

**A CONDITIONAL-HETEROSKEDASTICITY-ROBUST CONFIDENCE
INTERVAL FOR THE AUTOREGRESSIVE PARAMETER**

BY

Donald Andrews and Patrik Guggenberger

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**COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281**

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NOTE

A CONDITIONAL-HETEROSKEDASTICITY-ROBUST CONFIDENCE INTERVAL FOR THE AUTOREGRESSIVE PARAMETER

Donald W. K. Andrews and Patrik Guggenberger*

Abstract—This paper introduces a new confidence interval (CI) for the autoregressive parameter (AR) in an AR(1) model that allows for conditional heteroskedasticity of a general form and AR parameters that are less than or equal to unity. The CI is a modification of Mikusheva's (2007a) modification of Stock's (1991) CI that employs the least squares estimator and a heteroskedasticity-robust variance estimator. The CI is shown to have correct asymptotic size and to be asymptotically similar (in a uniform sense). It does not require any tuning parameters. No existing procedures have these properties. Monte Carlo simulations show that the CI performs well in finite samples in terms of coverage probability and average length, for innovations with and without conditional heteroskedasticity.

I. Introduction

WE consider confidence intervals (CIs) for the autoregressive parameter (AR) ρ in a conditionally heteroskedastic AR(1) model in which ρ may be close to or equal to 1. The observed time series $\{Y_i : i = 0, \dots, n\}$ is based on a latent no-intercept AR(1) time series $\{Y_i^* : i = 0, \dots, n\}$:

$$\begin{aligned} Y_i &= \mu + Y_i^*, \\ Y_i^* &= \rho Y_{i-1}^* + U_i \text{ for } i = 1, \dots, n, \end{aligned} \quad (1)$$

where $\rho \in [-1 + \varepsilon, 1]$ for some $0 < \varepsilon < 2$, $\{U_i : i = \dots, 0, 1, \dots\}$ are stationary and ergodic under the distribution F , with conditional mean 0 given a σ -field \mathcal{G}_{i-1} for which $U_j \in \mathcal{G}_i$ for all $j \leq i$, conditional variance $\sigma_i^2 = E_F(U_i^2 | \mathcal{G}_{i-1})$, and unconditional variance $\sigma_U^2 \in (0, \infty)$. The distribution of Y_0^* is the distribution that yields strict stationarity for $\{Y_i^* : i \leq n\}$ when $\rho < 1$; that is, $Y_0^* = \sum_{j=0}^{\infty} \rho^j U_{-j}$ when $\rho < 1$, and when $\rho = 1$, Y_0^* is arbitrary.

Models of this sort are applicable to exchange rate and commodity and stock prices (see Kim & Schmidt, 1993). Simulations in Mikusheva (2007b, table II) show that CIs that are not designed to handle conditional heteroskedasticity may perform poorly in terms of coverage probabilities when conditional heteroskedasticity is present. In fact, most have an incorrect asymptotic size in this case.¹

For the case of conditional homoskedasticity, several CIs with correct asymptotic size have been introduced, including those in Stock (1991), Andrews (1993), Andrews and Chen (1994), Nankervis and

Savin (1996), Hansen (1999), Elliot and Stock (2001), Romano and Wolf (2001), Chen and Deo (2007), and Mikusheva (2007a).² Of these CIs, the only one that has correct asymptotic size in the presence of conditional heteroskedasticity is the symmetric two-sided subsampling CI of Romano and Wolf (2001).³ The latter CI has the disadvantages that it is not asymptotically similar, requires a tuning parameter (the subsample size), and is far from being equal-tailed when ρ is near 1.⁴

The first CIs that were shown to have correct asymptotic size under conditional heteroskedasticity and an AR parameter that may be close to, or equal to, unity were introduced in Andrews and Guggenberger (2009, hereafter referred to as AG09).⁵ These CIs are based on inverting a t -statistic constructed using a feasible quasi-generalized least squares (FQGLS) estimator of ρ . AG09 shows that equal-tailed and symmetric two-sided CIs based on hybrid (fixed/subsampling) critical values have correct asymptotic size.⁶ These CIs are robust to misspecification of the form of the conditional heteroskedasticity. However, they are not asymptotically similar and require the specification of a tuning parameter—the subsample size.

The contribution of this paper is to introduce a CI that (a) has correct asymptotic size for a parameter space that allows for general forms of conditional heteroskedasticity and for an AR parameter close to, or equal to, unity; (b) is asymptotically similar; and (c) does not require any tuning parameters.

The CI is constructed by inverting tests constructed using a t -statistic based on the LS estimator of ρ and a heteroskedasticity-consistent (HC) variance matrix estimator. For the latter, we use a variant of the HC3 version defined in MacKinnon and White (1985), which we call HC5. It employs an adjustment that improves the finite-sample coverage probabilities. This t -statistic is asymptotically nuisance-parameter-free under the null hypothesis under drifting sequences of null parameters ρ , whether or not these parameters are local to unity. In consequence, critical values can be obtained by matching the given null value of ρ and sample size n with a local-to-unity parameter $h = n(1 - \rho)$. Then one

² The CI of Stock (1991) needs to be modified as in Mikusheva (2007a) to have correct asymptotic size.

³ The correct asymptotic size of this CI is established in the appendix. The equal-tailed subsampling CI of Romano and Wolf (2001) does not have correct asymptotic size under homoskedasticity or heteroskedasticity; see Mikusheva (2007a) and Andrews and Guggenberger (2009).

⁴ Lack of asymptotic similarity implies that the CI overcovers asymptotically for some sequences of ρ values. This may yield a longer CI than is possible.

⁵ Gonçalves and Kilian (2004, 2007) also consider inference in autoregressive models with conditional heteroskedasticity using bootstrap methods. Their results do not allow for unit roots or roots near unity. Kuersteiner (2001) provides some related results for stationary models with conditional heteroskedasticity. Cavaliere and Taylor (2009) provide unit root tests in models with conditional heteroskedasticity using a recursive wild bootstrap. Inoue and Kilian (2002) consider bootstrap methods for autoregressions with unit roots, but their results apply only to non-unit root parameters and they do not allow for conditional heteroskedasticity.

⁶ AG09 also introduces several other CIs that have correct asymptotic size under conditional heteroskedasticity using size-corrected fixed critical values and size-corrected subsampling critical values (for equal-tailed CIs). The performance of these CIs is not as good as that of the FQGLS-based hybrid CI, so we do not discuss these further here.

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¹ Throughout this paper, we use the term *asymptotic size* to mean the limit as $n \rightarrow \infty$ of the finite-sample size. Uniformity in the asymptotics is built into this definition because finite-sample size is a uniform concept. By the definition of asymptotic size, the infimum of the coverage probability over different values of ρ and different innovation distributions is taken before the limit as $n \rightarrow \infty$ is taken.

uses the quantile(s) from the corresponding local-to-unity asymptotic distribution, which depends on h . This method is employed by Stock (1991), Andrews and Chen (1994), and Mikusheva (2007a) in her modification of Stock's CI.⁷ The resulting CI is the same as Mikusheva's (2007a) modification of Stock's (1991) CI applied to the LS estimator of ρ , except that we use the HC5 variance estimator in place of the homoskedastic variance estimator and use a stationary initial condition rather than a 0 initial condition.⁸ We refer to the new CI as the CHR CI (which abbreviates "conditional-heteroskedasticity-robust").

The use of the LS estimator, rather than the FQGLS estimator, is important because the latter has an asymptotic distribution in the local-to-unity case that is a convex combination of a random variable with a unit-root distribution and an independent standard normal random variable with coefficients that depend on the strength of the conditional heteroskedasticity (see Seo, 1999; Guo & Phillips, 2001; Andrews & Guggenberger, 2012). Hence, a nuisance parameter appears in the asymptotic distribution of the FQGLS estimator that does not appear with the LS estimator. This yields a trade-off when constructing a CI between using a more efficient estimator (FQGLS) combined with critical values that do not lead to an asymptotically similar CI and using a less efficient estimator (LS) with critical values that yield an asymptotically similar CI.

The use of an HC variance matrix estimator with the new CHR CI is important to obtain a (nuisance-parameter free) standard normal asymptotic distribution of the t -statistic when the sequence of true ρ parameters converges to a value less than 1 as $n \rightarrow \infty$ and conditional heteroskedasticity is present. It is not needed to yield a nuisance parameter-free asymptotic distribution when ρ converges to unity (either at a $O(n^{-1})$ rate or more slowly).⁹ This follows from results in Seo (1999; also see Guo & Phillips, 2001; Cavaliere & Taylor, 2009; Andrews & Guggenberger, 2012).

Simulations indicate that the CHR CI has good finite-sample coverage probabilities and shorter average lengths—often noticeably shorter—than the hybrid CI of AG09 (based on the FQGLS estimator) for a variety of GARCH(1, 1) processes whose parameterizations are empirically relevant. When no conditional heteroskedasticity is present, the CHR CI performs very well in finite samples relative to CIs that are designed for the i.i.d. innovation case. Hence, there is little cost to achieving robustness to conditional heteroskedasticity.

The asymptotic size and similarity results for the new CI are obtained by employing the asymptotic results of Andrews and Guggenberger (2012) for FQGLS estimators under a drifting sequence of distributions, which include LS estimators as a special case, combined with the generic uniformity results in Andrews, Cheng, and Guggenberger (2009).

⁷ As in Mikusheva's (2007a) modification of Stock's CI, we invert the t statistic that is designed for a given value of ρ , not the t statistic for testing $H_0 : \rho = 1$, which Stock (1991) employs. This is necessary to obtain correct asymptotic coverage when ρ is not $O(n^{-1})$ local to unity.

⁸ Mikusheva's (2007a) results do not cover the new CI because she does not consider innovations that have conditional heteroskedasticity and even in the i.i.d. innovation case, the t statistic considered here does not lie in the class of test statistics that she considers.

The use of a stationary initial condition when $\rho < 1$, rather than a zero initial condition, is not crucial to obtaining robustness to conditional heteroskedasticity. Our results also apply to the case of a zero initial condition, in which case the second command of $I_h^*(r)$ in equation (2.5) below is deleted.

⁹ That is, when ρ converges to unity, one obtains the same asymptotic distribution whether an HC or a homoskedastic variance estimator is employed.

The CHR CI yields a unit root test that is robust to conditional heteroskedasticity. One rejects a unit root if the CI does not include unity. Seo (1999), Guo and Phillips (2001), and Cavaliere and Taylor (2009) also provide unit root tests that are robust to conditional heteroskedasticity.

The CHR CI for ρ can be extended to give a CI for the sum of the AR coefficients in an AR(k) model when all but one root is bounded away from the unit circle, as in Andrews and Chen (1994) and Mikusheva (2007a). In this case, the asymptotic distributions (and hence the CHR critical values) are unchanged. (See the end of section II for more details.) The CHR CI for ρ also can be extended to models with a linear time trend.¹⁰ In this case, the asymptotic distributions are given in equation (7.7) of Andrews and Guggenberger (2009) with $h_{2,7} = 1$. Extending the proof of theorem 1 below for these two cases requires additional detailed analysis, as in Mikusheva (2007a). For brevity, we do not provide such proofs here.

The paper is structured as follows. Section II defines the new CI and establishes its large sample properties. Section III provides tables of critical values. Section IV contains a Monte Carlo study. An appendix (Andrews & Guggenberger, 2013) that is available on the *Review of Economics and Statistics* website provides (a) the local asymptotic false coverage probabilities of the CHR CI; (b) asymptotic and finite-sample assessments of the price the CHR CI pays in the i.i.d. case for obtaining robustness to conditional heteroskedasticity; (c) probabilities of obtaining disconnected CHR CIs; (d) definitions, tables of critical values, and simulation results for symmetric two-sided CHR CIs; (e) details concerning the simulations; (f) description of a recursive residual-based wild bootstrap version of the CHR CI; (g) proofs of the asymptotic results for the CHR CI; and (h) a proof that the symmetric two-sided subsampling CI of Romano and Wolf (2001) has correct asymptotic size under conditional heteroskedasticity.

II. The CHR CI for the AR Parameter

For the exposition of the theory, we focus on equal-tailed two-sided CIs for ρ .¹¹ The CI is obtained by inverting a test of the null hypothesis that the true value is ρ . The model, equation (1), can be rewritten as $Y_i = \tilde{\mu} + \rho Y_{i-1} + U_i$, where $\tilde{\mu} = \mu(1 - \rho)$ for $i = 1, \dots, n$. We use the t -statistic

$$T_n(\rho) = \frac{n^{1/2}(\hat{\rho}_n - \rho)}{\hat{\sigma}_n}, \quad (2)$$

where $\hat{\rho}_n$ is the LS estimator from the regression of Y_i on Y_{i-1} and 1 and $\hat{\sigma}_n^2$ is the (1, 1) element of the HC5 heteroskedasticity-robust variance estimator, defined below, for the LS estimator in the preceding regression. More explicitly, let Y, U, X_1 , and X_2 be n -vectors with i th elements given by Y_i, U_i, Y_{i-1} , and 1, respectively. Let $X = [X_1 : X_2]$, $P_X = X(X'X)^{-1}X'$, and $M_X = I_n - P_X$. Let \hat{U}_i denote the i th element of the residual vector $M_X Y$. Let p_{ii} denote the i th diagonal element of P_X . Let $p_{ii}^* = \min\{p_{ii}, n^{-1/2}\}$. Let Δ be the diagonal $n \times n$ matrix with

¹⁰ Note that there is a one-to-one mapping between the sum of the AR coefficients and the cumulative impulse response. Hence, a CI for the former yields a CI for the latter. See Andrews and Chen (1994) for a discussion of the advantages of the sum of the AR coefficients over the largest AR root as a measure of the long-run dynamics of an AR(k) process.

¹¹ Symmetric two-sided and one-sided CIs can be handled in a similar fashion; see the appendix for details.

We prefer equal-tailed CIs over symmetric CIs in the AR(1) context because the latter can have quite unequal coverage probabilities for missing the true value above and below when ρ is near unity, which is a form of biasedness, due to the lack of symmetry of the near-unit root distributions.

ith diagonal element given by $\widehat{U}_i/(1 - p_{ii}^*)$.¹² Then the LS estimator of ρ and the HC5 estimator $\widehat{\sigma}_n^2$ are

$$\begin{aligned} \widehat{\rho}_n &= (X_1' M_{X_2} X_1)^{-1} X_1' M_{X_2} Y, \text{ and} \\ \widehat{\sigma}_n^2 &= (n^{-1} X_1' M_{X_2} X_1)^{-1} (n^{-1} X_1' M_{X_2} \Delta^2 M_{X_2} X_1) (n^{-1} X_1' M_{X_2} X_1)^{-1}. \end{aligned} \tag{3}$$

Equivalently, $\widehat{\sigma}_n^2$ is the (1, 1) element of $n(X'X)^{-1} X' \Delta^2 X(X'X)^{-1}$. The parameter space for (ρ, F) is given by:

$$\begin{aligned} \Lambda &= \{\lambda = (\rho, F) : \rho \in [-1 + \varepsilon, 1], \{U_i : i = 0, \pm 1, \pm 2, \dots\} \\ &\text{are stationary and strong mixing under } F \text{ with} \\ &E_F(U_i | \mathcal{G}_{i-1}) = 0 \text{ a.s., } E_F(U_i^2 | \mathcal{G}_{i-1}) = \sigma_i^2 \text{ a.s.,} \\ &\text{where } \mathcal{G}_i \text{ is some nondecreasing sequence of } \sigma\text{-fields} \\ &\text{for } i = \dots, 1, 2, \dots \text{ for which } U_j \in \mathcal{G}_i \text{ for all } j \leq i, \\ &\text{the strong-mixing numbers } \{\alpha_F(m) : m \geq 1\} \text{ satisfy} \\ &\alpha_F(m) \leq C m^{-3\zeta/(\zeta-3)} \forall m \geq 1, \sup_{i,s,t,u,v,A} E_F \left| \prod_{a \in A} a \right|^\zeta \leq M, \\ &\text{where } 0 \leq i, s, t, u, v < \infty, \text{ and } A \text{ is any nonempty subset of} \\ &\{U_{i-s}, U_{i-t}, U_{i+1}^2, U_{-u}, U_{-v}, U_1^2\}, \text{ and } E_F U_1^2 \geq \delta\}, \end{aligned} \tag{4}$$

for some constants $0 < \varepsilon < 2, \zeta > 3, C < \infty$, and $\delta > 0$. Next, we define the critical values used in the construction of the CI. They are based on the asymptotic distributions of the test statistic under drifting sequences $\{\lambda_n = (\rho_n, F_n) : n \geq 1\}$ of AR parameters ρ_n and distributions F_n , when $n(1 - \rho_n) \rightarrow h \in [0, \infty)$. When F_n depends on $n, \{U_i : i \leq n\}$ for $n \geq 1$ form a triangular array of random variables and $U_i = U_{n,i}$. To describe the asymptotic distribution, let $W(\cdot)$ be a standard Brownian motion on $[0, 1]$. Let Z_1 be a standard normal random variable that is independent of $W(\cdot)$. Define

$$\begin{aligned} I_h(r) &= \int_0^r \exp(-(r-s)h) dW(s), \\ I_h^*(r) &= I_h(r) + \frac{1}{\sqrt{2h}} \exp(-hr) Z_1 \text{ for } h > 0, \\ I_h^*(r) &= W(r) \text{ for } h = 0, \text{ and} \\ I_{D,h}^*(r) &= I_h^*(r) - \int_0^1 I_h^*(s) ds. \end{aligned} \tag{5}$$

Andrews and Guggenberger (2012, theorem 1), with a minor adjustment for the p_{ii}^* term in Δ , show that under any sequence $\lambda_n = (\rho_n, F_n) \in \Lambda$ such that $n(1 - \rho_n) \rightarrow h \in [0, \infty]$,

$$T_n(\rho_n) \rightarrow_d J_h, \tag{6}$$

where J_h is defined as follows. For $h = \infty, J_h$ is the $N(0, 1)$ distribution, and for $h \in [0, \infty), J_h$ is the distribution of

$$\int_0^1 I_{D,h}^*(r) dW(r) / \left(\int_0^1 I_{D,h}^*(r)^2 dr \right)^{1/2}. \tag{7}$$

¹² The quantity p_{ii}^* used in HC5 is a finite-sample adjustment to the standard HC variance estimator. In contrast, the HC3 variance estimator uses p_{ii} in the definition of Δ . The use of p_{ii}^* guarantees that the finite-sample adjustment does not affect the asymptotics. When $n(1 - \rho_n) \rightarrow h < \infty$, it is straightforward to show that the use of p_{ii} is valid asymptotically. In other cases, it is more difficult to do so. However, the finite-sample results reported below are essentially the same whether p_{ii}^* or p_{ii} is used. Users may find it more convenient to use the HC3 version because it is computable in STATA using the linear regression option `vce(hc3)`. Note that the asymptotic results given in the paper also hold if one sets $p_{ii}^* = 0$, which yields the standard HC variance estimator.

For $\alpha \in (0, 1)$, let $c_h(1 - \alpha)$ denote the $(1 - \alpha)$ -quantile of J_h . The second command of $I_h^*(r)$ in equation (5) is due to the stationary start-up of the AR(1) process when $\rho < 1$, as in Elliott (1999), Elliott and Stock (2001), Müller and Elliott (2003), and Andrews and Guggenberger (2009, 2012). Giraitis and Phillips (2006) and Phillips and Magdalinos (2007) provide similar results for the LS estimator for the case $h = \infty$ under assumptions that do not allow for conditional heteroskedasticity.

The new nominal $1 - \alpha$ equal-tailed two-sided CHR CI for ρ is

$$CI_{CHR,n} = \{\rho \in [-1 + \varepsilon, 1] : c_h(\alpha/2) \leq T_n(\rho) \leq c_h(1 - \alpha/2) \text{ for } h = n(1 - \rho)\}. \tag{8}$$

The CI $CI_{CHR,n}$ can be calculated by taking a fine grid of values $\rho \in [-1 + \varepsilon, 1]$ and comparing $T_n(\rho)$ to $c_h(\alpha/2)$ and $c_h(1 - \alpha/2)$, where $h = n(1 - \rho)$. Tables of values of $c_h(\alpha/2)$ and $c_h(1 - \alpha/2)$ are given in section III. Given these values, calculation of $CI_{CHR,n}$ is simple and fast.¹³

One could replace the asymptotic quantiles $c_h(\alpha/2)$ and $c_h(1 - \alpha/2)$ in equation (8) by recursive residual-based wild bootstrap quantiles and the CI would still have correct asymptotic size. (For brevity, we do not prove this claim.) The resulting CI is a grid bootstrap, as in Hansen (1999), but it is designed to allow for conditional heteroskedasticity. Note that the bootstrap needs to be defined carefully. (See the appendix for its definition.)¹⁴ The bootstrap version of the CI is much less convenient computationally because one cannot use tables of critical values. Rather, one has to compute bootstrap critical values for each value of ρ to determine whether ρ is in the CI.

The main theoretical result of this paper shows that $CI_{CHR,n}$ has correct asymptotic size for the parameter space Λ and is asymptotically similar. Let P_λ denote probability under $\lambda = (\rho, F) \in \Lambda$.

Theorem 1. *Let $\alpha \in (0, 1)$. For the parameter space Λ , the nominal $1 - \alpha$ confidence interval $CI_{CHR,n}$ for the AR parameter ρ satisfies*

$$AsySz \equiv \liminf_{n \rightarrow \infty} \inf_{\lambda=(\rho,F) \in \Lambda} P_\lambda(\rho \in CI_{CHR,n}) = 1 - \alpha.$$

Furthermore, $CI_{CHR,n}$ is asymptotically similar; that is,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \inf_{\lambda=(\rho,F) \in \Lambda} P_\lambda(\rho \in CI_{CHR,n}) \\ = \limsup_{n \rightarrow \infty} \sup_{\lambda=(\rho,F) \in \Lambda} P_\lambda(\rho \in CI_{CHR,n}). \end{aligned}$$

Theorem 2 in the appendix establishes the local asymptotic false coverage probabilities of the CHR CI, which are directly related to their length.

As noted in section I, the CHR CI for ρ can be extended to give a CI for the sum of the AR coefficients in an AR(k) model when all but one

¹³ Note that it is possible for $CI_{CHR,n}$ to consist of two disconnected intervals of the form $[a, b] \cup [c, 1]$, where $-1 + \varepsilon \leq a < b < c \leq 1$. This occurs with very low probability in most cases and low probability in all cases; see the appendix for details.

¹⁴ For example, the fixed-design wild bootstrap and the pairs bootstrap considered in Gonçalves and Kilian (2004, 2007) for stationary observations are not suitable; see the appendix. A suitable bootstrap is similar to, but different from, the recursive wild bootstrap considered by Cavaliere and Taylor (2009), which is designed for unit root tests, and the recursive wild bootstrap considered by Gonçalves and Kilian (2004), which is designed for stationary observations; see the appendix.

TABLE 1.—VALUES OF THE .025 AND .975 QUANTILES OF J_h FOR USE WITH 95% EQUAL-TAILED TWO-SIDED CONFIDENCE INTERVALS

Values of $c_h(.025)$, the .025 Quantile of J_h , for Use with 95% Equal-Tailed Two-Sided Confidence Intervals													
h	0	.2	.4	.6	.8	1.0	1.4	1.8	2.2	2.6	3.0	3.4	3.8
$c_h(.025)$	-3.13	-3.09	-3.06	-3.03	-3.00	-2.98	-2.93	-2.89	-2.85	-2.83	-2.80	-2.77	-2.75
h	4.2	4.6	5.0	6.0	7.0	8.0	9.0	10	11	12	13	14	15
$c_h(.025)$	-2.73	-2.71	-2.69	-2.65	-2.62	-2.59	-2.56	-2.54	-2.52	-2.50	-2.48	-2.47	-2.45
h	20	25	30	40	50	60	70	80	90	100	200	300	500
$c_h(.025)$	-2.39	-2.35	-2.32	-2.28	-2.24	-2.23	-2.21	-2.19	-2.18	-2.17	-2.11	-2.08	-2.05
Values of $c_h(.975)$, the .975 Quantile of J_h , for Use with 95% Equal-Tailed Two-Sided Confidence Intervals													
h	0	.2	.4	.6	.8	1.0	1.4	1.8	2.2	2.6	3.0	3.4	3.8
$c_h(.975)$.24	.31	.36	.41	.45	.50	.57	.64	.69	.74	.79	.84	.88
h	4.2	4.6	5.0	6.0	7.0	8.0	9.0	10	11	12	13	14	15
$c_h(.975)$.92	.95	.99	1.06	1.12	1.17	1.21	1.25	1.29	1.32	1.34	1.37	1.39
h	20	25	30	40	50	60	70	80	90	100	200	300	500
$c_h(.975)$	1.47	1.51	1.55	1.61	1.65	1.67	1.69	1.71	1.73	1.74	1.81	1.83	1.86

TABLE 2.—VALUES OF THE .05 AND .95 QUANTILES OF J_h FOR USE WITH 90% EQUAL-TAILED TWO-SIDED CONFIDENCE INTERVALS

Values of $c_h(.05)$, the .05 Quantile of J_h , for Use with 90% Equal-Tailed Two-Sided Confidence Intervals													
h	0	.2	.4	.6	.8	1.0	1.4	1.8	2.2	2.6	3.0	3.4	3.8
$c_h(.05)$	-2.87	-2.83	-2.79	-2.76	-2.73	-2.70	-2.65	-2.61	-2.57	-2.54	-2.51	-2.48	-2.46
h	4.2	4.6	5.0	6.0	7.0	8.0	9.0	10	11	12	13	14	15
$c_h(.05)$	-2.44	-2.42	-2.39	-2.35	-2.32	-2.29	-2.26	-2.23	-2.21	-2.19	-2.18	-2.16	-2.14
h	20	25	30	40	50	60	70	80	90	100	200	300	500
$c_h(.05)$	-2.09	-2.05	-2.01	-1.97	-1.93	-1.91	-1.89	-1.87	-1.86	-1.85	-1.79	-1.76	-1.74
Values of $c_h(.95)$, the .95 Quantile of J_h , for Use with 90% Equal-Tailed Two-Sided Confidence Intervals													
h	0	.2	.4	.6	.8	1.0	1.4	1.8	2.2	2.6	3.0	3.4	3.8
$c_h(.95)$	-.07	-.02	.04	.08	.13	.17	.25	.31	.37	.43	.48	.52	.57
h	4.2	4.6	5.0	6.0	7.0	8.0	9.0	10	11	12	13	14	15
$c_h(.95)$.61	.64	.68	.75	.81	.87	.91	.95	.98	1.01	1.03	1.05	1.08
h	20	25	30	40	50	60	70	80	90	100	200	300	500
$c_h(.95)$	1.15	1.20	1.24	1.30	1.34	1.36	1.39	1.40	1.42	1.43	1.49	1.52	1.55

root is bounded away from the unit circle. The AR(k) model written in augmented Dickey-Fuller form is

$$Y_i = \mu + Y_i^* \text{ and } Y_i^* = \rho Y_{i-1}^* + \sum_{j=1}^{k-1} \psi_j \delta Y_{i-j}^* + U_i \text{ for } i = 1, \dots, n, \tag{9}$$

where $\delta Y_{i-j}^* = Y_{i-j}^* - Y_{i-j-1}^*$. Here ρ equals the sum of the k AR coefficients. For this model, the estimator $\hat{\rho}_n$ of ρ that we consider is the LS estimator from the regression of Y_i on $Y_{i-1}, \delta Y_{i-1}, \dots, \delta Y_{i-k+1}$, and 1, where $\delta Y_{i-j} = Y_{i-j} - Y_{i-j-1}$. The estimator $\hat{\sigma}_n^2$ that we consider is the (1, 1) element of $n(X'X)^{-1}X'\Delta^2X(X'X)^{-1}$, where $X = [X_1 : X_2 : \dots : X_{k+1}]$, $X_1, X_2, \dots, X_k, X_{k+1}$ are the n -vectors with i th elements equal to $Y_{i-1}, \delta Y_{i-1}, \dots, \delta Y_{i-k+1}, 1$, respectively, and Δ is defined as in the paragraph containing equation (2), but with X defined as immediately above rather than as in the paragraph containing equation (2). The CHR CI for ρ in equation (9) is defined exactly as the CHR CI for ρ in equation (1) is defined, but with the definitions of $\hat{\rho}_n$ and $\hat{\sigma}_n^2$ given immediately above rather than just below equation (2).

III. Tables of Critical Values

Table 1 reports the quantiles $c_h(.025)$ and $c_h(.975)$ (for a broad range of values of h) used to calculate 95% equal-tailed CHR CIs. Table 2 reports analogous quantiles used to calculate 90% equal-tailed CHR CIs. These tables also can be used for 97.5% and 95% lower and upper one-sided CIs. (Section 9 in the appendix provides critical values for symmetric two-sided CHR CIs.)

For given α , $c_h(\alpha)$, the α -quantile of J_h in equation (4), is computed by simulating the asymptotic distribution J_h . To do so, 300,000 independent AR(1) sequences are generated from the model in equation (1) with innovations $U_i \sim iid N(0, 1)$, $\mu = 0$, stationary start-up, $n = 25,000$, and $\rho_h = 1 - h/n$. For each sequence, the test statistic $T_n(\rho_h)$, defined in equation (2) but using the homoskedastic variance estimator, is calculated. Then the simulated estimate of $c_h(\alpha)$ is the α -quantile of the empirical distribution of the 300,000 realizations of the test statistic.

In table 1, the critical values do not reach the $h = \infty$ values of -1.96 and 1.96 for $h = 500$. Larger values of h , which would be needed only in very large samples, yield the following: $c_{1,000}(.025) = -2.02$, $c_{5,000}(.025) = -1.98$, $c_{10,000}(.025) = -1.97$, $c_{1,000}(.975) = 1.90$, $c_{5,000}(.975) = 1.93$, and $c_{10,000}(.975) = 1.94$.

IV. Finite-Sample Simulation Results

Here we compare the finite-sample coverage probabilities (CPs) and average lengths of the new CHR CI and the hybrid CI of AG09.¹⁵ For brevity, we focus on nominal 95% equal-tailed two-sided CIs. Results for symmetric CIs, including the symmetric subsampling CI of Romano and Wolf (2001), are provided in the appendix.

We consider a wide range of ρ values: .99, .9, .5, .0, -.9. The innovations are of the form $U_i = \sigma_i \varepsilon_i$, where $\{\varepsilon_i : i \geq 1\}$ are i.i.d. standard

¹⁵ See MacKinnon and White (1985) and Long and Ervin (2000) for simulation results concerning the properties of the HC3 estimator in the standard linear regression model with i.i.d. observations.

The hybrid CI is defined as in AG09 using the standard HC variance estimator with $\rho_{ii}^* = 0$, not the HC5 estimator.

TABLE 3.—COVERAGE PROBABILITIES AND (CP-CORRECTED) AVERAGE LENGTHS OF NOMINAL 95% EQUAL-TAILED TWO-SIDED CIs: CHR AND HYBRID

Innovations	CI	ρ :	Coverage Probabilities ($\times 100$)					Average Lengths ($\times 100$) (CP-Corrected)				
			.99	.9	.5	.0	-.9	.99	.9	.5	.0	-.9
a. GARCH(1, 1)- (.05, .9; .001)	CHR		94.2	94.7	94.8	94.5	94.4	8.5	19	33	37	17
	Hyb		98.5	98.3	96.5	95.2	95.6	9.6	21	46	47	19
b. GARCH(1, 1)- (.15, .8; .2)	CHR		94.2	94.6	94.7	94.1	94.2	8.8	20	37	43	18
	Hyb		98.0	97.9	96.0	94.3	95.0	9.8	22	49	52	21
c. i.i.d.	CHR		94.5	94.7	94.8	94.7	94.6	8.3	18	31	35	16
	Hyb		97.7	97.6	95.7	94.2	94.8	9.6	21	45	48	19
d. GARCH(1, 1)- (.25, .7; .2)	CHR		94.3	94.5	94.4	93.7	94.1	9.2	21	42	49	20
	Hyb		98.4	98.3	95.9	94.8	95.1	9.5	22	51	54	21
e. ARCH(4)- (.3, .2, .2, .2; .2)	CHR		94.5	94.3	93.9	93.2	94.0	9.6	23	48	56	22
	Hyb		98.5	98.2	95.9	94.2	95.4	9.1	22	53	57	21

normal and σ_i is the multiplicative conditional heteroskedasticity. Let GARCH($ma, ar; \psi$) denote a GARCH(1, 1) process with MA, AR, and intercept parameters ($ma, ar; \psi$), and let ARCH($ar_1, \dots, ar_4; \psi$) denote an ARCH(4) process with AR parameters (ar_1, \dots, ar_4) and intercept ψ . We consider five specifications for the conditional heteroskedasticity of the innovations: (a) GARCH-(.05, .9; .001), (b) GARCH-(.15, .8; .2), (c) i.i.d., (d) GARCH-(.25, .7; .2), and (e) ARCH-(.3, .2, .2, .2; .2). The first three specifications are the most relevant ones empirically.¹⁶ The last two specifications are included for purposes of robustness. They exhibit stronger conditional heteroskedasticity than in the first three cases. In the first four cases, the hybrid CI has an unfair advantage over the CHR CI because it uses a GARCH(1, 1) model, which is correctly specified in these cases. The results are invariant to the choice of μ .

We consider a sample size of $n = 130$. The hybrid CI is based on a GARCH(1, 1) specification.¹⁷ The hybrid critical values use subsamples of size $b = 12$, as in AG09.

We report average lengths of CP-corrected CIs. A CP-corrected CI equals the actual nominal 95% CI if its CP is at least .95 (for the given data-generating process), but otherwise it equals the CI implemented at a nominal CP that makes the finite-sample CP equal to .95.¹⁸ All simulation results are based on 30,000 simulation repetitions.

Table 3 reports the results. CHR denotes the CI in equation (8). Hyb denotes the hybrid CI of AG09. The new CHR CI has very good finite-sample coverage probabilities. Specifically, its CPs ($\times 100$) are in the range [94.1, 94.8] for all values of ρ in cases a to d. For cases d and e, the range is [93.2, 94.5]. The hybrid CI has CPs in the range [94.2, 98.5] for cases a to c and [93.9, 98.5] for cases d and e. These CPs reflect the fact that the hybrid CI is not asymptotically similar due to its reliance on subsampling.

The average length results of table 3 (CP-corrected) show that the CHR CI is shorter than the hybrid CI for all values of ρ in cases a to d. The greatest length reductions are for $\rho = .5, .0$, where the CHR CI is from .69 to .83 times the length of the hybrid CI in cases a to c. For $\rho = .99, .9$, it is from .86 to .91 times the length of the hybrid CI in cases a to c. In cases d and e, the CHR and hybrid CIs have similar lengths for

$\rho = .99, .9$. In cases d and e, for $\rho = .5, .0$, the CHR CI is from .82 to .98 times the length of the hybrid CI. In conclusion, in an overall sense, the CHR CI outperforms the hybrid CI in terms of average length by a noticeable margin in the cases considered.¹⁹

Simulations for the symmetric two-sided subsampling CI of Romano and Wolf (2001) given in the appendix show that the latter CI undercovers substantially in some cases (for example, its CPs ($\times 100$) are 88.9, 88.3, 86.7 for $b = 8, 12, 16$, where b is the subsample size in case b with $\rho = .0$). It is longer than the symmetric and equal-tailed CHR CIs when $\rho = .99$ in cases a to e and has similar average length (CP-corrected) in other cases. Hence, the CHR CIs outperform the Romano and Wolf (2001) CI in the finite-sample cases considered.

Results reported in the appendix compare the CHR CI in the i.i.d. case with the analogous CI that employs the homoskedastic variance estimator. The use of the HC5 variance matrix estimator²⁰ increases the deviations of the CPs ($\times 100$) from 95.0 compared to the homoskedastic variance estimator somewhat, but even so, the deviations for the equal-tailed CIs are only .3 on average over the five ρ values. It has no impact on the average lengths except when $\rho = .99$, in which case, the impact is very small: 8.3 for the equal-tailed CHR CI versus 8.1 for the equal-tailed homoskedastic variance CI. Hence, the CHR CI pays a very small price in the i.i.d. case for its robustness to conditional heteroskedasticity.

¹⁹ The CHR CI also outperforms the hybrid CI based on the infeasible QGLS estimator; see the appendix. The CP ($\times 100$) results of table 3 using p_{ii} rather than p_{ii}^* are the same in all cases except case a, $\rho = .99$; case d, $\rho = .5, .0$; and case e, $\rho = .99$; where the differences are .1% (for example, 94.2% versus 94.3%), and case v, $\rho = .5, .0$, where the differences are .2% and .3%, respectively. There are no differences in the average lengths. For the symmetric two-sided CHR CI, the CP results and the average length results compared to the hybrid CI are similar to those in table 3, although slightly better in both dimensions; see the appendix.

²⁰ The latter CI is Mikusheva's (2007a) modification of Stock's (1991) CI applied to the LS estimator of ρ , but with a stationary initial condition when $\rho < 1$, rather than a zero initial condition.

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- ¹⁶ For example, see Bollerslev (1987), Engle, Ng, and Rothschild (1990), and, for more references, Ma, Nelson, and Startz (2007).
- ¹⁷ See section 10 of the appendix for more details concerning the definition and computation of the hybrid CI. Note that the hybrid CI has correct asymptotic size whether or not the GARCH(1, 1) specification is correct.
- ¹⁸ When calculating the average length of a CI, we restrict the CI to the interval $[-1, 1]$. The search to find the nominal significance level such that the actual finite-sample CP ($\times 100$) equals 95.0 is done with step size .025. In the case of a disconnected CI, the gap in the CI is not included in its length.
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