S1. PROOF OF THE ASYMPTOTIC SIZE RESULT FOR GMS

In this section, we reiterate and prove Theorem 1 of the main paper. In addition, at the end of this section, the proof of Lemma 1 is given.

The following lemmas are used in the proof of Theorem 1. The first lemma uses the following notation. Suppose \( \pi = (\pi_1, \pi_2) \in R_{+, \infty}^p \times R_{[1, \infty)}^q \), where \( \pi_2 = (\pi_{2,1}, \pi_{2,2}) \) and \( \pi_{2,2} = \text{vech}(\Omega_{\pi_{2,2}}) \) for some \( k \times k \) correlation matrix \( \Omega_{\pi_{2,2}} \). Given \( \pi \), define \( \pi^* = (\pi_{1,j}^*, \pi_{2,j}^*) \) for some \( j = 1, \ldots, p \), \( \pi_{1,j}^* = \pi_{1,j} + \frac{\sqrt{n}}{\sqrt{\gamma n}} \), and let \( \pi^* = (\pi_{1,1}^*, \ldots, \pi_{1,p}^*) \). Define \( \pi' = (\pi_1', \pi_2) \) and let \( c_{\pi'}(1 - \alpha) \) denote the \( 1 - \alpha \) quantile of \( S(\Omega_{\pi_{2,2}}^{1/2} Z^* + (\pi_1', 0_v), \Omega_{\pi_{2,2}}) \), where \( Z^* \sim N(0_k, I_k) \) and, by definition, if \( \pi_{1,j}^* = \infty \), then the \( j \)th element of \( \Omega_{\pi_{2,2}}^{1/2} Z^* + (\pi_1', 0_v) \) equals \( \infty \) for \( j = 1, \ldots, p \). Because there is a one-to-one mapping between \( (\theta, F) \in F \) and \( \gamma \in \Gamma \), we write \( E_F \) and \( E_{\gamma} \) interchangeably.

LEMMA 2: Suppose Assumptions 1–3, GMS1, and GMS2 hold and \( 0 < \alpha < 1/2 \). Let \( \{\gamma_{n,h} = (\gamma_{n,h,1}, \gamma_{n,h,2}, \gamma_{n,h,3}) : n \geq 1\} \) be a sequence of points in \( \Gamma \) that satisfies (i) \( n^{1/2}/\gamma_{n,h,1} \rightarrow h_1 \) for some \( h_1 \in R_{+, \infty}^p \), (ii) \( \kappa_n n^{1/2}/\gamma_{n,h,1} \rightarrow \pi_1 \) for some
\(\pi_1 \in R^{p}_{+}, \infty,\) and (iii) \(\gamma_{n,h,2} \to h_2\) for some \(h_2 \in R^d_{(\leq \infty)}\). Let \(h = (h_1, h_2), \pi = (\pi_1, \pi_2),\) and \(\pi_2 = h_2.\) Then (a)–(c) hold:

(a) \(\widehat{c}_n(\theta_{n,h}, 1 - \alpha) \geq c^*_n \) a.s. for all \(n\) for a sequence of random variables \(\{c^*_n : n \geq 1\}\) that satisfies \(c^*_n \to c_{\pi,1}(1 - \alpha)\) under \(\{\gamma_{n,h} : n \geq 1\}\), where \(\gamma_{n,h,2} = (\theta_{n,h}, \gamma_{n,h,2}).\)

(b) \(\lim_{n \to \infty} P_{\gamma_{n,h}}(T_n(\theta_{n,h}) \leq \widehat{c}_n(\theta_{n,h}, 1 - \alpha)) \geq 1 - \alpha.\)

(c) For any subsequence \(\{w_n : n \geq 1\}\) of \(\{n\}\), the results of parts (a) and (b) hold with \(w_n\) in place of \(n\) provided conditions (i)–(iii) above hold with \(w_n\) in place of \(n\).

**Lemma 3:** Suppose Assumptions 1–3, 7, and GMS1–GMS4 hold and \(0 < \alpha < 1/2.\) Let \((\theta^*, F^*)\) be an element of \(\mathcal{F}\) for which Assumption 7 applies, let \(\gamma^*\) be the value in \(\Gamma\) that corresponds to \((\theta^*, F^*) \in \mathcal{F}\), and let \(h^* = (h^*_1, h^*_2)\) be defined by \(h^*_j = (h^*_1, \ldots, h^*_p),\) where \(h^*_j, j = \infty\) if \(\gamma_{1,j} > 0\) and \(h^*_j = 0\) if \(\gamma_{1,j} = 0\) for \(j = 1, \ldots, p),\) and \(h^*_2 = \gamma^*_{2}.\) Let \(c_{h^*}(1 - \alpha)\) denote the \(1 - \alpha\) quantile of the distribution of \(S(\Omega^0_{\delta,2}Z^* + (h^*_1, 0), \Omega^0_{\delta,2}).\) When the true distribution is determined by \(\gamma^*\) for all \(n,\) the following results hold:

(a) \(\widehat{c}_n(\theta^*, 1 - \alpha) \to_{P} c_{h^*}(1 - \alpha).\)

(b) \(\lim_{n \to \infty} P_{\gamma^*}(T_n(\theta^*) \leq \widehat{c}_n(\theta^*, 1 - \alpha)) = 1 - \alpha.\)

**Proof of Theorem 1:** First, we prove part (a). Let \(CP_n(\gamma) = P_{\gamma}(T_n(\theta) \leq \widehat{c}_n(\theta, 1 - \alpha))\), where \(\gamma = (\gamma_1, \gamma_2, \gamma_3), \gamma_2 = (\gamma_{2,1}, \gamma_{2,2}),\) and \(\gamma_{2,1} = \theta.\) Let \(\gamma^* = (\gamma_{n,1}, \gamma_{n,2}, \gamma_{n,3}) \in \Gamma : n \geq 1\) be a sequence such that \(\lim_{n \to \infty} CP_n(\gamma^*_n) = \lim_{n \to \infty} \inf_{\gamma \in \Gamma} CP_n(\gamma) (= \text{AsyCS}).\) Such a sequence always exists. Let \(\{u_n : n \geq 1\}\) be a subsequence of \(\{n\}\) such that \(\lim_{n \to \infty} CP_{u_n}(\gamma^*_n)\) exists and equals \(\lim_{n \to \infty} CP_n(\gamma^*_n) = \text{AsyCS}.\) Such a subsequence always exists.

Let \(\gamma^*_{n,1,j}\) denote the \(j\)th component of \(\gamma^*_n,\) for \(j = 1, \ldots, p.\) Either

(i) \(\limsup_{n \to \infty} u_{n,1,1,j} < \infty\) or

(ii) \(\limsup_{n \to \infty} u_{n,1,1,j} = \infty.\)

If (i) holds, then for some subsequence \(\{w_n\}\) of \(\{u_n\},\)

\[
\kappa_{\pi_1}^{-1}w_{n,1/2}^{1/2}\gamma_{\pi_1,1,j} \to 0, \\
w_{n,1/2}^{1/2}\gamma_{\pi_1,1,j} \to h_{1,j} \text{ for some } h_{1,j} \in R_+. 
\]

If (ii) holds, then either

(i)(a) \(\limsup_{n \to \infty} \kappa_{\pi_1}^{-1}u_{n,1/2}\gamma_{u_1,1,j} < \infty\) or

(ii)(b) \(\limsup_{n \to \infty} \kappa_{\pi_1}^{-1}u_{n,1/2}\gamma_{u_1,1,j} = \infty.\)

If (ii)(a) holds, then for some subsequence \(\{w_n\}\) of \(\{u_n\},\)

\[
\kappa_{\pi_1}^{-1}w_{n,1/2}^{1/2}\gamma_{\pi_1,1,j} \to \pi_{1,j} \text{ for some } \pi_{1,j} \in R_+, \\
w_{n,1/2}^{1/2}\gamma_{\pi_1,1,j} \to h_{1,j} \text{ where } h_{1,j} = \infty. 
\]
If (ii)(b) holds, then for some subsequence \( \{w_n\} \) of \( \{u_n\} \),

\[
\begin{align*}
(S1.3) \quad & \kappa_{u_n}^{-1} w_n^{1/2} \gamma_{u_n,1,j}^* \to \pi_{1,j}, \text{ where } \pi_{1,j} = \infty, \\
& w_n^{1/2} \gamma_{u_n,1,j}^* \to h_{1,j}, \text{ where } h_{1,j} = \infty.
\end{align*}
\]

In addition, for some subsequence \( \{w_n\} \) of \( \{u_n\} \),

\[
(S1.4) \quad \gamma_{w_n,2}^* \to h_2 \quad \text{for some } h_2 \in \text{cl}(I_2).
\]

By taking successive subsequences over the \( p \) components of \( \gamma_{u_n,1}^* \) and \( \gamma_{u_n,2}^* \), we find that there exists a subsequence \( \{w_n\} \) of \( \{u_n\} \) such that for each \( j = 1, \ldots, p \), exactly one of the cases (S1.1)–(S1.3) applies and (S1.4) holds. In consequence, conditions (i)–(iii) of Lemma 2 hold. Hence,

\[
(S1.5) \quad \liminf_{n \to \infty} CP_{w_n}(\gamma_{w_n}^*) \geq 1 - \alpha
\]

by Lemma 2. Since \( \lim_{n \to \infty} CP_{w_n}(\gamma_{u_n}^*) = \text{AsyCS} \) and \( \{w_n\} \) is a subsequence of \( \{u_n\} \), we have \( \lim_{n \to \infty} CP_{w_n}(\gamma_{w_n}^*) = \text{AsyCS} \). This and (S1.5) yield the result of part (a).

Part (b) follows from part (a) and Lemma 3(b) because \( \text{AsyCS} \leq \lim_{n \to \infty} P_{\gamma^*}(T_n(\theta^*) \leq \hat{c}_n(\theta^*, 1 - \alpha)) \).

Now, we prove part (c) of the theorem. By assumption, \( v = 0 \). Under Assumption M, the sequence of constant true values \( \{(\theta^*, F^*) \in \mathcal{F} : n \geq 1\} \) satisfies \( n^{1/2} E_{\gamma^*}. m_j(W_i, \theta^*)/\sigma_{F^*}.(\theta^*) \to \infty \) for \( j = 1, \ldots, p \). Let \( \gamma^* = (\gamma_1^*, \gamma_2^*, F^*) \in \Gamma \) correspond to \( (\theta^*, F^*) \in \mathcal{F} \), where \( \gamma_2^* = (\theta^*, \gamma_{2,2}^*) \). Define \( h^* = (\infty^p, \gamma_2^*) \).

As in the proof of part (b) of Lemma 2 below, we have \( T_n(\theta^*) \to_d J_{h^*} \) under \( \{\gamma^*\} \), where \( J_{h^*} \) is the distribution of \( S(Z_{h_{2,2}^*}^* + (h_{1,2}^*, 0), \Omega_{h_{2,2}^*}) \), where \( Z_{h_{2,2}^*} \sim N(0_k, \Omega_{h_{2,2}^*}) \). (Note that \( J_{h^*} \) is the same as \( J_{h_{2,2}^*, a_p} \) defined in the text.) Furthermore, \( J_{h^*}(x) = 1 \) for \( x \geq 0 \) because \( S(Z_{h_{2,2}^*} + (\infty^p, \Omega_{h_{2,2}^*}) = S(\infty^p, \Omega_{h_{2,2}^*}) = 0 \) by Assumption 3. Using these results, we obtain

\[
(S1.6) \quad \text{AsyMaxCP} \geq \limsup_{n \to \infty} P_{\gamma^*}(T_n(\theta^*) \leq \hat{c}_n(\theta^*, 1 - \alpha)) \geq \limsup_{n \to \infty} P_{\gamma^*}(T_n(\theta^*) \leq 0) = J_{h^*}(0) = 1,
\]

where the first inequality follows from the definition of AsyMaxCP and the second inequality holds by Assumption 1(c). Q.E.D.

**PROOF OF LEMMA 2:** First, suppose \( c_\alpha^*(1 - \alpha) = 0 \). In this case, define \( c_\alpha^* = 0 \) and we have

\[
(S1.7) \quad c_n(\hat{\theta}_{n,h}, 1 - \alpha) \geq c_n^* \to_p c_\alpha^*(1 - \alpha),
\]

\[\text{Inference for Parameters}\]
where the inequality holds by Assumption 1(c), which establishes part (a) for this case.

Next, suppose \( c_\alpha \cdot (1 - \alpha) > 0 \). For \((\xi, \Omega) \in R^k \times \Psi\), let \( \varphi^*(\xi, \Omega) \) denote the \( k \)-vector whose \( j \)th element is

\[
\varphi_j^*(\xi, \Omega) = \begin{cases} 
\varphi_j(\xi, \Omega), & \text{if } \pi_{1,j} = 0 \text{ and } j = 1, \ldots, p, \\
\infty, & \text{if } \pi_{1,j} > 0 \text{ and } j = 1, \ldots, p, \\
0, & \text{if } j = p + 1, \ldots, k.
\end{cases}
\]

By construction,

\[
\varphi^*(\xi_n(\theta_n), \Omega_n(\theta_n)) \geq \varphi(\xi_n(\theta_n), \Omega_n(\theta_n)) \quad \text{a.s.} \quad [Z^*] \quad \text{for all } n.
\]

Let \( c_n^* \) denote the \( 1 - \alpha \) quantile of the df of \( S(\Omega_n^{1/2}(\theta_n, h) \varphi^*(\xi_n(\theta_n), \Omega_n(\theta_n)), \Omega_n(\theta_n)) \), where \( Z^* \) is random and \((\xi_n(\theta_n), \Omega_n(\theta_n))\) is fixed. Then \( c_n(\theta_n, h, 1 - \alpha) \geq c_n^* \) a.s. for all \( n \) by \((S1.9)\) and Assumption 1(a).

We now show that \( c_n^* \rightarrow p \) \( c_\alpha \cdot (1 - \alpha) > 0 \). By definition, \( \gamma_{n,h,1} = (\gamma_{n,h,1,1}, \ldots, \gamma_{n,h,1,p})' \) \( \in R^p \) satisfies

\[
\sigma_{F_n,h,j}^{-1}(\theta_n, h) E_{F_n,h,j} m_j(W_i, \theta_n) - \gamma_{n,h,1,j} = 0 \quad \text{for } j = 1, \ldots, p.
\]

Also by definition, \( \Gamma \) is such that under any sequence \( \{\gamma_{n,h} = (\gamma_{n,h,1}, (\theta_n, h, \text{vech}_n(\Omega_n, h)), F_n, h): n \geq 1\} \) of parameters in \( \Gamma \) that satisfies \( n^{1/2} \gamma_{n,h,1} \rightarrow h_1 \) and \((\theta_n, h, \text{vech}_n(\Omega_n, h)) \rightarrow h_2 = (h_{2,1}, h_{2,2}) \) for some \( h = (h_1, h_2) \in R_{+}\infty \times R_{[\pm \infty]}^q \), we have

\[
\begin{align*}
(A_n = (A_{n,1}, \ldots, A_{n,k})') & \rightarrow_d Z_{h_{2,2}} \sim N(0_k, \Omega_{h_{2,2}}) \quad \text{as } n \rightarrow \infty, \\
\text{where} \quad A_{n,j} & = n^{1/2}(m_{n,j}(\theta_n) - E_{F_n,h,j} m_{n,j}(\theta_n))/\sigma_{F_n,h,j}(\theta_n), \\
(\gamma_{n,h,j})/\sigma_{F_n,h,j}(\theta_n) & \rightarrow 1 \quad \text{as } n \rightarrow \infty \quad \text{for } j = 1, \ldots, k, \\
(\widehat{D}_{n,h}^{-1}(\theta_n, h) \widehat{\Sigma}_n(\theta_n, h) \widehat{D}_{n,h}^{-1}(\theta_n, h)) & \rightarrow p \Omega_{h_{2,2}} \quad \text{as } n \rightarrow \infty, \quad \text{and} \\
(\chi) & \text{ conditions (vii)--(ix) hold for all subsequences \{w_n\} in place of \{n\}.}
\end{align*}
\]

Now, under \( \{\gamma_{n,h}: n \geq 1\} \), we have

\[
\begin{align*}
\kappa_n^{-1} n^{1/2} \widehat{D}_{n,h}^{-1}(\theta_n, h) & \rightarrow D(\theta_n, h, F_n) A_n \left( n^{1/2} \gamma_{n,h,1,0} \right) \\
& = o_p(1) + (I_k + o_p(1))(\kappa_n^{-1} n^{1/2} \gamma_{n,h,1,0}) \rightarrow p (\pi_1, 0_v),
\end{align*}
\]

where the first equality holds by the definitions of \( \gamma_{n,h,1} \) and \( A_n \) in \((S1.10)\) and \((S1.11)\), \( D(\theta_n, h, F_n) = \text{Diag}\{\sigma_{F_n,h,1}^2(\theta_n, h), \ldots, \sigma_{F_n,h,k}^2(\theta_n, h)\} \) in the second
line, the second equality holds using \( \kappa_n \to \infty \) and conditions (vii) and (viii) of (S1.11), which apply by conditions (i) and (iii) of the lemma, and the convergence holds using condition (ii) of the lemma. This and condition (ix) of (S1.11) yield that under \( \{ \gamma_{n,h} : n \geq 1 \}, \)

\[(S1.13) \quad (\xi_n(\theta_{n,h}), \hat{\Omega}_n(\theta_{n,h})) \to_p ((\pi_1, 0_v), (\pi_{\pi_2^2})).\]

For notational simplicity, let \( \Omega_0 \) denote \( \Omega_{\pi_2^2} \). We now show that \( \varphi^*(\xi, \Omega) \to \varphi((\pi_1, 0_v), \Omega_0) \) for any sequence \( (\xi, \Omega) \) for which \( (\xi, \Omega) \to ((\pi_1, 0_v), \Omega_0) \). If \( \pi_{1,j} = 0 \) and \( j \leq p \), then

\[(S1.14) \quad \varphi^*_j((\pi, \Omega)) = \varphi_j((\pi, \Omega)) \to \varphi_j((\pi_1, 0_v), \Omega_0) = 0 \]

as \( (\xi, \Omega) \to ((\pi_1, 0_v), \Omega_0) \), where the first equality holds by (S1.8), the convergence holds by Assumption GMS1(a), and the last equality holds by Assumption GMS1(b). If \( \pi_{1,j} > 0 \), then \( \varphi^*_j((\pi, \Omega)) = \infty = \varphi_j((\pi_1, 0_v), \Omega_0) \) by the definition in (S1.8) and Assumption GMS3. For \( j = p + 1, \ldots, k \), \( \varphi^*_j((\pi, \Omega)) = 0 = \varphi_j((\pi_1, 0_v), \Omega_0) \) by the definition in (S1.8) and Assumption GMS1(c). These results, (S1.14), and Assumption 1(d) give, for \( x > 0 \), as \( (\xi, \Omega) \to ((\pi_1, 0_v), \Omega_0) \),

\[(S1.15) \quad S(\Omega^{1/2} Z^* + \varphi^*(\xi, \Omega), \Omega) \]

\[\to S(\Omega_0^{1/2} Z^* + \varphi^*((\pi_1, 0_v), \Omega_0), \Omega_0) \quad \text{a.s.} \quad [Z^*],\]

\[1(S(\Omega^{1/2} Z^* + \varphi^*(\xi, \Omega), \Omega) \leq x) \]

\[\to 1(S(\Omega_0^{1/2} Z^* + \varphi^*((\pi_1, 0_v), \Omega_0), \Omega_0) \leq x) \quad \text{a.s.} \quad [Z^*],\]

\[P(S(\Omega^{1/2} Z^* + \varphi^*(\xi, \Omega), \Omega) \leq x) \]

\[\to P(S(\Omega_0^{1/2} Z^* + \varphi^*((\pi_1, 0_v), \Omega_0), \Omega_0) \leq x).\]

The third convergence result of (S1.15) holds by the second result and the bounded convergence theorem. The second convergence result of (S1.15) follows from the first result provided

\[(S1.16) \quad P(S(Z + \varphi^*((\pi_1, 0_v), \Omega_0), \Omega_0) = x) = P(S(Z + (\pi^*_1, 0_v), \Omega_0) = x) = 0,\]

where \( Z = \Omega_0^{1/2} Z^* \sim N(0_k, \Omega_0) \). The first equality in (S1.16) holds because \( [(\pi^*_1, 0_v)] = \infty = \varphi^*_j((\pi_1, 0_v), \Omega_0) \) if \( \pi^*_1,j = \infty \) by definition and \( [(\pi^*_1, 0_v)] = 0 = \varphi_j((\pi_1, 0_v), \Omega_0) = \varphi^*_j((\pi_1, 0_v), \Omega_0) \) if \( \pi^*_1,j = \pi_1,j = 0 \) using Assumption GMS1(b) and (S1.8). The second equality in (S1.16) holds because the df of \( S(Z + (\pi^*_1, 0_v), \Omega_0) \) is continuous and strictly increasing for \( x > 0 \) by Assumption 2(a) and (b) unless \( \upsilon = 0 \) and \( \pi^*_1 = \infty^\upsilon \). The latter does not hold be-
Lemma 5 of Andrews and Guggenberger (2010b) (AG1), given that (S1.17)

\[ \pi^* = \infty^\theta, \text{ then } S(Z + (\pi^*_1, 0), \Omega_h) = 0 \] by Assumptions 1(c) and 3 and \( c_{\pi^*} (1 - \alpha) = 0 \), which contradicts the assumption that \( c_{\pi^*} (1 - \alpha) > 0 \).

In sum, (S1.15) shows that \( P(S(\Omega^{1/2}Z^* + \varphi^*(\xi, \Omega), \Omega) \leq x) \) is a continuous function of \((\pi, \Omega)\) at \((\pi_1, 0), \Omega_h)\). This, (S1.13), and Slutsky’s theorem combine to give, under \( \{\gamma_{n,h}: n \geq 1\} \),

\[
(S1.17) \quad L_n(x) = P(S(\hat{\Omega}^{1/2}_n(\theta_{n,h})Z^* + \varphi^*(\hat{\xi}_n(\theta_{n,h}), \hat{\Omega}_n(\theta_{n,h})), \hat{\Omega}_n(\theta_{n,h})) \leq x) \
\quad \rightarrow_p P(S(\Omega^{1/2}_0Z^* + \varphi^*((\pi_1, 0), \Omega_0), \Omega_0) \leq x) = L(x)
\]

for all \( x > 0 \), where \( P(\cdot) \) denotes conditional probability given \((\hat{\xi}_n(\theta_{n,h}), \hat{\Omega}_n(\theta_{n,h}))\) in (S1.17) and hence is a random probability. By definition, \( c_{\pi^*} \) is the \( 1 - \alpha \) quantile of \( L_n(x) \) and \( c_{\pi^*} (1 - \alpha) \) is the \( 1 - \alpha \) quantile of \( L(x) \). By Lemma 5 of Andrews and Guggenberger (2010b) (AG1), given that (S1.17) holds for all \( x \) in a neighborhood of \( c_{\pi^*} (1 - \alpha) > 0 \) and that \( L(x) \) is continuous and strictly increasing at \( x = c_{\pi^*} (1 - \alpha) \) (see the previous paragraph), we have \( c_{\pi^*} \rightarrow_p c_{\pi^*} (1 - \alpha) \). This completes the proof of part (a).

Part (b) is proved as follows. First, conditions (i) and (ii) of the lemma imply that if \( \pi_{1,j} > 0 \), then \( h_{1,j} = \infty \) and \( \pi_{1,j} = \infty \), and if \( \pi_{1,j} = 0 \), then \( h_{1,j} \geq 0 \) and \( \pi_{1,j} = 0 \). Thus, we have

\[
(S1.18) \quad \pi^*_1 \leq h_1,
\]

\[
S(\Omega_{h_{2,2}}^{1/2}Z^* + (\pi^*_1, 0), \Omega_{h_{2,2}}) \geq \Omega_{h_{2,2}}^{1/2}Z^* + (h_1, 0), \Omega_{h_{2,2}}),
\]

\[
c_{\pi^*} (1 - \alpha) \geq c_{h}(1 - \alpha),
\]

where \( c_{h}(1 - \alpha) \) denotes the \( 1 - \alpha \) quantile of \( S(\Omega_{h_{2,2}}^{1/2}Z^* + (h_1, 0), \Omega_{h_{2,2}}) \), the second inequality holds by the first inequality and Assumption 1(a), and the third inequality holds by the second.

Second, by the verification of Assumption B0 in Andrews and Guggenberger (2009b) (AG4), we have

\[
(S1.19) \quad T_n(\theta_{n,h}) \rightarrow_d J_h \quad \text{under} \quad \{\gamma_{n,h}\},
\]

where \( J_h \) is the distribution of \( S(\Omega_{h_{2,2}}^{1/2}Z^* + (h_1, 0), \Omega_{h_{2,2}}) \). This result is obtained by using Assumption 1(b) to write

\[
(S1.20) \quad T_n(\theta_{n,h}) = S(\hat{D}_n^{1/2}Z^* + \varphi^*(\hat{\xi}_n(\theta_{n,h}), \hat{\Omega}_n(\theta_{n,h})), \hat{\Omega}_n(\theta_{n,h}))
\]

If any element of \( h_1 \) equals \( \infty \), then it can be shown using (S1.11) that the corresponding element of \( \hat{D}_n^{1/2}(\theta_{n,h})n^{1/2}\hat{\sigma}_n(\theta_{n,h}) \) diverges in probability to \( \infty \). Hence, \( \hat{D}_n^{1/2}(\theta_{n,h})n^{1/2}\hat{\sigma}_n(\theta_{n,h}) \) does not converge in distribution to a proper finite random vector and the continuous mapping theorem cannot be applied to obtain the asymptotic distribution of the right-hand side of (S1.20). The
verification of Assumption B0 in AG4 avoids this problem by (i) considering a transformation of \( D_{n}^{1/2}(\theta_{n,h})n^{1/2}\overline{m}_{n}(\theta_{n,h}) \) that converges in distribution even if some elements of \( h_{1} \) equal \( \infty \), (ii) writing the right-hand side of (S1.20) as a continuous function of this transformation, and (iii) applying the continuous mapping theorem to the transformation.

We now have

\[
\lim_{n \to \infty} \inf P_{\gamma_{n,h}}(T_{n}(\theta_{n,h}) \leq \hat{c}_{n}(\theta_{n,h}, 1 - \alpha)) \\
\geq \lim_{n \to \infty} \inf P_{\gamma_{n,h}}(T_{n}(\theta_{n,h}) \leq c_{n}^*) \\
\geq J_{h}(c_{\pi^*}(1 - \alpha) -),
\]

where \( J_{h}(x- \) denotes the limit from the left of \( J_{h}(\cdot) \) at \( x \), the first inequality holds because \( \hat{c}_{n}(\theta_{n,h}, 1 - \alpha) \geq c_{n}^* \) a.s., and the second inequality holds by part (a) of the lemma and (S1.19).

Suppose \( c_{\pi^*}(1 - \alpha) > 0 \). Then

\[
J_{h}(c_{\pi^*}(1 - \alpha) -) = J_{h}(c_{\pi^*}(1 - \alpha)) \geq 1 - \alpha
\]

and part (b) of the lemma holds, where the equality holds because \( J_{h}(x) \) is continuous for all \( x > 0 \) by Assumption 2(a) and the inequality holds by (S1.18).

Next, suppose \( c_{\pi^*}(1 - \alpha) = 0 \). This implies that \( c_{h}(1 - \alpha) = 0 \) by (S1.18) and Assumption 1(c). The conditions \( c_{h}(1 - \alpha) = 0 \) and \( 0 < \alpha < 1/2 \) are consistent with Assumption 2(c) only if \( v = 0 \). Given \( v = 0 \), under \( \{\gamma_{n,h} : n \geq 1\} \), we have

\[
P_{\gamma_{n,h}}(T_{n}(\theta_{n,h}) \leq 0) \\
= P_{\gamma_{n,h}}(n^{1/2}\overline{m}_{n,j}(\theta_{n,h})/\sigma_{F_{n,h,j}}(\theta_{n,h}) \geq 0 \text{ for all } j = 1, \ldots, p) \\
= P_{\gamma_{n,h}}(A_{n,j} + n^{1/2}\gamma_{n,h,j} \geq 0 \text{ for all } j = 1, \ldots, p) \\
\to P(\sum_{j=1}^{p_{2,2}} \Omega^{1/2}_{h}(Z^*)_{j} + h_{1,j} \geq 0 \text{ for all } j = 1, \ldots, p) \\
= P(S(\sum_{j=1}^{p_{2,2}} \Omega^{1/2}_{h}(Z^*) + h_{1}, \Omega_{h_{2,2}}) \leq 0) \\
= J_{h}(0) \geq J_{h}(c_{h}(1 - \alpha)) \geq 1 - \alpha,
\]

where the first equality holds by Assumptions 1(b) and 3, the second equality and the convergence hold by (S1.11), the third equality holds by Assumption 3, the fourth equality holds by the definition of \( J_{h} \), the first inequality holds because \( c_{h}(1 - \alpha) = 0 \) (note that \( c_{h}(1 - \alpha) = 0 \) here, but the argument in (S1.23) is applied below to a case in which one only knows that \( c_{h}(1 - \alpha) \geq 0 \)), and the second inequality holds by the definition of \( c_{h}(1 - \alpha) \). This completes the proof of part (b).

The proof of part (c) is the same as that for parts (a) and (b) with \( w_{n} \) in place of \( n \).

Q.E.D.
Proof of Lemma 3: Conditions (i)–(iii) of Lemma 2 hold with $\gamma_{n,h} = \gamma^*$ for all $n$, $h = h^*$, and $\pi = h^*$ because $\kappa_n^{-1}h^{1/2} \to \infty$ by Assumption GMS4. Each element of $\pi_1$ is either zero or infinity. Thus, the vector $\pi^*$ that depends on $\pi$ and is defined preceding Lemma 2 equals $\pi$. Now, (S1.13) in the proof of Lemma 2 applies with $\theta_{n,h} = \theta^*$, $\pi_1 = h_1^*$, and $\Omega_{\pi_{2,2}} = \Omega_{h_{2,2}}$.

Equation (S1.14) applies (with the first quantity on the left-hand side deleted) for all $j = 1, \ldots, p$ for which $\pi_{1,j} = 0$. In addition, we have that as $(\xi, \Omega) \to ((\pi_1, 0_v), \Omega_{h_{2,2}})$, $\varphi(\xi, \Omega) \to \infty$ a.s. $[Z^*]$ for all $j = 1, \ldots, p$ for which $\pi_{1,j} = \infty$ by Assumption GMS3. Given these results, (S1.15) and (S1.16) hold with $\varphi^*(\xi, \Omega)$ and $S(\Omega_0^{1/2}Z^* + \varphi^*((\pi_1, 0_v), \Omega_0), \Omega_0)$ replaced by $\varphi(\xi, \Omega)$ and $S(\Omega_0^{1/2}Z^* + \varphi((h_1^*, 0_v), \Omega_{h_{2,2}}), \Omega_{h_{2,2}})$, respectively, and the second equality in (S1.16) holds because $c_{h^*}(1 - \alpha) = c_{\varphi}(1 - \alpha) > 0$. (The case $c_{h^*}(1 - \alpha) = 0$ does not occur because the df of $S(\Omega_{h_{2,2}}^{1/2}Z^* + (h_1^*, 0_v), \Omega_{h_{2,2}})$ at $x < 0$ is zero by Assumption 1(c), the df at $x = c_{h^*}(1 - \alpha) = 0$ is zero by continuity (Assumption 7), the latter implies that the df is less than $1 - \alpha$ for $x > 0$, and the latter implies that $c_{h^*}(1 - \alpha) > 0$.) The remainder of the proof of part (a) is the same as that given in (S1.17) but with $\widehat{c}_n(\theta^*, 1 - \alpha)$ in place of $c_{h^*}$.

To prove part (b), we note that the asymptotic distribution of $T_n(\theta^*)$ is $S(\Omega_{h_{2,2}}^{1/2}Z^* + (h_1^*, 0_v), \Omega_{h_{2,2}})$ under $\{\gamma^*: n \geq 1\}$ by the verification of Assumption B0 in AG1, see (S1.19) and the discussion following it. The df of $S(\Omega_{h_{2,2}}^{1/2}Z^* + (h_1^*, 0_v), \Omega_{h_{2,2}})$ is continuous and strictly increasing at $c_{h^*}(1 - \alpha) > 0$ by Assumption 2(a) and (b) unless $v = 0$ and $h_1^* = \infty$. The latter does not hold by the argument given in the proof of Lemma 2 just below (S1.16) because $c_{h^*}(1 - \alpha) > 0$. These results and $\widehat{c}_n(\theta^*, 1 - \alpha) \to_p c_{h^*}(1 - \alpha)$ establish part (b).

We now restate and prove the following lemma.

Lemma 1: The functions $S_1(m, \Sigma)$–$S_3(m, \Sigma)$ satisfy Assumptions 1–6 with $\Psi = \Psi_1$ for $S_1(m, \Sigma)$ and $S_3(m, \Sigma)$, and with $\Psi = \Psi_2$ for $S_2(m, \Sigma)$.

Proof: Assumptions 1–4 hold by Lemma 1 of AG4. For $S_1$, Assumption 5(a) holds by the same arguments as for Assumption 2 given in the proof of Lemma 1 of AG4. Assumption 5(b) holds with a nonstrict inequality by Assumption 1(a) and the fact that $Z + (m_1, 0_v)$ is stochastically strictly increasing in $m_1 \in R_{+\infty}$. Assumption 5(b) holds for $S_1$ with a strict inequality because $S_1(Z + (m_1, 0_v), \Omega)$ is strictly stochastically less than $S_1(Z + (m_1, 0_v), \Omega)$ on $R_+$ for $m_1 > m_1^*$. Assumption 6 holds immediately for $S_1$ with $\chi = 2$.

For $S_2$, Assumption 5(a)(i) and (ii) hold by the same arguments as for Assumption 2(a) and (b) in the proof of Lemma 1 of AG4. Assumption 5(b) holds for $S_2$ by the same argument as for $S_1$. Assumption 6 holds immediately for $S_2$ with $\chi = 2$. The verification of Assumptions 1–6 for $S_3$ is essentially the same as that for $S_1$.
S2. PROOFS OF RESULTS FOR LOCAL ALTERNATIVES

In this section, we restate and prove the following two theorems.

THEOREM 2: Under Assumptions 1–5, LA1, and LA2, the following statements hold:

(a) \[ \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,*}) > \hat{c}_n(\theta_{n,*}, 1 - \alpha)) = 1 - \frac{1}{2} \frac{J_{h_1,\lambda}(c_{\pi_1}(\varphi, 1 - \alpha))}{\lambda(c_{\pi_1}(\varphi, 1 - \alpha))} \]

provided Assumptions GMS2, GMS3, LA4, and LA5 hold.

(b) \[ \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,*}) > c_n(\theta_{n,*}, 1 - \alpha)) = 1 - \frac{1}{2} \frac{J_{h_1,\lambda}(c_{g_1,0}(1 - \alpha))}{\lambda(c_{g_1,0}(1 - \alpha))} \]

provided Assumption LA6 holds.

(c) \[ \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,*}) > c(\hat{\Omega}_n(\theta_{n,*}), 1 - \alpha)) = 1 - \frac{1}{2} \frac{J_{h_1,\lambda}(c_{0,p,0}(1 - \alpha))}{\lambda(c_{0,p,0}(1 - \alpha))} \]

THEOREM 3: Under Assumptions 1–5, LA1–LA4, LA6, GMS2–GMS3, and GMS5–GMS6, the following assertions are valid:

(a) \[ \liminf_{n \to \infty} P_{F_n}(T_n(\theta_{n,*}) > \hat{c}_n(\theta_{n,*}, 1 - \alpha)) \geq \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,*}) > c_{n,b}(\theta_{n,*}, 1 - \alpha)) \]

with strict inequality whenever \( g_{1,j} < \infty \) and \( \pi_{1,j} = \infty \) for some \( j = 1, \ldots, p \) and \( c_{g_1,0}(1 - \alpha) > 0. \)

(b) \[ \liminf_{n \to \infty} P_{F_n}(T_n(\theta_{n,*}) > \hat{c}_n(\theta_{n,*}, 1 - \alpha)) \geq \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,*}) > c(\hat{\Omega}_n(\theta_{n,*}), 1 - \alpha)) \]

with strict inequality whenever \( \pi_{1,j} = \infty \) for some \( j = 1, \ldots, p \).

(c) \[ \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,*}) > c_{n,b}(\theta_{n,*}, 1 - \alpha)) \geq \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,*}) > c(\hat{\Omega}_n(\theta_{n,*}), 1 - \alpha)) \]

with strict inequality whenever \( g_1 > 0_p \), where Assumptions GMS2, GMS3, GMS5, GMS6, and LA4 are not needed for this result.

Theorem 2 follows immediately from Lemmas 4 and 5 below. Theorem 3(a) and (c) do likewise from Lemmas 5–8. Theorem 3(b) follows from Lemmas 6 and 7, where one takes \( g_1 = 0_p \) in Lemma 7 and one notes that \( c_{0,p,0}(1 - \alpha) > 0 \) by Assumption 2(c) and \( \alpha \in (0, 1/2) \).

In each of Lemmas 4–8, the parameter space \( \mathcal{F} \) for \( (\theta, F) \) is assumed to satisfy

\[(S2.1) \quad \begin{align*}
(\text{i}) & \quad \theta \in \Theta, \\
(\text{ii}) & \quad E_F m_j(W_i, \theta) \geq 0 \quad \text{for} \quad j = 1, \ldots, p, \\
(\text{iii}) & \quad E_F m_j(W_i, \theta) = 0 \quad \text{for} \quad j = p + 1, \ldots, k, \\
(\text{iv}) & \quad \{W_i: i \geq 1\} \text{ are i.i.d. under } F, \\
(\text{v}) & \quad \sigma^2_{F,j}(\theta) = \text{Var}_F(m_j(W_i, \theta)) \in (0, \infty) \quad \text{for} \quad j = 1, \ldots, k, \\
(\text{vi}) & \quad \text{Corr}_F(m(W_i, \theta)) \in \Psi, \\
(\text{vii}) & \quad E_F |m_j(W_i, \theta)/\sigma_{F,j}(\theta)|^{2+\delta} \leq M \quad \text{for} \quad j = 1, \ldots, k.
\end{align*}\]

In the lemmas that involve subsampling, it is assumed that \( b \to \infty \) and \( b/n \to 0 \) as \( n \to \infty \). Let \( Z \sim N(0_k, \Omega_0) \).
LEMMA 4: Under Assumptions 1–3, 5(a), GMS2, GMS3, LA1, LA2, LA4, and LA5, the following equations hold:
(a) \( T_n(\theta_{n*,s}) \to_d S(Z + (h_1, 0_v) + \Pi_0 \lambda, \Omega_0^2) \sim J_{h_1, \lambda} \).
(b) \( \hat{c}_{n}(\theta_{n*,s}, 1 - \alpha) \to_p c_{\pi_1}(\varphi, 1 - \alpha) \).
(c) \( \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n*,s}) > \hat{c}_{n}(\theta_{n*,s}, 1 - \alpha)) = 1 - J_{h_1, \lambda}(c_{\pi_1}(\varphi, 1 - \alpha)) \).

LEMMA 5: Under Assumptions 1–3, 5(a), LA1, LA2, and LA6, the following results hold:
(a) \( T_n(\theta_{n*,s}) \to_d S(Z + (h_1, 0_v) + \Pi_0 \lambda, \Omega_0^2) \sim J_{h_1, \lambda} \).
(b) \( c_{n,b}(\theta_{n*,s}, 1 - \alpha) \to c_{b_1,0_d}(1 - \alpha) \).
(c) \( \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n*,s}) > c_{n,b}(\theta_{n*,s}, 1 - \alpha)) = 1 - J_{h_1, \lambda}(c_{b_1,0_d}(1 - \alpha)) \).

LEMMA 6: Under Assumptions 1, 4, 5(a), LA1, and LA2, the following results hold:
(a) \( T_n(\theta_{n*,s}) \to_d S(Z + (h_1, 0_v) + \Pi_0 \lambda, \Omega_0^2) \sim J_{h_1, \lambda} \).
(b) \( c_{\Omega_n}(\theta_{n*,s}, 1 - \alpha) \to c_{0,0_d}(1 - \alpha) \).
(c) \( \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n*,s}) > c_{\hat{\Omega}_n}(\theta_{n*,s}, 1 - \alpha)) = 1 - J_{h_1, \lambda}(c_{0,0_d}(1 - \alpha)) \).

The next lemma uses the following notation. Let \( g_1 = (g_{1,1}, \ldots, g_{1,p})' \) be as in Assumption LA6. Let \( \pi_{1*, s} = \infty \) if \( \pi_{1,j} = \infty \) and let \( \pi_{1*, s} = 0 \) if \( \pi_{1,j} < \infty \) for \( j = 1, \ldots, p \). As defined, \( \pi_{1*, s} = (\pi_{1,1, s}, \ldots, \pi_{1,p, s})' \leq h_1 \). Let \( \pi_{2*, s} = (\pi_{2,1, s}, \pi_{2,2, s})' \in R_{+}^{q} \times R_{+}^{q} \), where \( \pi_{2,1} = (\pi_{2,1, s}, \pi_{2,2, s}) \), \( \pi_{2,1} = \theta_0 \), where \( \theta_0 \) is as in Assumption LA1(a), and \( \pi_{2,2} = \text{vech}_k(\Omega_0) \) for the \( k \times k \) correlation matrix \( \Omega_0 = \Omega(\theta_0, F_0) \) determined by Assumption LA1(a). Let \( c_{\pi_{2, 1}}(1 - \alpha) \) denote the \( 1 - \alpha \) quantile of \( S(\Omega_0^{1/2}Z^* + (\pi_{1*, s}, 0_v), \Omega_0) \), where \( Z^* \sim N(0_k, I_k) \) and, by definition, if \( \pi_{1,j} = \infty \), then the \( j \)th element of \( \Omega_0^{1/2}Z^* + (\pi_{1*, s}, 0_v) \) equals \( \infty \) for \( j = 1, \ldots, p \).

LEMMA 7: Under Assumptions 1–3, 5(a), LA1–LA4, LA6, GMS2, GMS3, GMS5, and GMS6, the following statements hold:
(a) \( c_{\pi_{1, s}}(1 - \alpha) > 0 \), \( \hat{c}_{n}(\theta_{n*, 1 - \alpha}) \leq c_{\pi_{1, s}} \) a.s. for all \( n \) for a sequence of random variables \( \{c_{\pi_{1, s}}: n \geq 1\} \) that satisfies \( c_{\pi_{1, s}} \to_p c_{\pi_{1, s}}(1 - \alpha) \).
(b) \( \liminf_{n \to \infty} P_{F_n}(T_n(\theta_{n*, s}) > \hat{c}_{n}(\theta_{n*, 1 - \alpha})) \geq 1 - J_{h_1, \lambda}(c_{\pi_{1, s}}(1 - \alpha)) \).
(c) \( c_{g_{1,0_d}(1 - \alpha)}(1 - \alpha) \geq c_{\pi_{1, s}}(1 - \alpha) \) with strict inequality whenever \( g_{1,j} < \infty \) and \( \pi_{1,j} = \infty \) for some \( j = 1, \ldots, p \) and \( c_{g_{1,0_d}(1 - \alpha)}(1 - \alpha) > 0 \).
(d) \( 1 - J_{h_1, \lambda}(c_{\pi_{1, s}}(1 - \alpha)) \geq 1 - J_{h_1, \lambda}(c_{g_{1,0_d}(1 - \alpha)}) \) with strict inequality whenever \( g_{1,j} < \infty \) and \( \pi_{1,j} = \infty \) for some \( j = 1, \ldots, p \) and \( c_{g_{1,0_d}(1 - \alpha)}(1 - \alpha) > 0 \).

LEMMA 8: Under Assumptions 1–5, LA1–LA3, and LA6, assertions (a)–(d) are valid:
(a) \( c_{0,0_d}(1 - \alpha) \geq c_{g_{1,0_d}(1 - \alpha)} \).
(b) \( J_{h_1, \lambda}(c_{g_1, 0}\sigma(1 - \alpha)) \geq J_{h_1, \lambda}(c_{0, 0}\sigma(1 - \alpha)) \).
(c) \( c_{0, 0}\sigma(1 - \alpha) > c_{g_1, 0}\sigma(1 - \alpha) \) unless \( g_1 = 0_p \).
(d) \( 1 - J_{h_1, \lambda}(c_{g_1, 0}\sigma(1 - \alpha)) > 1 - J_{h_1, \lambda}(c_{0, 0}\sigma(1 - \alpha)) \) unless \( g_1 = 0_p \).

**Proof of Lemma 4:** To prove part (a), by using element-by-element mean-value expansions about \( \theta = \theta_n \) and Assumptions LA1 and LA2, we obtain

\[
\begin{align*}
\text{(S2.2)} & \quad D^{-1/2}(\theta_{n, s}, F_n)E_{F_n}m(W_i, \theta_{n, s}) \\
& = D^{-1/2}(\theta_n, F_n)E_{F_n}m(W_i, \theta_n) + \Pi(\hat{\theta}_n, F_n)(\theta_{n, s} - \theta_n), \\
& = n^{1/2}D^{-1/2}(\theta_{n, s}, F_n)E_{F_n}m(W_i, \theta_{n, s}) \rightarrow (h_1, 0_{v}) + \Pi_0\lambda,
\end{align*}
\]

where \( D(\theta, F) = \text{Diag}\{\sigma_{F,1}^2(\theta), \ldots, \sigma_{F,k}^2(\theta)\} \), \( \hat{\theta}_n \) may differ across rows of \( \Pi(\hat{\theta}_n, F_n) \), \( \hat{\theta}_n \) lies between \( \theta_{n, s} \) and \( \theta_n \), \( \hat{\theta}_n \rightarrow 0 \), and \( \Pi(\hat{\theta}_n, F_n) \rightarrow \Pi_0 \).

Next, under \( \{(\theta_n, F_n) \in F : n \geq 1\} \) as in Assumption LA1, we have

\[
\begin{align*}
\text{(S2.3)} & \quad (i) \quad A_n^0 = (A_{n,1}^0, \ldots, A_{n,k}^0) \rightarrow d Z \sim N(0_k, \Omega_0) \quad \text{as} \quad n \rightarrow \infty, \\
& \quad \quad \text{where} \quad A_{n,j}^0 = n^{1/2}(\bar{m}_{n,j}(\theta_{n, s}) - E_{F_n}\bar{m}_{n,j}(\theta_{n, s}))/\sigma_{F_n,j}(\theta_{n, s}), \\
& \quad (ii) \quad \bar{\sigma}_{n,j}(\theta_{n, s})/\sigma_{F_n,j}(\theta_{n, s}) \rightarrow p 1 \quad \text{as} \quad n \rightarrow \infty \quad \text{for} \quad j = 1, \ldots, k, \\
& \quad (iii) \quad \hat{D}_n^{-1/2}(\theta_{n, s})\hat{\Sigma}_n(\theta_{n, s})\hat{D}_n^{-1/2}(\theta_{n, s}) \rightarrow p \Omega_0 \quad \text{as} \quad n \rightarrow \infty,
\end{align*}
\]

where result (i) holds by the Cramér–Wold device and the Liapounov triangular array CLT for rowwise i.i.d. random variables with mean 0 and variance 1 using condition (iv) of (S2.1) and Assumptions LA1(a) and LA1(c), and results (ii) and (iii) hold by standard arguments using a weak law of large numbers for rowwise i.i.d. random variables with variance 1 by condition (iv) of (S2.1) and Assumptions LA1(a) and LA1(c) and Assumptions LA1(a) and LA1(c). Note that results (i)–(iii) of (S2.3) do not hold by (S1.11) because the functions are evaluated at \( \theta_{n, s} \) but the true value is \( \theta_n \).

For the same reason as described above following (S1.20), to obtain the asymptotic distribution of \( T_n(\theta_{n, s}) \), we use the same type of argument as in the verification of Assumption B0 in AG4. Let \( G(\cdot) \) be a strictly increasing continuous df on \( R \), such as the standard normal df. Using (S2.2) and (S2.3), for \( j = 1, \ldots, k \), we have

\[
\begin{align*}
\text{(S2.4)} & \quad G_{n,j}^0 = G(\hat{\sigma}_{n,j}^{-1}(\theta_{n, s})n^{1/2}\bar{m}_{n,j}(\theta_{n, s})) \\
& = G(\hat{\sigma}_{n,j}^{-1}(\theta_{n, s})\sigma_{F_n,j}(\theta_{n, s})[A_{n,j}^0 + n^{1/2}\sigma_{F_n,j}^{-1}(\theta_{n, s})E_{F_n}m_j(W_i, \theta_{n, s})]), \\
& = \begin{cases} 
\rightarrow p 1, & \text{if} \quad j \leq p \quad \text{and} \quad h_{1,j} = \infty, \\
\rightarrow d G(Z_j + h_{1,j} + \Pi_{0,j}^0(\lambda)), & \text{if} \quad j \leq p \quad \text{and} \quad h_{1,j} < \infty, \\
\rightarrow d G(Z_j + \Pi_{0,j}^0(\lambda)), & \text{if} \quad j = p + 1, \ldots, k,
\end{cases}
\end{align*}
\]
\[ G_n^0 = (G_{n,1}^0, \ldots, G_{n,k}^0) \]
\[ \to_d G_\infty^0 = (G(Z_1 + h_{1,1} + \Pi_{0,1}'), \ldots, G(Z_k + \Pi_{0,k}')'), \]
where \( Z = (Z_1, \ldots, Z_k) \) and \( Z_j + h_{1,j} + \Pi_{0,j} = \infty \) by definition if \( h_{1,j} = \infty \).
Now, the same argument as in the verification of Assumption B0 in AG4 gives
\[ T_n(\theta_n, \star) \to_d S(Z + (h_1, 0_v) + \Pi_0 \lambda, \Omega_0) \sim J_{h_1, \lambda}. \]
In short, the idea behind the argument is to write the right-hand side of (S1.20) as a continuous function of \( G_0^0 \) and \( \hat{\Sigma}_n(\theta_n, \star) \) and apply the continuous mapping theorem. This completes the proof of part (a).

To prove part (b), by the mean-value expansions in (S2.2), Assumptions LA1(a), LA2, and \( \kappa_n \to \infty \), we obtain
\[ \kappa_n^{-1} n^{1/2} D^{-1/2}(\theta_{n,s}, F_n) E_{F_t} m(W_t, \theta_{n,s}) \to (\pi_1, 0_v). \]
This leads to
\[ \kappa_n^{-1} n^{1/2} D^{-1/2}(\theta_{n,s}, \hat{\Omega}_n(\theta_{n,s})) \]
\[ = \kappa_n^{-1} D^{-1/2}(\theta_{n,s}, F_n) \]
\[ \times (A_n^0 + n^{1/2} D^{-1/2}(\theta_{n,s}, F_n) E_{F_t} m(W_t, \theta_{n,s})) \]
\[ \to_p (\pi_1, 0_v), \]
where the equality holds by the definition of \( A_n^0 \) in (S2.3), and the convergence holds by (S2.6), conditions (i) and (ii) of (S2.3), and \( \kappa_n \to \infty \). Equation (S2.7) and condition (iii) of (S2.3) yield
\[ (\xi_n(\theta_{n,s}), \hat{\Omega}_n(\theta_{n,s})) \to_p ((\pi_1, 0_v), \Omega_0). \]

For \( j = 1, \ldots, k \), as \( (\xi, \Omega) \to ((\pi_1, 0_v), \Omega_0) \),
\[ \varphi_j(\xi, \Omega) \to \varphi_j((\pi_1, 0_v), \Omega_0), \]
because by Assumption LA5(a), \( \pi_1 \in C(\varphi) \), which yields (S2.9) by Assumption GMS3 if \( \pi_{1,j} = \infty \) and yields (S2.9) by the definition of \( C(\varphi) \) otherwise.
Assumption 1(d) and (S2.9) give that for \( x \) in a neighborhood of \( c_{\pi_1}(\varphi, 1 - \alpha) \), as \( (\xi, \Omega) \to ((\pi_1, 0_v), \Omega_0) \),
\[ S(\Omega^{1/2} Z^* + \varphi(\xi, \Omega), \Omega) \]
\[ \to S(\Omega_0^{1/2} Z^* + \varphi((\pi_1, 0_v), \Omega_0), \Omega_0) \quad \text{a.s.} \ [Z^*], \]
\[ 1(S(\Omega^{1/2} Z^* + \varphi(\xi, \Omega), \Omega) \leq x) \]
\[ \to 1(S(\Omega_0^{1/2} Z^* + \varphi((\pi_1, 0_v), \Omega_0), \Omega_0) \leq x) \quad \text{a.s.} \ [Z^*], \]
The third convergence result of (S2.10) holds by the second result and the bounded convergence theorem, and the second convergence result of (S2.10) follows from the first result provided \( P(S(\Omega_{1/2}Z^* + \varphi((\pi_1, 0_v), \Omega_0), \Omega_0) \leq x) = 0 \), which holds by Assumption LA5(b).

Given (S2.8) and (S2.10), the remainder of the proof of part (b) is the same as that given in the paragraph containing (S1.17) using Lemma 5 of AG1.

Part (c) of the lemma holds by parts (a) and (b) and Assumption LA5(b). Q.E.D.

PROOF OF L EMMA 5: Part (a) holds by Lemma 4(a).

To prove part (b), by the mean-value expansions in (S2.2), Assumptions LA1(a), LA2, and LA6, and \( b/n \to 0 \), we obtain

\[
(S2.11) \quad b^{1/2}D^{-1/2}(\theta_{n,*}, F_n)E_{F_n}m(W_i, \theta_{n,*}) \to (g_1, 0_v).
\]

Using (S2.11) and an analogous argument to that given in the proof of Lemma 4(a) with \( n^{1/2} \) replaced by \( b^{1/2} \) in (S2.4), we have

\[
(S2.12) \quad T_b(\theta_{n,*}) \to_d S(Z + (g_1, 0_v), \Omega_0) \sim J_{g_1,0_d},
\]

which proves part (b).

To establish Lemma 5(c) and (d), we apply Lemma 5 of AG1. We verify conditions (i)–(iii) of Lemma 5 of AG1 as follows. Lemma 5(a) of the present paper implies condition (ii). To verify condition (i), Lemma 5(b) of the present paper and identical distributions for \( \{W_i : i \leq n\} \) imply that

\[
(S2.13) \quad E_{F_n}U_{n,b}(\theta_{n,*}, x) = P_{F_n}(T_b(\theta_{n,*}) \leq x) \to J_{g_1,0_d}(x)
\]

for all continuity points \( x \) of \( J_{g_1,0_d} \). In addition, \( \text{Var}_{F_n}(U_{n,b}(\theta_{n,*}, x)) \to 0 \) by a \( U \)-statistic inequality of Hoeffding, as in Politis, Romano, and Wolf (1999, p. 44), using the i.i.d. property of \( \{W_i : i \leq n\} \) and the boundedness of \( U_{n,b}(\theta_{n,*}, x) \). This and (S2.13) give

\[
(S2.14) \quad U_{n,b}(\theta_{n,*}, x) \to_p J_{g_1,0_d}(x)
\]

for all continuity points \( x \) of \( J_{g_1,0_d} \), which verifies condition (i) of Lemma 5 of AG1.

To verify condition (iii) of Lemma 5 of AG1, we need to show

\[
(S2.15) \quad J_{g_1,0_d}(c_{g_1,0_d}(1 - \alpha) + \varepsilon) > 1 - \alpha \quad \text{for all } \varepsilon > 0.
\]

When \( v = 0 \) and \( g_1 = \infty^v \), \( S(Z + (g_1, 0_v), \Omega_0) = S(\infty^v, \Omega_0) = 0 \) using Assumption 3. In consequence, \( J_{g_1,0_d}(x) = 1 \) for all \( x > 0 \), \( c_{g_1,0_d}(1 - \alpha) = 0 \), and
(S2.15) holds for $\alpha > 0$. Now, suppose $v \geq 1$ or $g_1 \neq \infty^p$. Then, by Assumption 2(b), $J_{g_1,0}\theta(x)$ is strictly increasing for $x > 0$. Using this, we have (i) if $c_{g_1,0}\theta(1-\alpha) > 0$, then $J_{g_1,0}\theta(x)$ is strictly increasing at $x = c_{g_1,0}\theta(1-\alpha)$ and (S2.15) holds, (ii) if $c_{g_1,0}\theta(1-\alpha) = 0$, then $J_{g_1,0}\theta(0) \geq 1-\alpha$ (by the definition of $c_{g_1,0}\theta(1-\alpha)$), (iii) if $c_{g_1,0}\theta(1-\alpha) = 0$ and $J_{g_1,0}\theta(0) \geq 1-\alpha$, then $J_{g_1,0}\theta(x) > 1-\alpha$ for all $x > 0$ and (S2.15) holds (otherwise, $J_{g_1,0}\theta(x) = 1-\alpha$ for some $x > 0$ and $J_{g_1,0}\theta(x/2) = 1-\alpha$ since $J_{g_1,0}\theta$ is nondecreasing, which contradicts the fact that $J_{g_1,0}\theta(x)$ is strictly increasing for $x > 0$). Hence, (S2.15) holds.

Lemma 5 of AG1 establishes Lemma 5(c) of the present paper and shows that
\[
\lim_{n \to \infty} P_{\theta} (T_n(\theta) > c_{n,h}(\theta, 1-\alpha)) \in [1 - J_{h,1,\alpha}(c_{g_1,0}\theta(1-\alpha)), 1 - J_{h,1,\alpha}(c_{g_1,0}\theta(1-\alpha) - \alpha)].
\]
If $c_{g_1,0}\theta(1-\alpha) > 0$, then by Assumption 5(a)(i), $J_{h,1,\alpha}$ is continuous at $c_{g_1,0}\theta(1-\alpha)$ and the result of Lemma 5(d) holds. Assumption 1(c) implies that $c_{0,p,0}\theta(1-\alpha) \geq 0$. The conditions $c_{g_1,0}\theta(1-\alpha) = 0$ and $0 < \alpha < 1/2$ are consistent with Assumption 2(c) only if $v = 0$. Given $v = 0$ and $c_{g_1,0}\theta(1-\alpha) = 0$, we use the argument given in (S1.23) to establish Lemma 5(d) with $\theta, P_{n,k}, S(\Omega_{h2,2}, Z^* + h_1, \Omega_{h2,2}), J_{h}$, and $c_{h}(1-\alpha)$ replaced by $\theta, P_{n}, S(Z + h_1 + \Pi_{0}\lambda, \Omega_0), J_{h,1,\alpha}$, and $c_{h,1,\alpha}(1-\alpha)$, respectively. Q.E.D.

PROOF OF LEMMA 6: Part (a) holds by Lemma 4(a) because Assumptions 2 and 3 are not used in the proof of Lemma 4(a).

By standard arguments using a weak law of large numbers for rowwise i.i.d. triangular arrays and Assumption LA1(c), we have
\[
\begin{align*}
(D2.16)\quad D^{-1/2}(\theta_{n,*}, F_n) &\tilde{\Sigma}_n(\theta_{n,*})D^{-1/2}(\theta_{n,*}, F_n) \\
& - D^{-1/2}(\theta_{n,*}, F_n) \text{Var}_{F_n}(m(W_1, \theta_{n,*}))D^{-1/2}(\theta_{n,*}, F_n) \to_p 0_{k \times k}, \\
&\widehat{D}^{-1/2}(\theta_{n,*}, F_n)D^{1/2}(\theta_{n,*}, F_n) \to_p 0_{k \times k}.
\end{align*}
\]
In consequence, $\widehat{\Omega}_n(\theta_{n,*}) - \Omega(\theta_{n,*}, F_n) \to_p 0_{k \times k}$. This and Assumptions LA1(a) and LA1(c) give $\widehat{\Omega}_n(\theta_{n,*}) \to_p \Omega_0$. The latter and Assumption 4(b) yield $c(\widehat{\Omega}_n(\theta_{n,*}), 1-\alpha) \to_p c(\Omega_0, 1-\alpha)$. This establishes Lemma 6(b) because $c(\Omega_0, 1-\alpha) = c_{0,p,0}\theta(1-\alpha)$ by definition.

If $c_{0,p,0}\theta(1-\alpha) > 0$, Lemma 6(c) holds by parts (a) and (b) and Assumption 5(a)(i). Assumption 1(c) implies that $c_{0,p,0}\theta(1-\alpha) = 0$. The conditions $c_{0,p,0}\theta(1-\alpha) = 0$ and $0 < \alpha < 1/2$ are consistent with Assumption 2(c) only if $v = 0$ (because $c_{0,p,0}\theta(1-\alpha)$ is the $1-\alpha$ quantile of $S(Z, \Omega_0)$). Given $v = 0$ and $c_{0,p,0}\theta(1-\alpha) = 0$, Lemma 6(c) holds by the same argument as used to prove Lemma 5(d) when $c_{g_1,0}\theta(1-\alpha) = 0$. Q.E.D.

PROOF OF LEMMA 7: First we prove part (a). By assumption, $c_{n}^{-1}(1-\alpha) > 0$. For $(\xi, \Omega) \in R^k \times \Psi$, let $\varphi^*(\xi, \Omega)$ denote the $k$-vector whose $j$th element is
\[
\varphi^*_j(\xi, \Omega) = \begin{cases} 
0, & \text{if } \pi_{1,j} < \infty \text{ and } j = 1, \ldots, p, \\
\varphi_j(\xi, \Omega), & \text{if } \pi_{1,j} = \infty \text{ and } j = 1, \ldots, p, \\\n0, & \text{if } j = p+1, \ldots, k. 
\end{cases}
\]
By Assumption GMS6, \( \varphi_j(\xi, \Omega) \geq 0 \) for \( j \leq p \). Hence, \( \varphi^*(\xi, \Omega) \leq \varphi(\xi, \Omega) \) and

\[
(2.18) \quad \varphi^*(\xi_n(\theta_n), \hat{\Omega}_n(\theta_n)) \leq \varphi(\xi_n(\theta_n), \hat{\Omega}_n(\theta_n)) \quad \text{a.s.}[Z^*] \text{ for all } n.
\]

Let \( c_n^* \) denote the \( 1 - \alpha \) quantile of the conditional df of \( S(\hat{\Omega}_n^{1/2}(\theta_n)Z^* + \varphi^*(\xi_n(\theta_n), \hat{\Omega}_n(\theta_n), \hat{\Omega}_n(\theta_n)) \) given \( (\xi_n(\theta_n), \hat{\Omega}_n(\theta_n)) \). Then \( c_n(\theta_n, 1 - \alpha) \leq c_n^* \) a.s. for all \( n \) by (2.18) and Assumption 1(a). By the same argument as in the proof of Lemma 4, (2.8) holds.

For \( j = 1, \ldots, p \) with \( \pi_{1,j} < \infty \), \( \varphi_j^*(\xi, \Omega) = 0 \). For \( j = 1, \ldots, p \) with \( \pi_{1,j} = \infty \), as \( (\xi, \Omega) \rightarrow ((\pi_{1}, 0), \Omega_0) \), we have \( \varphi_j(\xi, \Omega) \rightarrow \infty \) a.s. \([Z^*]\) by Assumption GMS3. These results can be written as

\[
(2.19) \quad \varphi_j^*(\xi, \Omega) \rightarrow \pi_{1,j}^* \quad \text{a.s.}[Z^*]
\]

for \( j = 1, \ldots, p \) by the definition of \( \pi_{1,j}^* \).

Assumption 1(d) and (2.19) give that for \( x \) in a neighborhood of \( c_{\pi_{1,j}}^*(1 - \alpha) \), as \( (\xi, \Omega) \rightarrow ((\pi_{1}, 0), \Omega_0) \),

\[
(2.20) \quad S(\Omega^{1/2}Z^* + \varphi^*(\xi, \Omega), \Omega) \rightarrow S(\Omega^{1/2}Z^* + (\pi_{1,j}^*, 0), \Omega_0) \quad \text{a.s.}[Z^*],
\]

\[
1(S(\Omega^{1/2}Z^* + \varphi^*(\xi, \Omega), \Omega) \leq x)
\]

\[
\rightarrow 1(S(\Omega^{1/2}Z^* + (\pi_{1,j}^*, 0), \Omega_0) \leq x) \quad \text{a.s.}[Z^*],
\]

\[
P(S(\Omega^{1/2}Z^* + \varphi^*(\xi, \Omega), \Omega) \leq x)
\]

\[
\rightarrow P(S(\Omega^{1/2}Z^* + (\pi_{1,j}^*, 0), \Omega_0) \leq x).
\]

The third convergence result of (2.20) holds by the second result and the bounded convergence theorem. The second convergence result of (2.20) follows from the first result provided \( P(S(\Omega^{1/2}Z^* + (\pi_{1,j}^*, 0), \Omega_0) = x) = 0 \), which holds because \( c_{\pi_{1,j}}^*(1 - \alpha) > 0 \) for the same reason that the second equality in (1.16) holds.

Given (2.8) and (2.20), the remainder of the proof of part (a) is the same as that given in the paragraph containing (1.17) using Lemma 5 of AG1.

Now we prove part (b). If \( c_{\pi_{1,j}}^*(1 - \alpha) > 0 \), part (b) of the lemma holds by part (a), Lemma 4(a) (i.e., \( T_n(\theta_n, \pi_1) \rightarrow_d J_{h_1,\lambda} \)), and Assumption 5(a)(i).

Next, we prove part (b) for the case where \( c_{\pi_{1,j}}^*(1 - \alpha) = 0 \). We have

\[
(2.21) \quad \lim \inf_{n \rightarrow \infty} P_{\gamma_n}(T_n(\theta_n) \leq c_n^*) \geq \lim \inf_{n \rightarrow \infty} P_{\gamma_n}(T_n(\theta_n) \leq 0)
\]

because \( c_n^* \geq 0 \) by Assumption 1(c). By the definition of \( \pi_{1,j}^* \), we have \( \pi_{1,j}^* \leq h_1 \).

As in (1.18), this implies that \( c_h(1 - \alpha) \leq c_{\varphi^*}^*(1 - \alpha) \) and hence \( c_h(1 - \alpha) = 0 \) using Assumption 1(c). The conditions \( c_h(1 - \alpha) = 0 \) and \( 0 < \alpha < 1/2 \) are consistent with Assumption 2(c) only if \( v = 0 \). Given \( v = 0 \), we use the same
When the inequality is not strict, part (c) holds because (i) \( \pi_{1}^{**} \geq g_{1} \), which holds because if \( \pi_{1,j} = \infty \), then \( \pi_{1,j} = \infty \), and if \( \pi_{1,j} < \infty \) then \( g_{1,j} = 0 \) by Assumptions LA4 and GMS5, and \( \pi_{1,j}^{**} = 0 \) by definition; (ii) \( S(\Omega_{0}^{1/2}Z^{*} + (\pi_{1}^{**}, 0, \nu), \Omega_{0}) \leq S(Z + (g_{1,0}, \nu), \Omega_{0}) \) a.s. by (i) and Assumption 1(a), and (iii) the corresponding quantiles satisfy \( c_{\pi_{1}^{**}}(1 - \alpha) \leq c_{g_{1,0}}(1 - \alpha) \) by (ii).

Next, we show part (c) holds with a strict inequality when \( c_{g_{1,0}}(1 - \alpha) > 0 \), \( g_{1,j} < \infty \), and \( \pi_{1,j} = \infty \) for some \( j = 1, \ldots, p \). The latter implies that \( \pi_{1}^{**} > g_{1} \).

Given \( \pi_{1}^{**} > g_{1} \) and \( c_{g_{1,0}}(1 - \alpha) > 0 \), Assumption 5(b) implies that

\[
(S2.22) \quad P(S(Z + (\pi_{1}^{**}, 0, \nu), \Omega) \leq c_{g_{1,0}}(1 - \alpha)) > P(S(Z + (g_{1,0}, \nu), \Omega) \leq c_{g_{1,0}}(1 - \alpha)) \geq 1 - \alpha,
\]

where \( Z \sim N(0_{k}, \Omega) \). If “\( v = 0 \) and \( \pi_{1}^{**} = \infty^{0} \)” does not hold, then the df of \( S(Z + (\pi_{1}^{**}, 0, \nu), \Omega) \) is strictly increasing for \( x > 0 \) by Assumption 2(b). This and (S2.22) imply that \( c_{\pi_{1}^{**}}(1 - \alpha) < c_{g_{1,0}}(1 - \alpha) \). If \( v = 0 \) and \( \pi_{1}^{**} = \infty^{0} \), then \( S(Z + (\pi_{1}^{**}, 0, \nu), \Omega) = S(Z + \infty^{0}, \Omega) = 0 \) by Assumption 1(c) and \( c_{\pi_{1}^{**}}(1 - \alpha) = 0 < c_{g_{1,0}}(1 - \alpha) \), and the proof of part (c) is complete.

Part (d) follows immediately from part (c) when the inequality is not strict. When \( c_{g_{1,0}}(1 - \alpha) > 0 \), \( g_{1,j} < \infty \), and \( \pi_{1,j} = \infty \) for some \( j = 1, \ldots, p \), part (c) holds with a strict inequality. The latter, \( c_{\pi_{1}^{**}}(1 - \alpha) > 0 \) (which holds by Assumption 1(c)), and \( J_{h_{1,\nu}}(x) \) is strictly increasing for \( x > 0 \) (which holds by Assumption 5(a)(ii)) because the caveat in Assumption 5(a)(ii) that “\( v = 0 \) and \( \ell = \infty^{0} \) does not occur” holds by Assumption LA3) imply that part (d) holds with a strict inequality.

**Proof of Lemma 8:** Part (a) holds because for \( 0_{p} \leq g_{1} \in R_{+, \infty}^{p} \), we have

\[
(S2.23) \quad S(Z + (0_{p}, 0, \nu), \Omega_{0}) \geq S(Z + (g_{1,0}, \nu), \Omega_{0})
\]

by Assumption 1(a). Part (b) follows from part (a). To prove Lemma 8(c), note that \( c_{0_{p},0_{\nu}}(1 - \alpha) > 0 \) by Assumption 2(c) and \( \alpha \in (0, 1/2) \). This, Assumptions 2(a) and 5(b), and \( g_{1} > 0_{p} \) imply that

\[
(S2.24) \quad 1 - \alpha = P(S(Z + (0_{p}, 0, \nu), \Omega_{0}) \leq c_{0_{p},0_{\nu}}(1 - \alpha)) < P(S(Z + (g_{1,0}, \nu), \Omega_{0}) \leq c_{0_{p},0_{\nu}}(1 - \alpha)),
\]

where \( Z \sim N(0_{k}, \Omega_{0}) \). The latter and Assumption 2(a) prove part (c).

Lemma 8(d) holds by part (c), \( c_{g_{1,0}}(1 - \alpha) \geq 0 \) (which holds by Assumption 1(c)), and Assumption 5(a)(ii) (because the caveat in Assumption 5(a)(ii) that “\( v = 0 \) and \( \ell = \infty^{0} \) does not occur” holds by Assumption LA3).

**Q.E.D.**
S3. PROOF OF RESULT FOR DISTANT ALTERNATIVES

In this section, we restate and prove Theorem 4.

THEOREM 4: Under Assumptions 1, 3, 6, and DA, the following results hold:
(a) \( \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,\ast}) > c_{n,h}(\theta_{n,\ast}, 1 - \alpha)) = 1 \) provided Assumption GMS7 holds.
(b) \( \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,\ast}) > c_{n,h}(\theta_{n,\ast}, 1 - \alpha)) = 1. \)
(c) \( \lim_{n \to \infty} P_{F_n}(T_n(\theta_{n,\ast}) > c(\hat{\Omega}_n(\theta_{n,\ast}), 1 - \alpha)) = 1. \)

PROOF: It suffices to show that for any subsequence \( \{t_n\} \) of \( \{n\} \) there exists a sub-subsequence \( \{s_n\} \) such that \( \lim_{n \to \infty} P_{F_n}(T_{s_n}(\theta_{n,\ast}) > c_{1-n}) = 1 \), where \( c_{1-n} = c_n(\theta_{n,\ast}, 1 - \alpha), c_{n,h}(\theta_{n,\ast}, 1 - \alpha) \), or \( c(\hat{\Omega}_n(\theta_{n,\ast}), 1 - \alpha) \). We can take the subsequence \( \{s_n\} \) to be such that \( m_{s_{n,j}}^*/\beta_{s_n} \to e_j \) for some \( e_j \in [-1, \infty] \) for \( j = 1, \ldots, k \) because \( \{m_{s_{n,j}}^*/\beta_{s_n}: n \geq 1\} \) is a sequence of points in the set \([-1, \infty]\) by the definition of \( \beta_{s_n} \). For notational simplicity, we establish the former result with \( s_n \) replaced by \( n \) and by a subsequence argument we assume without loss of generality (wlog) that

(S3.1) \( m_{s_{n,j}}^*/\beta_n \to e_j \) for some \( e_j \in [-1, \infty] \) for \( j = 1, \ldots, k \).

The following equation is used in the proofs of parts (a)–(c). We have

(S3.2) \[
\left( n^{1/2}/\beta_n \right)^{-x} T_n(\theta_{n,\ast}) \\
= \left( n^{1/2}/\beta_n \right)^{-x} \\
\times S(\hat{D}_n^{1/2}(\theta_{n,\ast}) n^{1/2}m_n(\theta_{n,\ast}), \hat{D}_n^{1/2}(\theta_{n,\ast}) \hat{\Sigma}_n(\theta_{n,\ast}) \hat{D}_n^{1/2}(\theta_{n,\ast})) \\
= \left( n^{1/2}/\beta_n \right)^{-x} S(\hat{D}_n^{1/2}(\theta_{n,\ast}) D^{1/2}(\theta_{n,\ast}, F_n)(A_n^0 + n^{1/2}m_n^*), \Omega_1 + o_p(1)) \\
\to_p S(e, \Omega_1) > 0,
\]

where \( A_n^0 \) is defined in (S2.3), \( m_n^* = (m_{n,1}^*, \ldots, m_{n,k}^*)' \), \( e = (e_1, \ldots, e_k)' \), the first equality uses Assumption 1(b), the second equality holds by the definitions of \( A_n^0, m_n^* \), and \( D(\theta_{n,\ast}, F_n) \) and by (S2.3) (with \( \Omega_1 \) in place of \( \Omega_0 \) and with Assumption DA(b) used in place of Assumption LA1(a) in the proof of (S2.3)), the third equality holds by Assumptions 6 and DA(a) and (S2.3) (with the same adjustments as above), the convergence holds by Assumption 1(d) and (S3.1), and the inequality holds by Assumption 3 because for some \( j^* \leq k \) the \( j^* \)th element of \( e \) is \( e_{j^*} \), has absolute value equal to 1 and is negative if \( j^* \leq p \), which implies that \( e_{j^*} < 0 \) if \( j^* \leq p \) and \( e_{j^*} \neq 0 \) if \( j^* \geq p + 1 \).

We prove part (b) first. By another subsequence argument, we can assume wlog that \( \lim_{n \to \infty} b_1^{1/2}/\beta_n \) exists and (S3.1) holds. We consider two cases:
(i) \( \lim_{n \to \infty} b^{1/2} \beta_n = \infty \) and (ii) \( \lim_{n \to \infty} b^{1/2} \beta_n \in [0, \infty) \). When case (i) holds, the same argument used to show (S3.2) gives

\[
\text{(S3.3)} \quad (b^{1/2} \beta_n)^{-x} T_b(\theta_{n,*}) \to_p S(e, \Omega_1),
\]

where \( \beta_n \) appears, not \( \beta_b \), because \( m_n^*/\beta_n \to e \) under \( \{F_n:n \geq 1\} \). Equation (S3.3) and \( b/n \to 0 \) imply that \( T_{n,b}^\dagger(\theta_{n,*}) = (n^{1/2} \beta_n)^{-x} T_b(\theta_{n,*}) \to_p 0 \).

Define \( U_{n,b}(\theta_{n,*}, x) \) as \( U_{n,b}(\theta_{n,*}, x) \) is defined but with \( T_{n,b,j}^\dagger(\theta_{n,*}) = (n^{1/2} \beta_n)^{-x} T_{n,b,j}(\theta_{n,*}) \) in place of \( T_{n,b,j}(\theta_{n,*}) \). Using the result of the previous paragraph, we have \( E_{F_n} U_{n,b}^\dagger(\theta_{n,*}, x) = P_{F_n}(T_{n,b}^\dagger(\theta_{n,*}) \leq x) \to 0 \) for \( x < 0 \) and \( \to 1 \) for \( x > 0 \). In addition, \( \text{Var}_{F_n}(U_{n,b}^\dagger(\theta_{n,*}, x)) \to 0 \) by Hoeffding’s \( U \)-statistic inequality for bounded i.i.d. random variables; see Politis, Romano, and Wolf (1999, p. 44). Hence, \( U_{n,b}^\dagger(\theta_{n,*}, x) \to_p 0 \) for \( x < 0 \) and \( \to_p 1 \) for \( x > 0 \). This and Lemma 5(a) of AG1 imply that \( c_{n,b}(\theta_{n,*}, 1 - \alpha) \to_p 0 \), where \( c_{n,b}(\theta_{n,*}, 1 - \alpha) \) is the \( 1 - \alpha \) quantile of the rescaled subsample statistics \( \{T_{n,b,j}^\dagger(\theta_{n,*}): j = 1, \ldots, q_n\} \). The latter result and (S3.2) give

\[
\text{(S3.4)} \quad P_{F_n}(T_n(\theta_{n,*}) > c_{n,b}(\theta_{n,*}, 1 - \alpha)) = P_{F_n}\left((n^{1/2} \beta_n)^{-x} T_n(\theta_{n,*}) > (n^{1/2} \beta_n)^{-x} c_{n,b}(\theta_{n,*}, 1 - \alpha)\right) = P_{F_n}\left((n^{1/2} \beta_n)^{-x} T_n(\theta_{n,*}) > c_{n,b}^\dagger(\theta_{n,*}, 1 - \alpha)\right) \to P(S(e, \Omega_1) > 0) = 1,
\]

where the second equality holds because \( c_{n,b}^\dagger(\theta_{n,*}, 1 - \alpha) = (n^{1/2} \beta_n)^{-x} c_{n,b}(\theta_{n,*}, 1 - \alpha) \) by the scale equivariance of quantiles and the last equality holds because \( S(e, \Omega_1) > 0 \) by (S3.2).

Next, suppose case (ii) holds. Then the same argument as used to show (S3.2) but with \( (n^{1/2} \beta_n)^{-x} \) deleted gives

\[
\text{(S3.5)} \quad T_b(\theta_{n,*}) = S(O_p(1) + (b^{1/2} \beta_n)^{-1} m_n^*, \Omega_1 + o_p(1)) = O_p(1),
\]

where the second equality uses Assumption 1(a). Hence, \( T_{n,b}^\dagger(\theta_{n,*}) = (n^{1/2} \beta_n)^{-x} T_b(\theta_{n,*}) \to_p 0 \). Given this, the remainder of the proof is the same as in case (i).

Next, we prove part (c). We have \( \widehat{\Omega}_n(\theta_{n,*}) \to_p \Omega_1 \) because (S2.16) of the paper holds by the argument given for (S2.16) but using condition (vii) of (2.2) and Assumption DA(b). This and Assumption 4(b) imply that \( c(\widehat{\Omega}_n(\theta_{n,*}), 1 - \alpha) \to_p c(\Omega_1, 1 - \alpha) \). Combining the latter with (S3.2) and (S3.4), with \( c_{n,b}(\theta_{n,*}, 1 - \alpha) \) replaced by \( c(\widehat{\Omega}_n(\theta_{n,*}), 1 - \alpha) \) in (S3.4), gives the desired result.

Finally, we prove part (a). By (S3.2) and the first equality of (S3.4) with \( c_{n,b}(\theta_{n,*}, 1 - \alpha) \) replaced by \( \widehat{c}_n(\theta_{n,*}, 1 - \alpha) \), it suffices to show that \( (n^{1/2} \beta_n)^{-x} \widehat{c}_n(\theta_{n,*}, 1 - \alpha) = o_p(1) \).
Let $A_n^0$ be defined as in (S2.3). We have

\begin{align}
(n^{1/2} \beta_n)^{-1} \kappa_n^{-1} n^{1/2} \hat{D}_n^{-1/2}(\theta_{n,\ast}) \bar{m}_n(\theta_{n,\ast}) \\
= (n^{1/2} \beta_n)^{-1} \kappa_n^{-1} (\hat{D}_n^{-1/2}(\theta_{n,\ast}) D_n^{1/2}(\theta_{n,\ast}, F_n)) (A_n^0 + n^{1/2} m_n^*) \\
= o_p(1) + \kappa_n^{-1} (m_n^*/\beta_n) (1 + o_p(1)),
\end{align}

where the second equality uses (S2.3) with $\Omega_0$ replaced by $\Omega_1$ (using Assumption DA(b)), $\kappa_n \to \infty$, and $n^{1/2} \beta_n \to \infty$. By the definition of $\beta_n$, $m_n^*/\beta_n \in [-1, \infty)$ for $j = 1, \ldots, k$ for all $n$. By a subsequence argument, wlog we assume $\kappa_n^{-1} m_{n,j}^*/\beta_n \to \eta_j \in [0, \infty]$ for $j = 1, \ldots, k$. This and (S3.6) give

\begin{align}
(n^{1/2} \beta_n)^{-1} \xi_n(\theta_{n,\ast}) &\to_p \eta = (\eta_1, \ldots, \eta_k) \in \mathbb{R}^p_{+\infty} \times \mathbb{R}^v, \\
\Phi_{n,1} &\equiv (n^{1/2} \beta_n)^{-1} \min(\xi_n, \xi_n, 0), \ldots, \min(\xi_n, \eta, 0), 0, \ldots, 0) \\
&\to_p 0_k,
\end{align}

where $\xi_n(\theta_{n,\ast}) = (\xi_n, 1(\theta_{n,\ast}), \ldots, \xi_n, k(\theta_{n,\ast}))'$.

Using Assumption 6, we have

\begin{align}
(n^{1/2} \beta_n)^{-1} S(\hat{\Omega}_n^{1/2}(\theta_{n,\ast}) Z^* + \varphi(\hat{\Omega}_n(\theta_{n,\ast}), \hat{\Omega}_n(\theta_{n,\ast})) \\
= S((n^{1/2} \beta_n)^{-1} \hat{\Omega}_n^{1/2}(\theta_{n,\ast}) Z^* + \varphi(\hat{\Omega}_n(\theta_{n,\ast}), \hat{\Omega}_n(\theta_{n,\ast}))) \\
\leq S(\Phi_{n,2} Z^* + \Phi_{n,1}, \hat{\Omega}_n(\theta_{n,\ast})),
\end{align}

where $\Phi_{n,2} \equiv (n^{1/2} \beta_n)^{-1} \hat{\Omega}_n^{1/2}(\theta_{n,\ast}) (\in \mathbb{R}^{k \times k})$ and the inequality holds by Assumptions 1(a) and GMS7. We have $\Phi_{n,2} = o_p(1)$ by (S2.3) and Assumption DA(a). Let $\tilde{\eta}_n$ denote the $1 - \alpha$ quantile of $S(\Phi_{n,2} Z^* + \Phi_{n,1}, \hat{\Omega}_n(\theta_{n,\ast}))$ in (S3.8).

By (S3.8), $(n^{1/2} \beta_n)^{-1} \tilde{\eta}_n(1 - \alpha) \leq \tilde{\eta}_n$. Hence, it suffices to show that $\tilde{\eta}_n = o_p(1)$. To do so, we use a similar argument to that in (S1.15). For $x > 0$, as $(\xi, \Omega_a, \Omega_b) \to (0_k, 0_k \times k, \Omega_1)$, we have

\begin{align}
S(\Omega_a^{1/2} Z^* + \xi, \Omega_b) &\to S(0_k, \Omega_1) = 0 \text{ a.s. } [Z^*], \\
1(S(\Omega_a^{1/2} Z^* + \xi, \Omega_b) \leq x) &\to 1 (0 \leq x) \text{ a.s. } [Z^*], \\
P(S(\Omega_a^{1/2} Z^* + \xi, \Omega_b) \leq x) &\to 1,
\end{align}

where the equality in the first line uses Assumption 3, the second convergence result follows from the first result for $x > 0$, and the third convergence result holds by the second result and the bounded convergence theorem. The third
result of (S3.9), $(\Phi_{n,1}, \Phi_{n,2}, \hat{\Omega}_n(\theta_{n,\pi})) \to_p (0_k, 0_{k \times k}, \Omega_1)$ (which uses (S3.7)), and Slutsky’s theorem give

$$(S3.10) \quad P(S(\Phi_{n,2} \mathbf{Z}^* + \Phi_{n,1}, \hat{\Omega}_n(\theta_{n,\pi})) \leq x) \to_p 1 \quad \text{for all } x > 0,$$

where $P(\cdot)$ denotes the distribution of $\mathbf{Z}^*$ conditional on $(\Phi_{n,1}, \Phi_{n,2}, \hat{\Omega}_n(\theta_{n,\pi}))$. By Assumption 1(c), the probability limit in (S3.10) is zero for all $x < 0$. These results and Lemma 5(a) of AG1 imply that $\tilde{c}_n \to_p 0$, where $\tilde{c}_n$ is the $1 - \alpha$ quantile of the (random) df in (S3.10). This completes the proof of part (a).

Q.E.D.

**S4. VERIFICATION OF ASSUMPTIONS GMS1, GMS3, GMS6, AND GMS7 FOR $\varphi^{(5)}$**

We now verify Assumptions GMS1, GMS3, GMS6, and GMS7 for $\varphi^{(5)}$. Assumption GMS1(b) holds for $\varphi^{(5)}$ if $c_j(\xi, \Omega) = 1$ whenever the $j$th element of $\xi$ equals 0 by the definition of $\varphi^{(5)}$. If the $j$th element of $\xi$ equals zero, $c_j(\xi, \Omega)$ does not enter the criterion function $S(-c \cdot \xi, \Omega) - \eta(|c|)$. In consequence, the criterion function is minimized by taking $c_j(\xi, \Omega) = 1$ because $\eta(\cdot)$ is strictly increasing. Hence, Assumption GMS1(b) holds for $\varphi^{(5)}$.

We show Assumption GMS1(a) holds for $\varphi^{(5)}$ (provided $S$ satisfies Assumption 1(d)) by showing that if $(\xi_{[r]}, \Omega_{[r]}) \to (\xi, \Omega)$ as $r \to \infty$ and $\xi_j = 0$, then $c_j(\xi_{[r]}, \Omega_{[r]}) = 1$ for $r$ sufficiently large. By Assumption 1(d), $S$ is continuous at $(\xi, \Omega)$. Hence, $\lim_{r \to \infty} S(-c \cdot \xi_{[r]}, \Omega_{[r]}) \to S(-c \cdot \xi, \Omega)$ as $r \to \infty$. The limit $S(-c \cdot \xi, \Omega)$ does not depend on $c_j$ because $\xi_j = 0$. Given $\epsilon > 0$, there exists an $r^*$ sufficiently large that $|S(-c \cdot \xi_{[r]}, \Omega_{[r]}) - S(-c \cdot \xi, \Omega)| \leq \epsilon$ for all $c \in C$ and all $r \geq r^*$. Hence, the first term of the selection criterion, $S(-c \cdot \xi, \Omega)$, is reduced by at most $\epsilon$ if $c_j$ is chosen from 1 to 0, where $c = (c_1, \ldots, c_k)^\prime$. On the other hand, the second term of the selection criterion, $-\eta(|c|)$, is increased by $\eta(|c| + 1) - \eta(|c|) > 0$. Taking $\epsilon < \inf_{c \in C}(\eta(|c| + 1) - \eta(|c|))$ implies that the selection criterion is minimized by $c_j(\xi_{[r]}, \Omega_{[r]}) = 1$ for all $r \geq r^*$. Hence, Assumption GMS1(a) holds for $\varphi^{(5)}$.

Next we verify Assumption GMS3 for $\varphi^{(5)}$ for all functions $S$ for which $S(-c \cdot \xi, \Omega) \to \infty$ as $(\xi, \Omega) \to (\xi_*, \Omega_*)$ whenever $c_j = 1$. For any $c_s \in C$ with $c_{\ell} = 0$ for all $\ell$ such that $\xi_{\ell} = \infty$ we have $S(-c_s \cdot \xi, \Omega) \leq K$ as $(\xi, \Omega) \to (\xi_*, \Omega_*)$ for some $K < \infty$ by Assumption 1(d). Hence, some $c_s = (c_{s1}, \ldots, c_{sk})^\prime \in C$ with $c_{sj} = 0$ is selected over any $c = (c_1, \ldots, c_k)^\prime \in C$ with $c_j = 1$ as $(\xi, \Omega) \to (\xi_*, \Omega_*)$. This gives $c_j(\xi, \Omega) = 0$ and $\varphi_j(\xi, \Omega) = \infty$ (using the definition of $\varphi^{(5)}$) as $(\xi, \Omega) \to (\xi_*, \Omega_*)$.

Assumptions GMS6 and GMS7 hold immediately for $\varphi^{(5)}$ by its definition.

**S5. MEAN VECTORS FOR SIMULATION RESULTS**

The main paper provides simulation results concerning the finite-sample average power of PA, subsampling, and GMS tests. Here we define the sets of
mean vectors, denoted $\mathcal{M}_p(\Omega)$, for which the average power is calculated for $p = 2, 4$, and 10 and $\Omega_{\text{Neg}}$, $\Omega_{\text{Zero}}$, and $\Omega_{\text{Pos}}$. Each element of $\mathcal{M}_p(\Omega)$ is in the alternative hypothesis.

For $p = 2$, the $\mu$ vectors considered are

\begin{align}
\mathcal{M}_2(\Omega_{\text{Neg}}) &= \{(-1.001, 0), (-1.804, 1), (-2.303, 2), (-2.309, 3), \\
&\quad (-2.309, 4), (-2.309, 7), (-.5165, -.5165)\}, \\
\mathcal{M}_2(\Omega_{\text{Zero}}) &= \{(-2.309, 0), (-2.309, 1), (-2.309, 2), (-2.309, 3), \\
&\quad (-2.309, 4), (-2.309, 7), (-1.6263, -1.6263)\}, \\
\mathcal{M}_2(\Omega_{\text{Pos}}) &= \mathcal{M}_k(\Omega_{\text{Zero}}) \text{ except the last vector is } (-2.0040, -2.0040).
\end{align}

The finite-sample power envelope (for known $\Omega$) at each of these $\mu$ vectors is .75.

For $p = 4$, $\mathcal{M}_4(\Omega)$ includes 24 elements and is of the form

\begin{align}
\mathcal{M}_4(\Omega) &= \{(-\mu_1, -\mu_1, 1, 1), (-\mu_2, -\mu_2, 2, 2), (-\mu_3, -\mu_3, 3, 3), \\
&\quad (-\mu_4, -\mu_4, 4, 4), (-\mu_5, -\mu_5, 7, 7), (-\mu_6, -\mu_6, 1, 7), \\
&\quad (-\mu_7, -\mu_7, 2, 7), (-\mu_8, -\mu_8, 3, 7), (-\mu_9, -\mu_9, 4, 7), \\
&\quad (-\mu_{10}, 1, 1, 1), (-\mu_{11}, 2, 2, 2), (-\mu_{12}, 3, 3, 3), \\
&\quad (-\mu_{13}, 4, 4, 4), (-\mu_{14}, 7, 7, 7), (-\mu_{15}, 1, 1, 7), \\
&\quad (-\mu_{16}, 2, 2, 7), (-\mu_{17}, 3, 3, 7), (-\mu_{18}, 4, 4, 7), \\
&\quad (-\mu_{19}, -\mu_{19}, 0, 0), (-\mu_{20}, 0, 0, 0), (-\mu_{21}, 25, 25, 25), \\
&\quad (-\mu_{22}, -\mu_{22}, 25, 25), (-\mu_{23}, -\mu_{23}, -\mu_{23}, 25), \\
&\quad (-\mu_{24}, -\mu_{24}, -\mu_{24}, -\mu_{24})\},
\end{align}

where $\mu_j$ depends on $\Omega$ and is such that the finite-sample power envelope (for known $\Omega$) is .80 at each element of $\mathcal{M}_4(\Omega)$.

The $\mu$ vectors in $\mathcal{M}_4(\Omega_{\text{Neg}})$ are defined by (S5.2) and the following values: $\mu_1 = -.5505$, $\mu_j = -.5526$ for $j = 2, \ldots, 5$, $\mu_6 = -.5505$, $\mu_j = -.5526$ for $j = 7, 8, 9$, $\mu_{10} = -1.8814$, $\mu_{11} = -2.4283$, $\mu_j = -2.4705$ for $j = 12, 13, 14, 17, 18, 21$, $\mu_{15} = -1.8814$, $\mu_{16} = -2.4283$, $\mu_{19} = -3.176$, $\mu_{20} = -.8624$, $\mu_{22} = -.5526$, $\mu_{23} = -.2607$, and $\mu_{24} = -1.1756$.

The $\mu$ vectors in $\mathcal{M}_4(\Omega_{\text{Zero}})$ are defined by (S5.2) and the following values: $\mu_j = 1.7388$ for $j = 1, \ldots, 9, 19, 22$, $\mu_j = -2.4705$ for $j = 10, \ldots, 18, 20, 21$, $\mu_{23} = 1.4242$, and $\mu_{24} = 1.2350$.

The $\mu$ vectors in $\mathcal{M}_4(\Omega_{\text{Pos}})$ are defined by (S5.2) and the following values: $\mu_j = 2.4047$ for $j = 1, \ldots, 9, 19, 22$, $\mu_j = -2.4705$ for $j = 10, \ldots, 18, 20, 21$, $\mu_{23} = 2.2628$, and $\mu_{24} = -2.1293$. 

For $p = 10$, $M_{10}(\Omega)$ includes 40 vectors and is of the form:

(S5.3) \[ M_{10}(\Omega) = \{ (-\mu_1, -\mu_1, 1, \ldots, 1), (-\mu_2, -\mu_2, 2, \ldots, 2), \]
\[ (-\mu_3, -\mu_3, 3, \ldots, 3), (-\mu_4, -\mu_4, 4, \ldots, 4), \]
\[ (-\mu_5, -\mu_5, 7, \ldots, 7), (-\mu_6, -\mu_6, 1, 1, 1, 7, \ldots, 7), \]
\[ (-\mu_7, -\mu_7, 2, 2, 2, 7, \ldots, 7), (-\mu_8, -\mu_8, 3, 3, 3, 7, \ldots, 7), \]
\[ (-\mu_9, -\mu_9, 4, 4, 4, 7, \ldots, 7), \]
\[ (-\mu_{10}, -\mu_{10}, -\mu_{10}, -\mu_{10}, 1, \ldots, 1), \]
\[ (-\mu_{11}, -\mu_{11}, -\mu_{11}, -\mu_{11}, 2, \ldots, 2), \]
\[ (-\mu_{12}, -\mu_{12}, -\mu_{12}, -\mu_{12}, 3, \ldots, 3), \]
\[ (-\mu_{13}, -\mu_{13}, -\mu_{13}, -\mu_{13}, 4, \ldots, 4), \]
\[ (-\mu_{14}, -\mu_{14}, -\mu_{14}, -\mu_{14}, 7, \ldots, 7), \]
\[ (-\mu_{15}, -\mu_{15}, -\mu_{15}, -\mu_{15}, 1, 1, 1, 7, 7, 7), \]
\[ (-\mu_{16}, -\mu_{16}, -\mu_{16}, -\mu_{16}, 2, 2, 2, 7, 7, 7), \]
\[ (-\mu_{17}, -\mu_{17}, -\mu_{17}, -\mu_{17}, 3, 3, 3, 7, 7, 7), \]
\[ (-\mu_{18}, -\mu_{18}, -\mu_{18}, -\mu_{18}, 4, 4, 4, 7, 7, 7), (-\mu_{19}, 1, \ldots, 1), \]
\[ (-\mu_{20}, 2, \ldots, 2), (-\mu_{21}, 3, \ldots, 3), (-\mu_{22}, 4, \ldots, 4), \]
\[ (-\mu_{23}, 7, \ldots, 7), (-\mu_{24}, 1, 1, 1, 7, \ldots, 7), \]
\[ (-\mu_{25}, 2, 2, 2, 7, \ldots, 7), (-\mu_{26}, 3, 3, 3, 7, \ldots, 7), \]
\[ (-\mu_{27}, 4, 4, 4, 7, \ldots, 7), (-\mu_{28}, -\mu_{28}, 0, \ldots, 0), \]
\[ (-\mu_{29}, -\mu_{29}, -\mu_{29}, -\mu_{29}, 0, \ldots, 0), (-\mu_{30}, 0, \ldots, 0), \]
\[ (-\mu_{31}, 25, \ldots, 25), (-\mu_{32}, -\mu_{32}, 25, \ldots, 25), \]
\[ (-\mu_{33}, -\mu_{33}, -\mu_{33}, 25, \ldots, 25), \]
\[ (-\mu_{34}, -\mu_{34}, -\mu_{34}, -\mu_{34}, 25, \ldots, 25), \]
\[ (-\mu_{35}, -\mu_{35}, -\mu_{35}, -\mu_{35}, -\mu_{35}, 25, \ldots, 25), \]
\[ (-\mu_{36}, \ldots, -\mu_{36}, 25, 25, 25, 25), \]
\[ (-\mu_{37}, \ldots, -\mu_{37}, 25, 25, 25), (-\mu_{38}, \ldots, -\mu_{38}, 25, 25), \]
\[ (-\mu_{39}, \ldots, -\mu_{39}, 25), (-\mu_{40}, \ldots, -\mu_{40}) \}. \]

The $\mu$ vectors in $M_{10}(\Omega_{Ne8})$ are defined by (S5.3) and the following values:

$\mu_j = .6016$ for $j = 1, \ldots, 9$, $\mu_j = .3475$ for $j = 10, \ldots, 18$, $\mu_{19} = 1.9847$, $\mu_{20} = 2.5835$, $\mu_j = 2.6817$ for $j = 21, 22, 23, 26, 27, 31$, $\mu_{24} = 1.9847$, $\mu_{25} = 2.5835$. 
\[ \mu_{28} = .5341, \mu_{29} = .3322, \mu_{30} = 1.1551, \mu_{32} = .6016, \mu_{33} = .4195, \mu_{34} = .3475, \mu_{35} = .2985, \mu_{36} = .2674, \mu_{37} = .2430, \mu_{38} = .2254, \mu_{39} = .2106, \text{and } \mu_{40} = .1993. \]

The \( \mu \) vectors in \( M_10(\Omega_{\text{zere}}) \) are defined by (S5.3) and the following values:
\[ \mu_j = 1.8927 \text{ for } j = 1, \ldots, 9, 28, 32, \mu_j = 1.3360 \text{ for } j = 10, \ldots, 18, 29, 34, \mu_j = 2.6817 \text{ for } j = 19, \ldots, 27, 30, 31, \mu_{33} = 1.5463, \mu_{35} = 1.1963, \mu_{36} = 1.0893, \mu_{37} = 1.0099, \mu_{38} = 0.9645, \mu_{39} = 0.8882, \text{and } \mu_{40} = 0.8440. \]

The \( \mu \) vectors in \( M_10(\Omega_{\text{pos}}) \) are defined by (S5.3) and the following values:
\[ \mu_j = 2.6227 \text{ for } j = 1, \ldots, 9, \mu_j = 2.4676 \text{ for } j = 10, \ldots, 18, \mu_j = 2.6817 \text{ for } j = 19, \ldots, 27, \mu_{29} = 2.6227, \mu_{30} = 2.6817, \mu_{31} = 2.6817, \mu_{32} = 2.6227, \mu_{33} = 2.5401, \mu_{34} = 2.4676, \mu_{35} = 2.4005, \mu_{36} = 2.3140, \mu_{37} = 2.2846, \mu_{38} = 2.2565, \mu_{39} = 2.2343, \text{and } \mu_{40} = 2.2066. \]

For \( p = 10 \), the finite-sample power envelope (for known \( \Omega \)) at each of the \( \mu \) vectors is .85.

**S6. MONTE CARLO EXPERIMENTS I**

**S6.1. Introduction**

In this section, we report some additional simulation results. We use simulation to investigate the finite-sample properties of GMS CS’s and to compare them to some other methods in the literature. We consider the coverage probabilities (CP’s) of the CS’s for points in and not in the identified set. For points on the boundary of the identified set and for which all inequalities are binding (i.e., hold as equalities), the CP’s should be close to the nominal level \( 1 - \alpha \). For points on the boundary of the identified set and for which some inequality is not binding, the CP’s should be greater than or equal to \( 1 - \alpha \). Probabilities for these points indicate the nonsimilarity on the boundary of the CS’s. For points in the interior of the identified set, the CP’s should be greater than \( 1 - \alpha \). For points that are not in the identified set, the CP’s should be less than \( 1 - \alpha \)—the smaller, the better.

We consider two very simple models. The first is a particular case of the missing-data model considered in Imbens and Manski (2004) (IM). In this model, there is one parameter, two moment inequalities, and no moment equalities. We consider the GMS CS based on the MMM test statistic (i.e., the test function \( S_1 \)) with the MMSC function \( \varphi^{(5)} \) with \( \eta(x) = x \) and \( \kappa_n = (2.01 \ln \ln n)^{1/2} \) (i.e., the HQIC MMSC procedure). We compare the GMS CS to the CS introduced by IM for this model (see IM for its definition) and to the subsampling CS based on the MMM test statistic, and with subsample size \( b = \lfloor n^{1/2} \rfloor \), the integer part of \( n^{1/2} \). (We mention, but do not report, results for other values of \( b \).) Rosen (2008) provided a comparison of the finite-sample properties of his proposed CS with that of IM.

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1We take \( \tilde{\sigma}_{n,j}^2(\theta) = n^{-1} \sum_{i=1}^n m_j^2(W, \theta) \) for \( j = 1, \ldots, k \), rather than \( \tilde{\sigma}_{n,j}^2(\theta) = n^{-1} \sum_{i=1}^n (m_j(W_i, \theta) - \overline{m}_{n,j}(\theta))^2 \). Results for the latter are similar.
The second model considered is the interval-outcome regression model of Manski and Tamer (2002). In this model, there are two parameters, two moment inequalities, and no moment equalities. We compare the same GMS and subsampling procedures as defined above. (The IM CS does not apply to this model.)

All results reported are for CS’s with nominal level .95. For both models, we report results for $n = 100, 500$, and 1000. We take the number of simulation repetitions, $R$, to be 20,000 for the GMS and IM CS’s and 5000 for the subsampling CS’s. The reported CP’s are the relative frequencies of coverage over the $R$ repetitions.

S6.2. Missing-Data Model

In this model, $W_i = (Y_i, D_i)$ are i.i.d. for $i = 1, \ldots, n$ with $Y_i \sim U[0, 1]$, $D_i \sim \text{Bern}[.85]$, and $Y_i$ and $D_i$ independent. The observations are $\{(Y_i, D_i, D_i): i \leq n\}$. Thus, $Y_i$ is not observed when $D_i = 0$. The parameter of interest is $\theta = EY_i$.

Given that data are missing, the parameter $\theta$ is not identified. Two moment inequality functions used to bound $\theta$ are

$$m_1(W_i, \theta) = (\theta - Y_iD_i, (1 - \theta) - (1 - Y_i)D_i) .$$

When $\theta$ is the true parameter, we have $Em_1(W_i, \theta) = \theta - EY_iD_i \geq \theta - EY_i = 0$, where the inequality holds because $Y_i \geq 0$ and $D_i \leq 1$ and $Em_2(W_i, \theta) = (1 - \theta) - E(1 - Y_i)D_i \geq (1 - \theta) - E(1 - Y_i) = 0$, where the inequality holds because $1 - Y_i \geq 0$ and $D_i \leq 1$. Hence, the functions in (S6.1) satisfy two moment inequalities.

For the data-generating process above, the identified set $[\theta_L, \theta_U]$ is $[.425, .575]$. We consider the CP’s of the CS’s for the values $\theta_L = .425$, $\theta = .5$, and $\theta_U = .575$, which are in the identified set, and for the values $.9 \times \theta_L$ and $1.1 \times \theta_U$, which are not in the identified set.

Table S.I reports the CP’s of the GMS, IM, and subsampling CS’s with nominal level 95%. The table shows that for $\theta$ values in the identified set, the performance of the GMS and IM CS’s is excellent for all sample sizes. Probabilities for the GMS CS for boundary $\theta$ points range from .948 to .951. In contrast, the subsampling CS overcovers by a noticeable amount for all sample sizes. Probabilities for the subsampling CS for boundary $\theta$ points range from .971 to .990. (This overcoverage is a finite-sample phenomenon because the subsampling asymptotic CP is .95 at both boundaries.) All three CS’s cover $\theta = .5$, which

---

2The subsampling CS’s are more computationally intensive than the moment selection CS’s. Only 5000 repetitions are used for the moment selection results of Table III for $\theta$ not in the identified set.

3The identified set is determined by $\theta_L - EY_iD_i = 0$ (i.e., $\theta_L = .5 \times .85 = .425$) and $(1 - \theta_U) - E(1 - Y_i)D_i = 0$ (i.e., $\theta_U = 1 - E(1 - Y_i)ED_i = 1 - .5 \times .85 = .575$).
lies in the interior of the identified set and is far from either boundary, with probability 1. This is in accord with the asymptotic results.

For \( \theta \) points not in the identified set, we want the CP of a CS to be as close to zero as possible. (A lower CP for such points translates into a shorter and more informative CS.) Table S.I shows that the GMS CS covers points not in the identified set with substantially lower probability than the subsampling CS when \( n = 100 \) (viz., .619 versus .791 and .445 versus .667) and with slightly lower probability for \( n = 500 \) (viz., .095 versus .145 and .010 versus .030). This is consistent with the asymptotic power comparisons given in the main paper. For points not in the identified set, the GMS and IM CS’s have comparable CP’s. For \( n = 500 \) and 1000, the CP’s of all three CS’s are sufficiently low that the differences between them are small.

As has been reported in other scenarios, subsampling CP’s are sensitive to the choice of the subsample size \( b \). Additional simulation results not reported here show that for smaller \( b \), the subsampling CP’s for \( \theta \) in the identified set become slightly closer to the nominal level, while for larger subsample sizes, they become closer to 1. For \( \theta \) not in the identified set, smaller \( b \) reduces the subsampling CP’s slightly and larger \( b \) increases them slightly.

S6.3. Interval-Outcome Regression Model

This model is a regression model with unobserved dependent variable \( Y_i \),

\[
Y_i = \theta_1 + Z_i \theta_2 + U_i, \tag{S6.2}
\]
where \((Z_i, U_i)\) are i.i.d. for \(i = 1, \ldots, n\), \(Z_i \sim N(1, 1), U_i \sim N(0, 1)\), and \(\theta = (\theta_1, \theta_2)\). The observations are \(\{(Y_i^L, Y_i^H, Z_i) : i \leq n\}\), where \(Y_i^L\) equals the integer part of \(Y_i\), \(Y_i^H\) equals the smallest integer greater than or equal to \(Y_i\), and so \(Y_i^L \leq Y_i \leq Y_i^H\) a.s. The parameter \(\theta\) is not identified because \(Y_i\) is not observed. The two moment inequality functions are

\[
(S6.3) \quad \begin{pmatrix} m_1(W_i, \theta) \\ m_2(W_i, \theta) \end{pmatrix} = \begin{pmatrix} \theta_1 + Z_i \theta_2 - Y_i^L \\ (Y_i^H - \theta_1 - Z_i \theta_2) Z_i^2 \end{pmatrix},
\]

When \(\theta\) is the true parameter value, we have \(Em_1(W_i, \theta) = \theta_1 + Z_i \theta_2 - EY_i^L \geq \theta_1 + Z_i \theta_2 - EY_i = 0\) and \(Em_2(W_i, \theta) = E(Y_i^H - \theta_1 - Z_i \theta_2) Z_i^2 \geq E(Y_i - \theta_1 - Z_i \theta_2) Z_i^2 = 0.\) Thus, the functions in \((S6.3)\) satisfy two moment inequalities.

We consider the case where the true parameter is \(\theta = (1, 1)\). In this case, the identified set consists of the \((\theta_1, \theta_2)\) values that satisfy

\[
(S6.4) \quad \theta_1 + \theta_2 \geq 1.5 \quad \text{and} \quad 2\theta_1 + 4\theta_2 \leq 7.
\]

We consider the CP’s of the CS’s for the \(\theta\) values \((-0.5, 2), (1.5, 0), (1, 1.25),\) and \((1, 1)\), which are all in the identified set. The point \((-0.5, 2)\) is on the boundary of the identified set with both moment inequalities binding; \((1.5, 0)\) and \((1, 1.25)\) are on the boundary of the identified set with only one inequality binding in each case; and \((1, 1)\) is in the interior of the identified set. We also consider CP’s of the CS’s for the \(\theta\) values \((1.35, 0)\) and \((1, 1.375)\), which are not in the identified set. The point \((1.35, 0)\) violates the first inequality in \((S6.4)\) and satisfies the second. The reverse is true for the point \((1, 1.375)\).

Table S.II reports CP’s for the interval-outcome regression model. Table S.II shows that the GMS CS performs very well at \(\theta = (-0.5, 2)\) (at which both inequalities are binding) and at \(\theta = (1.5, 0)\) (at which only the first inequality is binding) with CP’s ranging between .948 and .953. Its CP’s at \(\theta = (1, 1.25)\) (at which only the second inequality is binding) are somewhat higher, with CP’s ranging between .957 and .963. Overcoverage in this case is not necessarily a finite-sample phenomenon because the CS is not asymptotically similar on the boundary of the identified set. For points on the boundary of the identified set, the CP’s of the subsampling CS are not quite as good as for the GMS CS. At \(\theta = (-0.5, 2)\) they vary between .940 and .963; at \(\theta = (1.5, 0)\), they vary between .972 and .986; at \(\theta = (1, 1.25)\) they are comparable to those of the GMS CS. Both CS’s cover the point \(\theta = (1, 1)\) (which is in the interior of the ident-

\[\text{In the second moment function, } Y_i^H - \theta_1 - Z_i \theta_2 \text{ is multiplied by } Z_i^2 \text{ to avoid perfect colinearity with } \theta_1 + Z_i \theta_2 - Y_i^L \text{ since } Y_i^H = Y_i^L + 1 \text{ by definition. We do not consider optimal choices of moment functions for this model because such choices are not known and the results are only illustrative anyway.}\]

\[\text{The identified set is determined by } \theta_1 + EZ_2 \theta_2 - EY_i^L \geq 0 \text{ (i.e., } \theta_1 + \theta_2 \geq 1.5\text{), where } EY_i^L \approx 1.5 \text{ (by numerical calculation), and by } E(Y_i^H - \theta_1 - Z_i \theta_2) Z_i^2 \geq 0 \text{ (i.e., } EY_i^H Z_i^2 - EZ_1^2 \theta_1 - EZ_2^2 \theta_2 \geq 0\text{), where } EY_i^H Z_i^2 \approx 7 \text{ (by numerical calculation), } EZ_1^2 = 2, \text{ and } EZ_2^2 = 4.\]
### TABLE S.II
**Interval-Outcome Regression Model: Finite-Sample Coverage Probabilities of Nominal 95% Confidence Sets for \((\theta_1, \theta_2)\)**

<table>
<thead>
<tr>
<th>(n)</th>
<th>Type of Confidence Interval</th>
<th>Coverage Probabilities for ((\theta_1, \theta_2)) in Identified Set</th>
<th>Coverage Probabilities for ((\theta_1, \theta_2)) Not in Identified Set</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>((-0.5, 2))</td>
<td>((1.5, 0))</td>
</tr>
<tr>
<td>100</td>
<td>GMS</td>
<td>.953</td>
<td>.948</td>
</tr>
<tr>
<td></td>
<td>Subsampling</td>
<td>.963</td>
<td>.986</td>
</tr>
<tr>
<td>500</td>
<td>GMS</td>
<td>.953</td>
<td>.950</td>
</tr>
<tr>
<td></td>
<td>Subsampling</td>
<td>.944</td>
<td>.980</td>
</tr>
<tr>
<td>1000</td>
<td>GMS</td>
<td>.951</td>
<td>.951</td>
</tr>
<tr>
<td></td>
<td>Subsampling</td>
<td>.940</td>
<td>.972</td>
</tr>
</tbody>
</table>

fied set and not close to a boundary) with probability 1. This is expected given that the asymptotic CP is 1.

Next, we consider \(\theta\) points not in the identified set. Table S.II shows that the GMS CS has noticeably lower CP at \(\theta = (1.35, 0)\) than the subsampling CS (viz., .719 versus .851 and .236 versus .375). For \(\theta = (1, 1.375)\), the two CS’s have comparable CP’s. These results are consistent with the power results given in the main paper which show that the GMS test has higher power at some points and equal power at other points compared to the subsampling test.

Similar comments regarding the sensitivity of the subsampling results to \(b\) apply in this model as in the missing-data model.

In sum, the simulation results of this section are in accord with the asymptotic results. They show that the GMS CS has advantages relative to the subsampling CS. The GMS CS has CP’s that are (i) closer to the nominal level and less nonsimilar on the boundary of the identified set, and (ii) lower for points outside the identified set.

### S7. Monte Carlo Experiment II: Subsampling with and Without Recentering

In this section, we provide finite-sample comparisons of the maximum null rejection probability (MNRP) over different null mean vectors of recentered and non-recentered subsampling tests based on the QLR test statistic (i.e., \(S^2\)). (MNRP is defined precisely below.) In short, the recentered subsampling test is found to out-perform the non-recentered subsampling test by a substantial margin in terms of the closeness of the nominal level and the finite-sample MNRP of the tests.

We consider the case in which no equalities arise (i.e., \(v = 0\)) and the number of inequalities, \(p\), is 2 or 4. For given \(\theta\), the null hypothesis is
\( H_0 : \text{Em}(W_i, \theta) \geq 0 \) for some given moment functions \( m(W_i, \theta) \) and the alternative hypothesis is that \( H_0 \) does not hold. The finite-sample properties of tests of \( H_0 \) depend on \( m(W_i, \theta) \) only through (i) \( \mu = \text{Em}(W_i, \theta) \), (ii) \( \Omega = \text{Corr}(m(W_i, \theta)) \), and (iii) the distribution of the mean zero, variance \( I_p \) random vector \( Z^\dagger = \text{Var}^{-1/2}(m(W_i, \theta))(m(W_i, \theta) - \text{Em}(W_i, \theta)) \). We consider the case in which \( Z^\dagger \sim N(0_p, I_p) \).

We consider three representative correlation matrices \( \Omega_{\text{Neg}} \), \( \Omega_{\text{Zero}} \), and \( \Omega_{\text{Pos}} \), which exhibit negative, zero, and positive correlations, respectively. Specifically, \( \Omega_{\text{Zero}} \) equals \( I_p \) for \( p = 2 \) and 4. The matrices \( \Omega_{\text{Neg}} \) and \( \Omega_{\text{Pos}} \) are Toeplitz matrices with correlations on the diagonals given by a \( p - 1 \) vector \( \rho \). For \( p = 2 \), \( \rho = -0.9 \) for \( \Omega_{\text{Neg}} \) and \( \rho = 0.5 \) for \( \Omega_{\text{Pos}} \). For \( p = 4 \), \( \rho = (-0.9, 0.7, -0.5) \) for \( \Omega_{\text{Neg}} \) and \( \rho = (0.9, 0.7, 0.5) \) for \( \Omega_{\text{Pos}} \).

By MNRP, we mean the maximum null rejection probability over all mean vectors in \( H_0 \) (i.e., all \( \mu = \text{Em}(W_i, \theta) \in R^p_+ \), where \( R^p_+ = \{ x \in R^p : x \geq 0 \} \) under the assumption of normally distributed moment inequalities (i.e., \( Z^\dagger \sim N(0_p, I_p) \)) and given the correlation matrix \( \Omega_{\text{Neg}}, \Omega_{\text{Zero}}, \) or \( \Omega_{\text{Pos}} \).

For the subsampling test without recentering, the subsample size is \( m = n^{1/2} \). (Better results for this test were not found by taking other values of \( m \).) For the recentered subsampling test, the subsample size is \( m = 0.75n^{2/3} \). (This choice is based on the simulation results reported in the main paper.) For the subsampling test without recentering, the simulation results are based on 5000 subsamples and 5000 simulation repetitions. For the recentered subsampling test, the simulation results are based on 2500 subsamples and 2500 simulation repetitions.

Table S.III reports the MNRP results for subsampling with and without re-

<table>
<thead>
<tr>
<th>Number of Moment Inequalities</th>
<th>Sample Size ( n )</th>
<th>MNRPb</th>
<th>( \Omega_{\text{Neg}} )</th>
<th>( \Omega_{\text{Zero}} )</th>
<th>( \Omega_{\text{Pos}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>250</td>
<td>Sub/Recenter</td>
<td>.050</td>
<td>.050</td>
<td>.051</td>
</tr>
<tr>
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<td>250</td>
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<td>.027</td>
<td>.020</td>
<td>.018</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>Sub/NoRecenter</td>
<td>.037</td>
<td>.027</td>
<td>.025</td>
</tr>
<tr>
<td></td>
<td>5000</td>
<td>Sub/NoRecenter</td>
<td>.044</td>
<td>.039</td>
<td>.037</td>
</tr>
<tr>
<td>4</td>
<td>250</td>
<td>Sub/Recenter</td>
<td>.046</td>
<td>.047</td>
<td>.051</td>
</tr>
<tr>
<td></td>
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<td>Sub/NoRecenter</td>
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<td>.030</td>
<td>.020</td>
</tr>
<tr>
<td></td>
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<td>Sub/NoRecenter</td>
<td>.036</td>
<td>.032</td>
<td>.028</td>
</tr>
<tr>
<td></td>
<td>5000</td>
<td>Sub/NoRecenter</td>
<td>.044</td>
<td>.041</td>
<td>.036</td>
</tr>
</tbody>
</table>

S.III | FINITE-SAMPLE MNRP’S OF NOMINAL .05 TESTSa

aTests are based on the quasi-likelihood ratio test statistic combined with recentered subsampling (Sub/Recenter) and non-recentered subsampling (Sub/NoRecenter) critical values for sample sizes \( n = 250, 1000, \) and 5000, three correlation matrices \( \Omega_{\text{Neg}}, \Omega_{\text{Zero}}, \) and \( \Omega_{\text{Pos}} \), and 2 and 4 moment inequalities.

bMNRP denotes the maximum null rejection probability over mean vectors in \( H_0 \) under the assumption of normally distributed moment inequalities and given the correlation matrix \( \Omega_{\text{Neg}}, \Omega_{\text{Zero}}, \) or \( \Omega_{\text{Pos}} \).
centering. The results show that subsampling without recentering leads to substantial underrejection of the null hypothesis for sample sizes $n = 250$ and 1000. For example, for $p = 2$ and $n = 250$, its MNRP ranges from .018 to .027 for nominal .05 tests. For $p = 2$ and $n = 1000$, it ranges from .025 to .037. Even for a sample size of 5000, the subsampling test without recentering underrejects the null hypothesis somewhat. In contrast, the subsampling test with recentering has good MNRP values for a sample size as small as 250. For $p = 2$ and $n = 250$, its MNRP ranges from .050 to .051.

REFERENCES


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