omega}_n^{-1}A_0 = A_0'M(M^{-1}\hat{\Omega}_n^{-1}M'^{-1})M'A_0 = H'\hat{\Omega}_{*n}^{-1}H. \tag{A.80}$$

Using the definition of T_n in (2.6), we have

$$\begin{aligned} T_n &= (Z'Z)^{-1/2}Z'(YM)(M^{-1}\hat{\Omega}_n^{-1}M'^{-1})(M'A_0)(A_0'\hat{\Omega}_n^{-1}A_0)^{-1/2} \\ &= (Z'Z)^{-1/2}Z'[Yb_0\hat{\sigma}_{gn}^{-1} : Y_2]\hat{\Omega}_{*n}^{-1}H(H'\hat{\Omega}_{*n}^{-1}H)^{-1/2}, \end{aligned} \tag{A.81}$$

where the second equality uses (A.80). The right-hand side of (A.81) is the expression in (3.6). ■

Asymptotic power calculations. Next, we describe the simulation method used to calculate the weak IV asymptotic power reported in Table 1. The first step is to compute $\xi(\varphi^{NS}, g)$ and $\xi(\varphi^{WS}, g)$ when g is the density of $u_i + \beta v_{2i}$ for v_{2i} defined in (4.18) and u_i and ε_i are independent with distribution F. The idea is to use the fact that the Hodges–Lehmann estimator of location based on φ (which is defined, e.g., in Hettmansperger, 1984, eqn. (2.8.12), p. 99) has asymptotic variance equal to $1/\xi(\varphi, g)$ (see Hettmansperger, 1984, Thm. 2.6.5 and eqn. (2.9.4), pp. 76, 105). We compute the Hodges–Lehmann estimators based on φ^{NS} and φ^{WS} for 30,000 independent samples of a location model with density g and sample size 100,000. This yields 30,000 Hodges–Lehmann estimates $\hat{\theta}^{NS}$ and $\hat{\theta}^{WS}$. The reciprocals of the sample variances of these estimates yield estimated values of $\xi(\varphi^{NS}, g)$ and $\xi(\varphi^{WS}, g)$, denoted $\tilde{\xi}(\varphi^{NS}, g)$ and $\tilde{\xi}(\varphi^{WS}, g)$.

The second step is to compute the matrices $\Omega_{\varphi^{NS}g}$, $\Omega_{\varphi^{WS}g}$, and Ω_g and the scalar σ_g^2 defined in (4.1) and (4.10). The df $G(x)$ is approximated by the empirical df of 100,000 i.i.d. observations with distribution G (independent of the preceding random variables), call it $\widetilde{G}(x)$. Using the same observations, σ_g^2 is estimated by the sample variance, denoted $\tilde{\sigma}_g^2$. Next, $\nu_{\varphi,g}$ is estimated by $\tilde{\nu}_{\varphi,g} = R^{-1}\sum_{i=1}^R(\tilde{v}_{2i} - \bar{\tilde{v}}_{2R})\varphi(\widetilde{G}(\widetilde{X}_i))c_\varphi^{-1/2}$, where $R = 40{,}000$ for all distributions except the uniform, $R = 100{,}000$ for the uniform distribution, $\bar{\tilde{v}}_{2R} = R^{-1}\sum_{i=1}^R \tilde{v}_{2i}$, $\tilde{v}_{2i} = (1 - \rho_{uv_2}^2)^{1/2}\tilde{\varepsilon}_i + \rho_{uv_2}\tilde{u}_i$, $\widetilde{X}_i = \tilde{u}_i + \beta\tilde{v}_{2i}$, $\tilde{u}_i$ and $\tilde{\varepsilon}_i$ are independent with distribution F, are independent of $\widetilde{G}(x)$, and are i.i.d. across $i = 1, \ldots, R$, and $\varphi = \varphi^{NS}, \varphi^{WS}$. The term ν_g is estimated by the sample covariance between $\tilde{v}_{2i}$ and $\widetilde{X}_i\tilde{\sigma}_g^{-1}$, denoted $\tilde{\nu}_g$, and ω_{22} is estimated by the sample variance of $\tilde{v}_{2i}$, denoted $\tilde{\omega}_{22}$. The matrices $\widetilde{\Omega}_{\varphi^{NS}g}$, $\widetilde{\Omega}_{\varphi^{WS}g}$, and $\widetilde{\Omega}_g$ are constructed using $\tilde{\nu}_{\varphi^{NS}g}$, $\tilde{\nu}_{\varphi^{WS}g}$, $\tilde{\nu}_g$, and $\tilde{\omega}_{22}$.

The third step is to compute 5,000 independent observations of (i) two independent k-variate normals $(\tilde{S}_\infty^\varphi, \widetilde{T}_\infty^\varphi)$ with covariance matrices equal to I_k and means given by $\lambda^{1/2}\beta\tilde{\xi}(\varphi, g)e_1$ and $\lambda^{1/2}(\beta\tilde{\xi}^{1/2}(\varphi, g), 1)\widetilde{\Omega}_{\varphi g}^{-1}e_2(e_2'\widetilde{\Omega}_{\varphi g}^{-1}e_2)^{-1/2}e_1$, respectively, for $\varphi = \varphi^{NS}, \varphi^{WS}$, where $e_1 = (1, 0, \ldots, 0)' \in R^k$ and $e_2 = (0, 1)'$, and (ii) two independent k-variate normals $(\tilde{S}_\infty, \widetilde{T}_\infty)$ with covariance matrices equal to I_k and means as in (i) but with $\tilde{\sigma}_g^{-1}$ in place of $\tilde{\xi}^{1/2}(\varphi, g)$. The same normal random variables were used for $(\tilde{S}_\infty^{\varphi^{WS}}, \widetilde{T}_\infty^{\varphi^{WS}})$, $(\tilde{S}_\infty^{\varphi^{NS}}, \widetilde{T}_\infty^{\varphi^{NS}})$, and $(\tilde{S}_\infty, \widetilde{T}_\infty)$—just the means are different.

The last step is to compare each of the 5,000 WS-RCLR, NS-RCLR, CLR, LM, and AR test statistics based on $(\tilde{S}_\infty^{\varphi^{WS}}, \widetilde{T}_\infty^{\varphi^{WS}})$, $(\tilde{S}_\infty^{\varphi^{NS}}, \widetilde{T}_\infty^{\varphi^{NS}})$, and $(\tilde{S}_\infty, \widetilde{T}_\infty)$ with the appropriate conditional critical value (determined by simulation) or unconditional critical value to determine whether the test rejects the null hypothesis. The fraction that rejects the null hypothesis is the reported power in Table 1.