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ON THE DISAGGREGATION OF EXCESS DEMAND FUNCTIONS

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We solve the problem of the restrictions imposed on the Jacobian A at prices $\bar{p}$ of the aggregate excess demand function $x(p)$ of m agents in an exchange economy with l commodities, under the assumption of individual rationality. Given an arbitrary differentiable function $x(p)$ satisfying homogeneity and Walras' law, we attribute rational individual excess demand functions $x^1(p), \ldots, x^m(p)$ to the m agents such that at any arbitrarily specified vector $\bar{p}$ aggregate excess demand is equal to $x(\bar{p})$ and the following condition is satisfied: There exists a subspace M of dimension m such that the Jacobian at $\bar{p}$ of $x(p)$ and the Jacobian at $\bar{p}$ of the aggregate excess demand function define the same linear function on M. If $x(\bar{p}) \neq 0$, M can be taken to have dimension $(m+1)$. As an immediate consequence of our proof for $m = 1$ we show that even if $\bar{p}$, $x(\bar{p})$, and $Dx(\bar{p})$ are known for the excess demand function of a single agent, the substitution effect and the income effect cannot be unambiguously determined without knowledge of the utility function.

We extend the results proved at a point to large open neighborhoods. We show that if $x(p)$ is an arbitrary function which is bounded from below and satisfies homogeneity and Walras' law, and if $x(\bar{p}) \neq 0$, then we can find an open neighborhood G of $\bar{p}$ and $(l-1)$ individually rational excess demand functions $x^1(p), \ldots, x^{l-1}(p)$, such that $\Sigma_{k=1}^{l-1} x^k(p) = x(p)$ everywhere on G.

1. INTRODUCTION

IN THE CONTEXT of a pure exchange economy, an agent is said to be rational if it expresses excess demand so as to maximize a preference preorder subject to the budget constraint. A group of agents is said to be rational if every member of the group is a rational agent. It is a question of methodological as well as empirical importance to characterize the class of functions that can arise as excess demand functions of rational agents. For instance, suppose an econometrician wants to estimate the Jacobian A at prices $\bar{p}$ of the aggregate excess demand function of m rational consumers (or m aggregate types of consumers) in an economy with l goods. What restrictions can be placed a priori on the form of A? Furthermore, suppose an economist hypothesizes that a function $x(p)$ is the aggregate excess demand function for prices fluctuating in some open set (in a short span of time prices may not change much) or even for all prices. Are there any further restrictions that can be stated a priori about $x(p)$ beyond those on the Jacobian A which must hold for the point case? In other words, does the rationality assumption allow us to be skeptical of any hypothesized $x(p)$ so long as $x(p)$ satisfies Walras' law and homogeneity?

Gerard Debreu solved the hardest case, showing that even with global information no restrictions may be placed on $x(p)$ when $m \geqslant l$. We solve in this paper the infinitesimal or point case for any m and l (of course for $m \geqslant l$ the point case follows immediately from Debreu's theorem). In addition we show that in the local case (for $x(p) \neq 0$) once there are $l-1$ agents the aggregate excess demand function $x(p)$ can be arbitrary except for Walras' law and homogeneity.

It is well known that when $m = 1$ the Jacobian A can be decomposed into a substitution and income effect. Our proof for the point case reveals that this decomposition cannot be determined without knowledge of the utility function.

Thus even if the econometrician knows A with certainty, he still cannot separate the income effect from the substitution effect without knowing the utility function of the agent as well. In the remainder of the introduction, we summarize our results and give the intuition that lies behind them.

Under the assumption of nonvanishing Gaussian curvature of the indifference hypersurfaces, the excess demand function of a single rational agent is differentiable and its Jacobian can be decomposed into the sum of two matrices, one symmetric and negative semidefinite (the Slutsky substitution matrix) and the other a matrix of rank one with rows colinear with the vector of excess demands (the income effect matrix). Conversely, we show that given an arbitrary price vector $\bar{p}$ and an arbitrary matrix A which can be expressed as the sum of a substitution matrix K and an income effect matrix $v\bar{x}^t$, $A = K - v\bar{x}^t$, there exists a locally quadratic utility function u and an endowment vector w such that the excess demand function $x(p)$ derived from (u, w) satisfies $x(\bar{p}) = \bar{x}$ and has Jacobian A at $\bar{p}$. This is the object of our Proposition 1 which thus completes the characterization—at a point—of the class of functions that can be derived as excess demand functions of a rational individual. Given any Jacobian $A = K - vx^t$ we can always find $\hat{K} \neq K$ and $\hat{v} \neq v$ such that $A = \hat{K} - \hat{v}x^t$ (take $\hat{K} = (k - \varepsilon xx^t)$ and $\hat{v} = v - \varepsilon x$ for some small positive number ε) and $\hat{K}$, $\hat{v}$ satisfy all the necessary conditions of Proposition 1. Thus either decomposition could arise from a rational consumer and the econometrician cannot know which one is correct without also knowing something about the utility function of the agent.

It was Sonnenschein's original argument [**11**, **13**] that Walras' law and homogeneity exhaust the restrictions that can be imposed on rationality grounds on an aggregate excess demand function. A subsequent series of papers, Debreu [**3**], McFadden, MasCollel, Mantel, and Richter [**6**], Mantel [**7**, **8**, **9**], and Sonnenschein [**12**], demonstrated that this is indeed the case: An arbitrary, continuous (or, at least, bounded from below) function satisfying homogeneity and Walras' law can be decomposed into rational individual excess demand functions. Furthermore, Debreu [**3**] gave an example to show that for the decomposition theorem to be true the number of rational individuals to be aggregated cannot be reduced below the number of goods in the economy. Consider the offer curve in Figure 1, and observe that it clearly violates the weak axiom of revealed preference. Consequently, no single rational individual can generate the offer curve in Figure 1. Finally, Diewert [**4**] derived the analogue of the Slutsky symmetry conditions for an aggregate excess demand function, as a function of the number of agents in the economy.

Apart from giving what we think are short and elementary proofs of known results we demonstrate that the decomposition of aggregate excess demand into rational individual excess demand functions can be carried out gradually. In an exchange economy with l commodities, given an arbitrary differentiable function $x(p)$ satisfying homogeneity and Walras' law, given an arbitrary price vector $\bar{p} \in \mathring{\mathscr{R}}^l_+$ and given any subspace M containing $\bar{p}$ (of dimension $m \leq l$) we can find m agents with utility functions $u^1, \ldots, u^m$ defined everywhere on $\mathring{\mathscr{R}}^l_+$ and quadratic near $\bar{p}$, and initial endowments $w^1, \ldots, w^m \in \mathring{\mathscr{R}}^l_+$ such that the derived excess

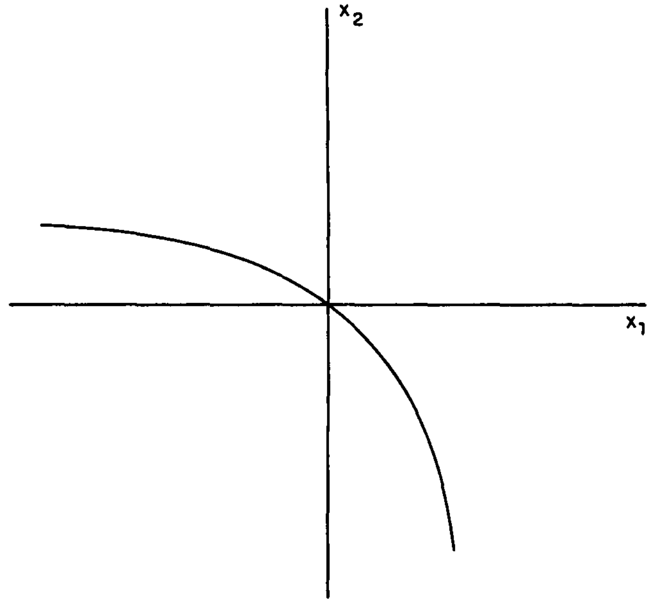

FIGURE 1

demands $x^1(p), \dots, x^m(p)$ satisfy: (a) $\pi_M(\Sigma_{k=1}^m x^k(\bar{p})) = \pi_M(x(\bar{p}))$ and (b) $D(\Sigma_{k=1}^m x^k(\bar{p}))$ and $Dx(\bar{p})$ define the same linear function on M. (By the symbol π_M we mean the projection onto M.) If $x(\bar{p}) \neq 0$, and M is not perpendicular to $x(\bar{p})$, we can get the same result using only $m-1$ agents.

This result follows easily from our main theorem (similar to a result already derived by Mantel), which shows that a slight strengthening of Diewert's conditions is a complete characterization of the Jacobian of the aggregate excess demand function of m rational agents in an economy with l goods. The Jacobian, $A = K - vx'$, of an individual rational excess demand function is symmetric and negative semidefinite on the orthogonal complement of the excess demand vector, $[x]^{\perp}$: This follows at once from the decomposition into a substitution matrix, K, and an income effect matrix, vx', the symmetry and negative semidefiniteness of the former, K, and the colinearity of the rows of the latter, vx', with the excess demand vector, x. Consequently, the Jacobian of the excess demand function of m agents is not arbitrary but must be symmetric and negative semidefinite on the orthogonal complement of the space spanned by the excess demand vectors of the m agents. Agreement with an arbitrary homogeneous function satisfying Walras' law and its Jacobian can thus only be attained, and we prove indeed *can* be attained, on the subspace spanned by the excess demand vectors of the m agents, and $\bar{p}$, the price vector, since, by homogeneity, $\bar{p}$ lies in the kernel of the Jacobian. If the arbitrary function is not equal to 0 at $\bar{p}$, $x(\bar{p}) \neq 0$, we can choose the excess demand vectors for the m agents so that the space $M = [x^1, \dots, x^m, \bar{p}]$ has dimension $(m+1)$. Thus if there is given an arbitrary function $x(p)$ satisfying homogeneity and Walras' law and a price vector $\bar{p}$ with $x(\bar{p}) = \bar{x} \neq 0$ such that the Jacobian $Dx(\bar{p})$ is negative definite and symmetric on a subspace N perpendicular to $\bar{x}$ of dimension $l - m - 1$, then there exist m rational individual agents (u^k, w^k), $k = 1, \ldots, m$, such that the derived excess demand functions $x^1(p), \ldots, x^m(p)$ satisfy $\Sigma_{k=1}^m x^k(\bar{p}) = \bar{x}$ and $\Sigma_{k=1}^m Dx^k(\bar{p}) = Dx(\bar{p})$. If $x(\bar{p}) = 0$, since $\Sigma_{k=1}^m x^k = x(\bar{p}) = 0$, N must have dimension $l - m$. This completes the characterization of aggregate excess demand as a function of the number of agents in the economy by demonstrating that a slightly stronger version of the symmetry conditions of Diewert [4] is not only necessary but also sufficient for rationality. Furthermore, our results preclude the discovery of any additional properties of the aggregate excess demand function of m rational agents for l goods.[1,2]

We show in Part 3 that the results proved at a point can be extended to large open neighborhoods (in a sense to be made precise).

2. THE POINT CASE

We consider a pure exchange economy with l commodities indexed by a subscript i, $i = 1, \ldots, l$. An agent is characterized by its consumption set X, a convex subset of $\mathcal{R}^l$, its endowment vector w, a point in $\mathcal{R}^l$, and its utility function u defined on X. We shall make the following assumptions:

ASSUMPTION 1: $X = \mathring{\mathcal{R}}^l_+$.

ASSUMPTION 2: $w \in X$.

ASSUMPTION 3: The utility function u is twice continuously differentiable and strictly quasi-concave. For all $x \in X$, $Du(x) > 0$; furthermore, the indifference hypersurface through x has no-where vanishing Gaussian curvature, and its closure with respect to $\mathcal{R}^l$ is contained in X.

In the discussion to follow we hold the consumption sets of agents fixed. Consequently, an agent can be characterized as an ordered pair (u, w). The excess demand function of an agent (u, w) is derived as the solution to the following problem:[3]

$$(1) \qquad \max_{x \in X - \{w\}} u(w + x) \quad \text{subject to } p^t x = 0.$$

By Assumption 2, given $p \in \mathring{\mathcal{R}}^l_+$, there exists a unique $x(p) \in X - \{w\}$ which solves (1). Furthermore, by the Kuhn-Tucker Theorem, $x(p)$ solves (1) if and only if

[1] We shall not concern ourselves with problems that arise if the aggregate endowment vector is observed (see [5] for a discussion); neither shall we consider the case of market demand functions—i.e., demand functions with prices and nominal income as independent variables (see [11] for a discussion).

[2] We use the term "homogeneous" to refer to homogeneity of degree zero.

[3] All vectors are column vectors. A superscript "t" denotes the transpose.

there exists $\lambda(p)>0$ such that

$$(2)\qquad \begin{aligned} &Du(w+x(p))-\lambda(p)p=0,\\ &p^t x(p)=0. \end{aligned}$$

By Assumption 2 and the argument in Debreu [**2**], $x(p)$ and $\lambda(p)$ are continuously differentiable functions of p on $\mathring{\mathscr{R}}^l_+$. Their Jacobians can be computed by totally differentiating (2), giving

$$\begin{pmatrix} D^2u(w+x(p)) & -p \\ -p^t & 0 \end{pmatrix}\begin{pmatrix} dx \\ d\lambda \end{pmatrix}=\begin{pmatrix} \lambda I \\ x^t \end{pmatrix}dp.$$

Setting

$$(3)\qquad \begin{pmatrix} S(p) & -v(p) \\ -v(p)^t & e(p) \end{pmatrix}=\begin{pmatrix} D^2u(w+x(p)) & -p \\ -p^t & 0 \end{pmatrix}^{-1}$$

which exists by Assumption 3, where $v(p)\in\mathscr{R}^l$ and $e(p)\in\mathscr{R}$ and

$$(4)\qquad K(p)=\lambda(p)S(p),$$

we obtain:

$$(5)\qquad Dx(p)=K(p)-v(p)x(p)^t,$$

$$(6)\qquad D\lambda(p)=-\lambda(p)v(p)^t+e(p)x(p)^t.$$

The following is well known:

PROPOSITION A: *Let $x(p)$ be the excess demand function of an agent (u, w) satisfying Assumptions 2 and 3. Then, everywhere on $\mathring{\mathscr{R}}^l_+$, $p^t x(p)=0$, $x(p)$ is continuously differentiable and homogeneous and $Dx(p)=K(p)-v(p)x(p)^t$ such that* (i) *$K(p)$ is symmetric and negative semidefinite*; (ii) *rank $(K(p))=l-1$*; (iii) *$p^tK(p)=K(p)p^t=0$*; (iv) *$p^tv=1$.*

Proposition A leads to the following:

COROLLARY A: *Restricted to the orthogonal complement[4,5] of $x(p)$, $[x(p)]^\perp$, $Dx(p)$ is symmetric and negative semidefinite. Furthermore, on $[x(p), p]^\perp$ $Dx(p)$ is symmetric and negative definite.*

PROOF: From (5), $Dx(p)=K(p)-v(p)x(p)^t$. Let $y\in[x(p)]^\perp$. Then $y^t[Dx(p)]y=y^t[K(p)]y-y^t[v(p)x(p)^t]y=y^t[K(p)]y\leq 0$. Symmetry follows from the symmetry of $K(p)$. Note that diagonalizing the quadratic form $K(p)$ shows that for any $y\notin[p]$, $y^tK(p)y<0$ since $K(p)$ has rank $l-1$. *Q.E.D.*

The results of Proposition A and Corollary A characterize the observable restrictions implied by utility maximization on the behavior of a single agent. We

[4] Given a set of vectors $\{y^1,\ldots,y^m\}$ in $\mathscr{R}^l$, $[y^1,\ldots,y^m]$ denotes their span, and $[y^1,\ldots,y^m]^\perp$ its orthogonal complement. Given a matrix M, $[M]$ denotes the subspace spanned by the columns of M.

[5] We use π_M to denote the projection to the subspace M.

shall now turn to the related question of the conditions satisfied by an aggregate excess demand function. Different agents are denoted by a superscript k, $k = 1, \ldots, m$. Agent k is an ordered pair (u^k, w^k) satisfying Assumptions 2 and 3, and its excess demand function is denoted by $x^k(p)$. The aggregate excess demand function is given by $x(p) = \Sigma_{k=1}^m x^k(p)$. Corollary A can now be employed to yield the following:

PROPOSITION B (Diewert): *Let $x(p)$ be the aggregate excess demand function of agents $\{(u^k, w^k), k = 1, \ldots, m\}$ satisfying Assumptions 2 and 3. Then everywhere on $\mathring{\mathscr{R}}^l_+$, $Dx(p)$ defines a symmetric, negative semidefinite quadratic form on $[x^1(p), \ldots, x^m(p)]^\perp$ and a symmetric, negative definite quadratic form on $[x^1(p), \ldots, x^m(p), p]^\perp$.*

PROOF: By definition, $Dx(p) = \Sigma_{k=1}^m Dx^k(p) = \Sigma_{k=1}^m K^k(p) - v^k(p)x^k(p)'$. Let $y \in [x^1(p), \ldots, x^m(p)]^\perp$. Then $y'[Dx(p)]y = y'[\Sigma_{k=1}^m K^k(p)]y + y'[\Sigma_{k=1}^m v^k(p)x^k(p)']y = \Sigma_{k=1}^m y'[K^k(p)]y \leq 0$. Symmetry follows from the symmetry of $K^k(p)$, $k = 1, \ldots, m$. For $y \in [x^1(p), \ldots, x^m(p), p]^\perp$ and $y \neq 0$, $y'K^k(p)y < 0$, $k = 1, \ldots, m$; hence $y'Dx(p)y < 0$. *Q.E.D.*

Propositions A and B and Corollary A cover one side of the characterization of the class of functions that arise as excess demand functions of individual agents (Proposition A and Corollary A), or of groups of agents (Proposition B): They give a set of necessary restrictions on the Jacobian matrix of an excess demand function. To complete the characterization, one would like to show that these restrictions are not only necessary but sufficient as well. That this is indeed the case will be demonstrated in the remainder of this section.

PROPOSITION 1: *Let A be an $(l \times l)$ matrix. Let $\bar{K}$ be an $(l \times l)$ matrix, $\bar{v}$ and $\bar{x}$ vectors in $\mathscr{R}^l$ and $\bar{p}$ a vector in $\mathring{\mathscr{R}}^l_+$ such that $A = \bar{K} - \bar{v}\bar{x}'$ and* (i) *$\bar{K}$ is symmetric, negative semidefinite, of rank $(l-1)$ and $\bar{p}'\bar{K} = \bar{K}\bar{p} = 0$;* (ii) *$\bar{p}'\bar{v} = 1$;* (iii) *$\bar{p}'\bar{x} = 0$. Then there exists an agent (u, w) satisfying Assumptions 2 and 3 whose excess demand function $x(p)$ satisfies the following: (a) $x(\bar{p}) = \bar{x}$, and (b) $Dx(\bar{p}) = \bar{K} - \bar{v}\bar{x}' = A$. Furthermore, u can be chosen to be quadratic on a neighborhood of $(\bar{x} + w)$.*

PROOF: Let e be an arbitrary real number, and consider the matrix

$$D = \begin{pmatrix} \bar{K} & -\bar{v} \\ -\bar{v}' & e \end{pmatrix}.$$

Observe that, since $\bar{p}'\bar{v} = 1$ while $\bar{p}'\bar{K} = 0$, $\bar{v} \notin [\bar{K}]$. Consequently, the $(l+1) \times l$ matrix

$$\begin{pmatrix} \bar{K} \\ -\bar{v}' \end{pmatrix}$$

has rank l and the $(l+1)\times(l+1)$ matrix D is invertible. Let

$$D^{-1}=\begin{pmatrix}\bar{U}'' & -\bar{q}\\ -\bar{q}' & \bar{z}\end{pmatrix};$$

the form of D^{-1} follows from the symmetry of D. Since $\bar{K}\bar{U}''+\bar{v}\bar{q}'=I$, premultiplying both sides of this equation by $\bar{p}'$ gives $\bar{q}'=\bar{p}'$; since $-\bar{K}\bar{q}-\bar{v}\bar{z}=0$, we have $\bar{v}z=0$, hence $\bar{z}=0$. We shall now show that restricted to $[p]^{\perp}$, $\bar{U}''$ defines a negative definite quadratic form. By a theorem of Debreu [**1**], it suffices to demonstrate that the quadratic form $[\bar{U}''-\mu pp']$ is negative definite for some $\mu\in\mathscr{R}$. Let $\mu=1-e$. Then $[\bar{U}''-\mu\bar{p}\bar{p}'][\bar{K}-\bar{v}\bar{v}']=\bar{U}''\bar{K}-\bar{U}''\bar{v}\bar{v}'+\mu\bar{p}\bar{v}'=I-\bar{p}\bar{v}'+e\bar{p}\bar{v}'-\mu\bar{p}\bar{v}'=I$. Since $\bar{K}$ is negative definite on $[p]^{\perp}$ and $\bar{v}\notin[\bar{K}]$, $[\bar{K}-\bar{v}\bar{v}']$ is negative definite. Since $[\bar{U}''-\mu\bar{p}\bar{p}']=[\bar{K}-\bar{v}\bar{v}']^{-1}$, $[\bar{U}''-\mu\bar{p}\bar{p}']$ is negative definite—as desired. Furthermore, since e is chosen arbitrarily, we may set it equal to 1. In this case $\mu=0$ and hence $\bar{U}''$ is negative definite. To complete the proof we shall demonstrate that there exists a utility function u satisfying Assumption 3 and $w\in\mathring{\mathscr{R}}^l_+$ such that the excess demand function $x(p)$ of the agent $(u,.w)$ satisfies (a) and (b). Choose w such that $(w+\bar{x})\gg 0$, and let $u(w+x)=\frac{1}{2}(x+w)'\bar{U}''(x+w)+(\bar{p}'-(\bar{x}+w)'\bar{U}'')(x+w)$. Then $D^2u(\bar{x}+w)=\bar{U}''$ and $Du(\bar{x}+w)=(\bar{x}+w)'\bar{U}''+\bar{p}'-(\bar{x}+w)'\bar{U}''=\bar{p}'$. Consequently, if $x(p)$ is the excess demand function of the agent (u,w), $x(\bar{p})=\bar{x}$ and $Dx(\bar{p})=\bar{K}-\bar{v}\bar{x}'$. Finally it remains to demonstrate that the utility function u satisfies Assumption 3. There exists an open neighborhood of $\bar{x}$ in $\mathscr{R}^l$, v, such that everywhere on $V+\{w\}$, the quadratic utility $u(x+w)$ satisfies Assumption 3. We can choose a compact subset of V, $\bar{W}$, and modify u or $\mathscr{R}^l-\bar{w}$ so that Assumption 3 is satisfied everywhere on $\mathring{\mathscr{R}}^l_+$. *Q.E.D.*

REMARK: Observe that in the previous construction e was chosen arbitrarily. It is easy to see that if $e\geq 0$, U'' is negative semidefinite and hence corresponds to a concave utility function.

COROLLARY 1: *Let $\bar{K}$ be an $(l\times l)$ matrix, $\bar{x}$ a vector in $\mathscr{R}^l$, and $\bar{p}$ a vector in $\mathring{\mathscr{R}}^l_+$ such that* (i) *$\bar{K}$ is symmetric, negative semidefinite, of rank $(l-1)$, and $\bar{p}'\bar{K}=\bar{K}\bar{p}=0$*; (ii) $\bar{p}'\bar{x}=0$. *Then there exists a vector $\bar{v}$ in $\mathscr{R}^l$ and an agent (u,w) satisfying Assumptions* 2 *and* 3 *whose excess demand function $x(p)$ satisfies the following*: (*a*) *$x(\bar{p})=\bar{x}$, and* (*b*) *$Dx(\bar{p})=\bar{K}-\bar{v}\bar{x}'$. Furthermore, u can be chosen to be quadratic on a neighborhood of $(\bar{x}+w)$.*

PROOF: Since rank $(\bar{K})=l-1$, there exists $v\in\mathscr{R}^l$ such that $v\notin[\bar{K}]$. Since $[\bar{K}]=[p]^{\perp}$, $\bar{p}'v\neq 0$. Consequently there exists $k\in\mathscr{R}$ such that $\bar{p}'(kv)=1$. Letting $\bar{v}=kv$, the result follows from Proposition 1. *Q.E.D.*

Proposition 1 and Corollary 1 have at least two important consequences. In estimating the Jacobian A of the excess demand function of a single rational agent, the econometrician cannot impose any more restrictions on the matrix A than that it has the form $A = \bar{K} - \bar{v}\bar{x}'$ with $\bar{K}$ and $\bar{v}$ as above. But worse still, *even if the Jacobian A is known with certainty at the point $\bar{p}$, $\bar{x}$, it is impossible to separate the substitution effects from the income effects without knowledge of the (unobservable) utility function.* For example, suppose $A = \bar{K} - \bar{v}\bar{x}'$. Let $\hat{K} = \bar{K} - \varepsilon\bar{x}\bar{x}'$ and $\hat{v} = \bar{v} - \varepsilon\bar{x}$ where ε is a small positive real number. Then K is the sum of symmetric and negative semidefinite matrices and so is itself symmetric and negative semidefinite. Furthermore, if we choose ε small enough the rank of $\hat{K}$ cannot be less than the rank of $\bar{K}$. To see this, note that since $\text{rank}(\bar{K}) = l-1$, we can find $l-1$ linearly independent vectors $y_1, \dots, y_{l-1}$ such that $\bar{K}y_i \neq 0$, $i = 1, \dots, l-1$. But then for ε small enough, $\hat{K}y_i = \bar{y}_1 - \varepsilon\bar{x}\bar{x}'y_i \neq 0$. On the other hand, $\hat{K}\bar{p} = \bar{K}\bar{p} - \bar{x}\bar{x}'\bar{p} = \bar{K}\bar{p} = 0$, so rank $\hat{K}$ is exactly $l-1$ and $\hat{K}$ satisfies all the assumptions of Proposition 1. Finally, $\bar{p}'\hat{v} = \bar{p}'\bar{v} - \varepsilon\bar{p}'\bar{x} = \bar{p}'\bar{v} = 1$ and $\hat{K} - \hat{v}\bar{x}' = \bar{K} - \varepsilon\bar{x}\bar{x}' - (\bar{v} - \varepsilon\bar{x})\bar{x}' = \bar{K} - \bar{v}\bar{x}' = A$. So by Proposition 1 we can find a rational agent with Slutsky substitution matrix $\hat{K}$ and income effect $\hat{v}$ whose Jacobian at $\bar{p}$ is also equal to A and who also demands $\bar{x}$ at $\bar{p}$; and another agent with Slutsky substitution matrix $\bar{K}$ and income effect $\bar{v}$ who demands $\bar{x}$ at $\bar{p}$ and whose excess demand has Jacobian A at $\bar{p}$. It is impossible to determine without information about the utilities of the two agents which agent one is observing.

Proposition 1 and Corollary 1 give sufficient conditions for the characterization of an individual excess demand function at a point $\bar{p}$. We shall now turn to the derivation of sufficient conditions for the characterization of aggregate excess demand functions at a point.

THEOREM: *Let $x(p)$ be the aggregate excess demand of m consumers in an exchange economy with l goods. Then for any $\bar{p} \in \mathcal{R}^l_{++}$, $\bar{p}'x(\bar{p}) = 0$, $\bar{p}'Dx(\bar{p}) = -x(\bar{p})$ and $Dx(\bar{p})\bar{p} = 0$. Moreover, we can find a subspace $\bar{N}$ of dimension at least $l-m-1$ such that $\bar{N} \perp [x(\bar{p}), \bar{p}]$ and on $\bar{N}$, $Dx(\bar{p})$ is a symmetric and negative definite quadratic form. On $N = [\bar{N}, \bar{p}]$, $Dx(\bar{p})$ is a symmetric and negative semi-definite quadratic form. If the individual excess demands $x_1(p), \dots, x_m(p)$ are observable, then $\bar{N} = [x_1(\bar{p}), \dots, x_m(\bar{p}), \bar{p}]^{\perp}$. Conversely, let A be an $(l \times l)$ matrix, $\bar{x}$ a vector in $\mathcal{R}^l$, and $\bar{p}$ a vector in $\mathcal{R}^l_{++}$ such that* (i) $\bar{p}'A = -\bar{x}'$, $A\bar{p} = 0$; (ii) $\bar{p}'\bar{x} = 0$; (iii) *A defines a symmetric, negative definite quadratic form on a subspace $\bar{N}$, $\bar{N} \perp [\bar{x}, \bar{p}]$ of dimension $l-m-1$ and therefore by* (i) *a negative semidefinite symmetric quadratic form on $N = [\bar{N}, \bar{p}]$.*

Then if $\bar{x} \neq 0$, we can find m agents $\{(u^k, w^k), k = 1, \dots, m\}$ with u^k quadratic on a neighborhood of $\bar{p}$, $k = 1, \dots, m$, such that the aggregate excess demand function $x(p) = \Sigma_{k=1}^m x^k(p)$, where $x^k(p)$ is derived by utility maximization of the agent (u^k, w^k), and satisfies: (a) $x(\bar{p}) = \bar{x}$, (b) $Dx(\bar{p}) = A$, and (c) $[x_1(\bar{p}), \dots, x_m(\bar{p}), \bar{p}]^{\perp} = \bar{N}$. If $\bar{x} = 0$, we need $m+1$ agents $\{(u^k, w^k), k = 0, 1, \dots, m\}$.

PROOF: The first half of the theorem follows from Walras' law, the homogeneity of $x(p)$, and Proposition B.

To prove the second half, choose an orthogonal basis $(\bar{q}^1, \dots, \bar{q}^{l-1}, \bar{p}/\|\bar{p}\|)$ for

$\mathcal{R}^l$ such that $\bar{N}=[\bar{q}^{m+1},\ldots,\bar{q}^{l-1}]$ and if $\bar{x}\neq 0$, $\bar{x}'\bar{q}^k\neq 0$, $k=1,\ldots,m$.[6] Let Q be the orthogonal transformation of the standard basis $(e^1,\ldots,e^l)$ of $\mathcal{R}^l$ to $(\bar{q}^1,\ldots,\bar{q}^{l-1},\bar{p}/\|\bar{p}\|)$;[7] a subscript Q denotes vectors and matrices expressed in the new basis. Let $A_Q=Q^{-1}AQ$, $\bar{p}_Q=Q^{-1}\bar{p}$, and $\bar{x}_Q=Q^{-1}\bar{x}$. Then $\bar{p}_Q=(0,\ldots,0\|\bar{p}\|)$, $\bar{x}_Q=(\hat{x},0)$, and

$$A_Q=\begin{bmatrix}\hat{A} & 0\\ \underbrace{-\hat{x}/\|p\|}_{l-1} & \underbrace{0}_{1}\end{bmatrix}.$$

The form of A follows from (1) and (2). But from (3) we can gain additional information about $\hat{A}$:

$$A_Q=\begin{bmatrix}\overbrace{A_Q^{11}}^{m} & \overbrace{A_Q^{12}}^{l-m-1} & \overbrace{0}^{1}\\ A_Q^{21} & K & 0\\ -\hat{x}_1\|p\| & 0 & 0\end{bmatrix}\begin{matrix}\}m\\ \}l-m-1,\\ \}1\end{matrix}$$

where K is the symmetric, negative defintive matrix defined by A on the subspace $\bar{N}$, expressed in the new basis $(\bar{q}_1,\ldots,\bar{q}_{l-1},\bar{p}/\|\bar{p}\|)$. K is an $(l-m-1)\times(l-m-1)$ matrix, $\bar{x}_Q=(\hat{x},0)=(\hat{x}_1,0,0)$, A_Q^{11} is an $(m\times m)$ matrix, A_Q^{12} is an $(m\times l-m-1)$ matrix, and A_Q^{21} is an $(l-m-1\times m)$ matrix.

Define

$$\bar{K}_Q=\begin{bmatrix}\overbrace{-\alpha I}^{m} & \overbrace{A_Q^{12}}^{l-m-1} & \overbrace{0}^{1}\\ (A_Q^{12})^t & K & 0\\ 0 & 0 & 0\end{bmatrix}\begin{matrix}\}m\\ \}l-m-1\\ \}1\end{matrix}$$

where α is a very large positive real number. We shall later let $(1/m)\bar{K}_Q$ be the Slutsky substitution effect for each of the m agents, $k=1,\ldots,m$.

Clearly $\bar{K}_Q$ is symmetric (since K is) and $\bar{K}_Q\bar{p}_Q=0$. If α is large enough, $\bar{K}_Q$ is negative semidefinite, and has rank $l-1$. To see this, recall K is negative definite and consider the product

$$(y_1^t,y_2^t)\begin{bmatrix}-\alpha I & A_Q^{12}\\ (A_Q^{12})^t & K\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=-\alpha\|y_1\|^2+2y_1^tA_Q^{12}y_2+y_2^tKy_2,$$

and assume $\|(y_1,y_2)\|=1$. Then we can find constants C and $\delta>0$ such that $2y_1^tA_Q^{12}y_2\leq\|y_1\|\|y_2\|C\leq\|y_1\|C$ and $y_2^tKy_2<-\delta\|y_2\|^2$ if $y_2\neq 0$. Notice that if $\|y_1\|$ is very small, $\|y_2\|$ must be close to 1 in order that $\|(y_1,y_2)\|=1$. Hence we can find a level of $\|y_1\|$, $\|\bar{y}_1\|$, such that $\|y_1\|C-\delta\|y_2\|^2<0$ for all y_1 with $\|y_1\|\leq\|\bar{y}_1\|$. Now choose α so big that $-\alpha\|y_1\|^2+\|y_1\|C<0$ for all y_1 satisfying $\|\bar{y}_1\|\leq\|y_1\|\leq 1$. This

[6] This is done as follows: Let $(\bar{e}^1,\ldots,\bar{e}^{l-1},\bar{p}/\|\bar{p}\|)$ be an orthogonal basis for $\mathcal{R}^l$—this is possible since $p\in\mathcal{R}^l_+$—and let ρ be a rotation taking $\bar{x}$ into $(e/\|e\|)\|x\|$, where $e=\Sigma_{k=1}^l e^l$. Then q^k can be defined as $\rho^{-1}(e^k)$.

[7] The matrix Q has columns $(\bar{q}^1,\ldots,\bar{q}^{l-1},p/\|p\|)$.

proves that

$$\begin{bmatrix} -\alpha I & A_Q^{12} \\ (A_Q^{12})' & K \end{bmatrix}$$

is negative definite, hence of rank $l-1$, and thus proves $\bar{K}_Q$ is negative semidefinite and of rank $l-1$.

Assume temporarily that $\bar{x} \neq 0$ and that therefore $x_k = \bar{x}'\bar{q}^k \neq 0$ for $k = 1, \dots, m$. Then define $\bar{K}_Q^k = (1/m)\bar{K}_Q$ and $B_Q = \bar{K}_Q - A_Q$. By construction

$$B_Q = \overset{\overbrace{l-1}\quad\overbrace{1}}{\begin{bmatrix} \hat{B} & 0 \\ & \\ \hat{x}/\|p\| & 0 \end{bmatrix}} \begin{matrix} \}l-1 \\ \\ \}1 \end{matrix} = \overset{\overbrace{m}\quad\overbrace{l-m-1}\quad\overbrace{1}}{\begin{bmatrix} B_Q^{11} & 0 & 0 \\ B_Q^{21} & 0 & 0 \\ \hat{x}_1/\|p\| & 0 & 0 \end{bmatrix}} \begin{matrix} \}m \\ \}l-m-1 \\ \}1 \end{matrix}$$

where $B_Q^{11} = -\alpha I - A_Q^{11}$ and $B_Q^{21} = (A_Q^{12})' - A_Q^{21}$.

Define $\bar{x}_Q^k = (0, \dots, 0, \hat{x}_k, 0, \dots, 0)$ for $k = 1, \dots, m$ and recall that $\hat{x}_k \neq 0$, $k = 1, \dots, m$, if $\bar{x} \neq 0$. If $\bar{x} = 0$ we will have to change our definition of $\bar{x}_Q^k$. But for now assume $\hat{x}_k \neq 0$, $k = 1, \dots, m$.

Now we shall define $\bar{v}_Q^k$, $k = 1, \dots, m$, such that $\bar{v}_Q^k(\bar{x}_Q^k)'$ is a matrix with kth column identical to that of B_Q and 0's everywhere else. Then it will follow that $\Sigma_{k=1}^m \bar{v}_Q^k(\bar{x}_Q^k)' = B_Q$; and since $\Sigma_{k=1}^m \bar{K}_Q^k = \bar{K}$, we will have $A_Q = \bar{K}_Q - B_Q = \Sigma_{k=1}^m (\bar{K}_Q^k - \bar{v}_Q^k(\bar{x}_Q^k)')$. Let

$$\bar{v}_Q^k = \begin{bmatrix} \hat{B}_{1,k}/\bar{x}_k^k \\ \vdots \\ \hat{B}_{l-1,k}/\bar{x}_k^k \\ 1/\|p\| \end{bmatrix}.$$

Observe that $\bar{v}_Q^k$ is well-defined under the assumption that $\bar{x}_k^k = \hat{x}_k \neq 0$. By construction, the following conditions are satisfied: (i) $\bar{K}_Q^k$ is symmetric, negative semidefinite, of rank $l-1$ for all k; (ii) $\bar{p}_Q'\bar{K}_Q^k = 0$ for all k; (iii) $\bar{p}_Q'\bar{v}_Q^k = 1$ for all k; (iv) $A_Q = \Sigma_{k=1}^m(\bar{K}_Q^k - \bar{v}_Q^k(\bar{x}_Q^k)')$. Consider now for each k, $k = 1, \dots, m$, the triplet $(\bar{K}^k, \bar{v}^k, \bar{x}^k)$ defined by $\bar{K}^k = Q\bar{K}_Q^k Q^{-1}$, $\bar{v}^k = Q\bar{v}_Q^k$, and $\bar{x}^k = Q\bar{x}_Q^k$. Since Q is an orthonormal matrix, $Q' = Q^{-1}$. Consequently, $\bar{K}^k$ is symmetric and has rank $l-1$ and is negative semidefinite. Since Q is orthonormal, it preserves inner products, hence $\bar{p}'\bar{K}^k = \bar{p}_Q^k\bar{K}_Q^k = 0$, $\bar{p}'\bar{x}^k = \bar{p}_Q'\bar{x}_Q' = 0$, and $\bar{p}'\bar{v}^k = \bar{p}_Q'\bar{v}_Q^k = 1$. Proposition 1 yields the existence of m agents $\{(u^k, w^k), k = 1, \dots, m\}$ satisfying Assumptions 2 and 3 whose excess demand functions $\{x^k(p), k = 1, \dots, m\}$ satisfy: (i) $x^k(p) = \bar{x}^k$, $k = 1, \dots, m$; (ii) $Dx^k(\bar{p}) = \bar{K}^k - \bar{v}^k(\bar{x}^k)'$.

This concludes the proof for the case $\bar{x} \neq 0$. If $\bar{x} = 0$, we proceed in exactly the same manner, except that we define

$$\bar{x}_Q^k = (0, \dots, 0, \overset{k}{\overbrace{1}}, 0, \dots, 0)$$

$k=1,\ldots,m$, and $\bar{x}_Q^0=(-1,-1,\ldots,-1,0,\ldots,0)$. Then $\bar{v}_Q^k$ becomes

$$\bar{v}_Q^k=\begin{bmatrix}\hat{B}_{1,k}\\ \vdots\\ \hat{B}_{l-1,k}\\ 1/\|p\|\end{bmatrix},\quad k=1,\ldots,m,\quad\text{and}\quad \bar{v}_Q^0=\begin{bmatrix}0\\ \vdots\\ 0\\ 1/\|p\|\end{bmatrix}. \qquad Q.E.D.$$

From the proof of the theorem, we can deduce several interesting consequences, showing how to decompose gradually an arbitrary $x(p)$.

Proposition 2: *Let A be an $(l\times l)$ matrix, $\bar{x}$ a vector in $\mathcal{R}^l$, $\bar{p}$ a vector in $\mathring{\mathcal{R}}^l_{+}$, and M a subspace of $\mathcal{R}^l$ of dimension $m\leq l$ such that* (i) $\bar{p}'A=-\bar{x}'$, $A\bar{p}=0$; (ii) $\bar{p}'\bar{x}=0$; (iii) $p\in M$. *Then there exists m agents $\{(u^k,w^k),\ k=0,1,\ldots,m-1\}$ with u^k quadratic on a neighborhood of $\bar{p}$, $k=0,1,\ldots,m$, such that the aggregate excess demand function $x(p)=\Sigma_{k=0}^{m-1}x^k(p)$ derived by maximizing the utility function satisfies: (a) $x(\bar{p})=\bar{x}$, and (b) $Dx(\bar{p})y=Ay\ \forall y\in M$. Furthermore, if* (iv) $\bar{x}\notin[M]^{\perp}$, *we need only $m-1$ agents. Also* (iii) *can be dispensed with by taking $\hat{M}=M+[p]$ and then using one more agent.*

Proof: Choose an orthogonal basis $(\bar{q}_1,\bar{q}_2,\ldots,\bar{q}_{l-1},\bar{p}/\|\bar{p}\|)$ such that $M=[\bar{q}_1,\ldots,\bar{q}_{m-1},\bar{p}/\|\bar{p}\|]$. If $\bar{x}\notin[M]^{\perp}$, the $\bar{q}_1,\ldots,\bar{q}_{m-1}$ can be chosen so that $\bar{x}'\bar{q}^k\neq 0$, $k=1,\ldots,m-1$.

Now proceed exactly as in the proof of the theorem except that, since we don't know that A is symmetric and negative definite on a subspace $\bar{N}$, we must define

$$\bar{K}_Q=\begin{bmatrix}\overbrace{-I}^{l-1} & \overbrace{0}^{1}\\ 0 & 0\end{bmatrix}\begin{matrix}\}l-1.\\ \}1\end{matrix}$$

Then

$$B_Q=\bar{K}_Q-A_Q=\begin{bmatrix}\overbrace{B_Q^{11}}^{m-1} & \overbrace{B_Q^{12}}^{l-m} & \overbrace{0}^{1}\\ B_Q^{21} & B_Q^{22} & 0\\ \hat{x}_1/\|x\| & \hat{x}/\|x\| & 0\end{bmatrix}\begin{matrix}\}m-1\\ \}l-m\\ \}1\end{matrix}$$

where

$$B_Q^{12}=-A_Q^{12};\quad B_Q^{22}=I-A_Q^{22};\quad B_Q^{11}=-I-A_Q^{11};\quad\text{and}\quad B_Q^{21}=-A_Q^{21}.$$

Proceeding exactly as in the previous case (and noting that $\bar{K}_Q$ is surely symmetric, negative semidefinite of rank $l-1$, and $\bar{p}_Q'\bar{k}_Q=0$) we can use $m-1$ agents to get agreement on the first $m-1$ columns of B_Q. Here only the last column of B_Q is necessarily 0 making a subspace M of dimension m. We need the last agent so that if $\hat{x}_k=0$, $k\leq m-1$, we can define

$$\bar{x}_Q^k=(0,\ldots,0,\overbrace{1}^{k},0,\ldots,0)$$

and $(\bar{x}_Q^0)_k = -1$. Choose any of the agents, say 1, and add $(0, x_{m-1}, \ldots, x_{l-1}, 0)$ to $\bar{x}_Q^1$. Then $\Sigma_{k=0}^{m-1} \bar{x}_Q^k = \bar{x}$ and the first $m-1$ columns of A_Q plus the last have been realized. *Q.E.D.*

COROLLARY 2: *Let A be an $(l \times l)$ matrix, $\bar{x}$ a vector in $\mathscr{R}^l$, and $\bar{p}$ a vector in $\mathring{\mathscr{R}}^l_+$, such that* (i) $\bar{p}'A = -\bar{x}'$, $A\bar{p} = 0$; (ii) $\bar{p}'\bar{x} = 0$. *Then there exists l agents $\{(u^k, w^k), k = 1, \ldots, l\}$, quadratic near $\bar{p}$, such that their aggregate excess demand function $x(p)$ satisfies: (a) $x(\bar{p}) = \bar{x}$; (b) $Dx(\bar{p}) = A$. If $\bar{x} \neq 0$, we can get the same result with $l-1$ agents.*

PROOF: Take $M = \mathscr{R}^l$ in Proposition 2 and note that unless $\bar{x} = 0$, $\bar{x} \notin [M]^\perp$.

This completes the characterization of the restrictions that can be derived in general for the Jacobian of the aggregate excess demand function of m agents in an economy with l goods, under the assumption of individual rationality. Also a technique for gradually decomposing the aggregate excess demand has been presented. We have tried to make explicit the difference between the equilibrium $\bar{x} = 0$ case and the disequilibrium $\bar{x} \neq 0$ case. This difference can be exploited to show that it is possible to decompose an arbitrary function $x(p)$ satisfying homogeneity and Walras' law into only $l-1$ rational individual agents on an open set not containing any equilibria, rather than just at a point.

3. THE LOCAL CASE

We now extend the results derived in the previous section for a point by showing that to decompose an arbitrary excess demand function globally it is sufficient (and in general necessary) to use l agents while to decompose an arbitrary excess demand function locally away from the aggregate no trade point, even on an entire quadrant, it suffices to consider $l-1$ consumers. Furthermore, it is clear from Proposition B that $l-1$ is a lower bound even for local decomposition.

PROPOSITION 3: *Let $\beta(p)$: $\mathring{\mathscr{R}}^l_+ \to \mathscr{R}$ be a positive, homogeneous, and twice differentiable function. Then for any $\mathscr{R}^l_\varepsilon \equiv \{p \in \mathring{\mathscr{R}}^l_+ (p_i/\|p\|) \geq \varepsilon\}$, the function $\beta(p)\pi_{T(p)}e^k$ is derivable from a monotonic, quasi-concave utility function u for all $p \in \mathscr{R}^l_\varepsilon$, $k = 1, \ldots, l$.*[8] *The same is true if we replace e^k with any $y \in \mathscr{R}^l$ and $\mathscr{R}^l_\varepsilon$ with $\{p \in \mathring{\mathscr{R}}^l_+ | p'y > 0, (p_i/\|p\|) \geq \varepsilon\}$.*

PROOF: Debreu [3] constructed indifference curves generating the excess demand $\beta(p)\pi_{T(p)}e^k$, and Geanakopolos [5] found a utility function giving rise to the same excess demand.

PROPOSITION 4: *Let $x(p)$ be a twice differentiable excess demand function. Given $\varepsilon > 0$ and $p^0 \in \mathring{\mathscr{R}}^l_+$ with $x(p^0) \neq 0$, there exists an open set H in $\mathring{R}^l_+$ containing p^0, and $l-1$ individual excess demand functions $x^1(p), \ldots, x^{l-1}(p)$ derivable from utility*

[8] $T(p) = \{y \in \mathscr{R}^l | p'y = 0\}$. $\pi_{T(p)}y$ = projection of y onto $T(p)$ in the direction p.

maximizing individuals $\{(u^k, w^k),\ k=1,\ldots,l-1\}$ *on* $\mathcal{R}^l_\varepsilon$ *such that* $x(p)=\sum_{k=1}^{l-1} x^k(p)$ *for all* $p\in H$. *The* $x^k(p)$ *can be chosen to be linearly independent. Furthermore, if* $x_i(p^0)<0$ *while* $x_j(p^0)>0$ *for all* $j\neq i$, H *can be taken to be the set of all* $p\in\mathcal{R}^l_+$ *such that* $x(p)$ *stays within the quadrant containing* $x(p^0)$ *and bounded away from its boundary.*

LEMMA 1: *Let* $p\in\mathcal{R}^l_+$, $x\in T(p)$, $x\neq 0$. *Then we can choose* $(l-1)$ *of the standard basis vectors* $e^{k_1},\ldots,e^{k_{l-1}}$ *and write* $x=\sum_{i=1}^{l-1}\beta_i[I-pp'/\|p\|^2]e^{k_i}$, *with* $\beta_i\geq 0$, $i=1,\ldots,l+1$. *If* x *is negative in only one coordinate, we can take* $\beta_i>0$, $i=1,\ldots,l-1$. *Note that* $\pi_{T(p)}y=[I-pp'/\|p\|^2]y$ *for any* $y\in\mathcal{R}^l$.

PROOF OF LEMMA 1: $x=(x_1,\ldots,x_k,\ldots,x_l)=\sum_{k=1}^l x_k e^k$. Projecting on $T(p)$ we get $x=\sum_{k=1}^l x_k[I-pp'/\|p\|^2]e^k$. Let $\bar{e}^k=[I-pp'\|p\|^2]e^k$, $k=1,\ldots,l$. Then $x=\sum_{k=1}^l x_k\bar{e}^k$. But

$$\sum_{k=1}^{l} p_k\bar{e}^k=\sum_{k=1}^{l} p_k[I-pp'/\|p\|^2]e^k$$

$$=\sum_{k=1}^{l} p_k\left(e^k-\frac{p_k p}{\|p\|^2}=p-p/\|p\|^2\right)\left(\sum_{k=1}^{l}p_k^2\right)=p-p=0.$$

Hence $-\bar{e}^k=\sum_{l\neq k}(p_l/p_k)\bar{e}^l$, $k=1,\ldots,l$. Consider now $x=\sum_{k=1}^l x_k\bar{e}^k$. Since $x\neq 0$, $p\in\mathcal{R}^l_+$, and $p'x=0$, we may, with no loss of generality, assume that $x_l<0$. Replace $x_l\bar{e}^l$ by $-x_l\sum_{j=1}^{l-1}(p_j/p_l)\bar{e}^j$. We now have x expressed as a linear combination of $(l-1)$ of the $\bar{e}^k$'s. If x_l were the only coordinate less than zero, x is now expressed as the strictly positive linear combination of $(l-1)$ of the $\bar{e}^k$'s. If $x_i<0$ as well $(i\neq l)$, it may be the case that, after $\bar{e}^l$ has been replaced, the coefficient on the ith is still negative—i.e., $x_i-x_l(p_i/p_l)<0$. This is no problem. Simply eliminate $\bar{e}^i$ in the same manner as above, reintroducing $\bar{e}^l$ (but with a positive coefficient), and thus adding a strictly positive amount to all the other coefficients. Repeat this process until all the coefficients are nonnegative. Note that since at every step one of the $\bar{e}^k$'s was entirely eliminated, no more than $(l-1)$ of the coefficients can be strictly positive. On the other hand, it may be that less than $(l-1)$ of the coefficients are strictly positive; for instance, if $x_i-x_l(p_i/p_l)=0$ for some $i\neq l$, both $\bar{e}^i$ and $\bar{e}^l$ drop out. *Q.E.D.*

PROOF OF PROPOSITION 4: We shall first consider the simple case where $x_l(p^0)<0$ while $x_1(p^0),\ldots,x_{l-1}(p^0)>0$. Define $G=\{x\in\mathcal{R}^l\,|\,x_k>0,\ k=1,\ldots,l-1,\ x_l<0,\ |x_k|>\varepsilon,\ k=1,\ldots,l\}$. Since G is open and $x(p)$ is continuous, $H=\{p\in\mathcal{R}^l_+\,|\,x(p)\in G\}$ is open and contains p^0. We now define $x^k(p)=\beta_k(p)[I-pp'/\|p\|^2]e^k$, $k=1,\ldots,l-1$, $p\in\mathcal{R}^l_\varepsilon$ where $\beta_k(p)=\max[x_k(p)-x_l(p)(p_k/p_l),\ \varepsilon]$, $k=1,\ldots,l-1$. Note that $\beta_k(p)$ is homogeneous, and $\beta_k(p)$ is strictly positive everywhere on $\mathcal{R}^l_\varepsilon$.[9] Thus, by Proposition 3, $x^k(p)$ is derivable from a monotonic, quasi-concave utility function u for all $p\in\mathcal{R}^l_\varepsilon$.

[9] $\beta_k(p)$ is not differentiable but by making the function a bit more complicated we could easily remedy this. The same remark applies to the $\gamma_i(p)$ we define later.

Moreover, for $p \in H$, $x(p) \in G$, $|x_p(p)| \geq \varepsilon$, $k = 1, \ldots, l$, and $\beta_k(p) = x_k(p) - x_l(p)(p_k/p_l)$, and so

$$\sum_{k=1}^{l-1} x_k(p)\bar{e}^k - \frac{x_l(p)}{p_l} \sum_{k=1}^{l-1} p_k \bar{e}^k = \sum_{k=1}^{l} x_k(p)\bar{e}_k = x(p),$$

for all $p \in H$. This completes the proof of the proposition for the special case where $x_k(p^0) > 0$, $k = 1, \ldots, l-1$, while $x_l(p^0) < 0$.

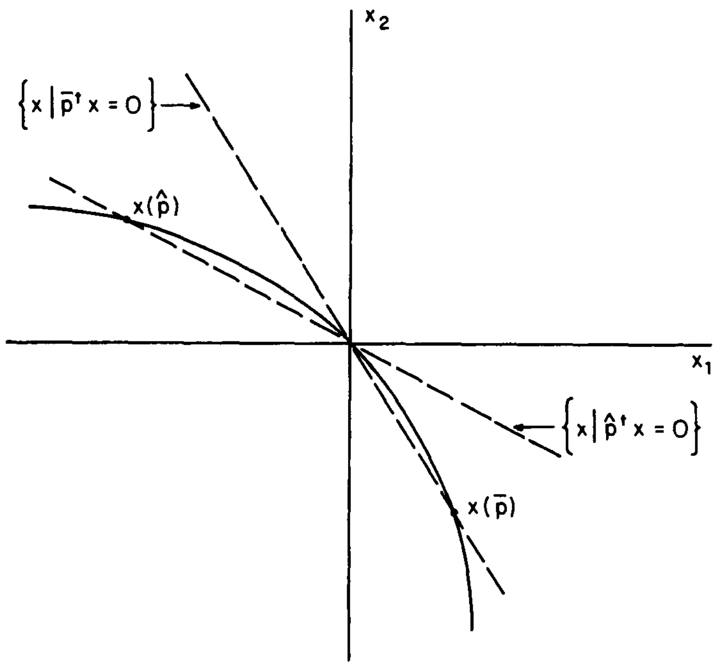

FIGURE 2

REMARK: In the case of a two good market, where except for the point $x = 0$ the conditions for the special case considered above are always fulfilled, given any p^0 such that $x(p^0) \neq 0$, the entire segment of the curve $x(p)$ can be rationalized as the offer curve of a single rational individual so long as it stays in a compact set in the open quadrant containing $x(p^0)$. In Figure 2, $x(p)$ *cannot* be globally explained as the excess demand function of a single rational agent since it violates the weak axiom of revealed preference: At prices $\hat{p}$, $x(\hat{p})$ is chosen though $\hat{p}'x(\bar{p}) < 0$, while at prices $\bar{p}$, $x(\bar{p})$ is chosen even though $\bar{p}'x(\hat{p}) < 0$. Nevertheless, the curve $x(p)$, so long as it stays in the same quadrant and bounded away from the coordinate axes, does satisfy the strong axiom of revealed preference and hence can be considered as part of the excess demand function of a single rational consumer.

To complete the proof of Proposition 4, we shall need the following.

LEMMA 2: *For* $p \in \mathring{\mathcal{R}}^l_{+}$, *any* $(l-1)$ *of the standard basis vectors of* $\mathcal{R}^l$, $e^{k_1}, \ldots, e^{k_{l-1}}$, *satisfy the property that their projections on* $T(p)$ *span* $T(p)$ *and are linearly independent. Consequently, any* $x \in T(p)$ *can be uniquely written as*

$$x = \sum_{i=1}^{l-1} \beta_{k_i}[I - pp'/\|p\|^2]e^{k_i} = \sum_{i=1}^{l-1} \beta_{k_i}\bar{e}^{k_i}.$$

PROOF OF LEMMA: Consider the linear map from $[e^{k_1}, \ldots, e^{k_i}, \ldots, e^{k_{l-1}}]$ to $T(p)$ given by

$$\sum_{i=1}^{l-1} \beta_{k_i}e^{k_i} \to \sum_{i=1}^{l-1} \beta_{k_i}[I - pp'/\|p\|^2]e^{k_i} = \sum_{i=1}^{l-1} \beta_{k_i}\bar{e}^{k_i};$$

it maps a space of dimension $(l-1)$ into $T(p)$ whose dimension is $(l-1)$ as well. But its kernel is simply $\{0\}$ since $p \notin [e^{k_1}, \ldots, e^{k_{l-1}}]$. Thus its range must have dimension $(l-1)$ and span $T(p)$; hence $\bar{e}^{k_1}, \ldots, \bar{e}^{k_{l-1}}$ are linearly independent and span $T(p)$. *Q.E.D.*

We can now complete the proof of Proposition 4. Proceed as in Lemma 1 to get $\beta_i(p) \geq 0$, $i = 1, \ldots, l-1$, and write

$$x(p^0) = \sum_{i=1}^{l-1} \beta_i(p^0)[I - p^0(p^0)'/\|p^0\|^2]e^{k_i} = \sum_{i=1}^{l-1} \beta_i(p^0)\bar{e}^{k_i}.$$

From Lemma 2, $\bar{e}^{k_1}, \ldots, \bar{e}^{k_{l-1}}$ are linearly independent and span $T(p^0)$. Now suppose, to begin with, that $\beta_i(p^0) > 0$, $i = 1, \ldots, l-1$. Then as p varies around p^0, since $x(p)$ is continuous so is $\beta_i(p)$, $i = 1, \ldots, l-1$, and hence, in some neighborhood of p^0, $\beta_i(p) > 0$, $i = 1, \ldots, l-1$. The continuity of $\beta_1(p)$ follows from the continuity of $x(p)$ and the linear independence of $\bar{e}^{k_1}, \ldots, \bar{e}^{k_{l-1}}$. We now define $x^i(p) = \gamma_i(p)[I - pp'/\|p\|^2]e^{k_i} \equiv \gamma_i(p)\bar{e}^{k_i}$, where $\gamma_i(p) = \beta_i(p) + \max(\delta - \beta_i(p), 0)$, with δ as small as we like in the open interval $(0, \min\{\beta_1(p^0), \ldots, \beta_{l-1}(p^0)\})$. Furthermore, $\gamma_i(p)$ is homogeneous and strictly positive on $\mathring{\mathcal{R}}^l_{+}$, $i = 1, \ldots, l-1$. Thus, by Proposition 3 each, $x^i(p)$, is the excess demand function of a rational agent. Furthermore, for all $p \in H$, $H = \{p \in \mathring{\mathcal{R}}_{+} | \beta_i(p) > \delta,\ i = 1, \ldots, l-1\}$,

$$\sum_{i=1}^{l-1} x^i(p) = \sum_{i=1}^{l-1} \gamma_i(p)\bar{e}^{k_i} = \sum_{i=1}^{l-1} \beta_i(p)\bar{e}^{k_i} \equiv x(p).$$

Note that the $\bar{e}^{k_i}$'s form the boundaries of a "quadrant" in $T(p)$. Consequently, we have shown that as long as $x(p)$ stays bounded away from the boundary inside the same quadrant, $x(p)$ can be explained as aggregate behavior of $(l-1)$ rational agents. Note, in addition, that the subset of the graph of $x(p)$ that can be rationalized using $(l-1)$ agents need not be connected.

Figure 2b shows the segment of the original offer curve in Figure 2a that can be attributed to $l-1$ (i.e., 1 in this case) rational agents.

Finally we consider the case where $x(p^0) = \sum_{i=1}^{l-1} \beta_i(p^0)\bar{e}^{k_i}$ and at least one of the $\beta_i(p^0)$'s is equal to zero. In that case consider $S^{l-1}_\varepsilon = \{p \in \mathcal{R}^l_\varepsilon | \|p\| = 1\}$. Then S^{l-1}_ε is

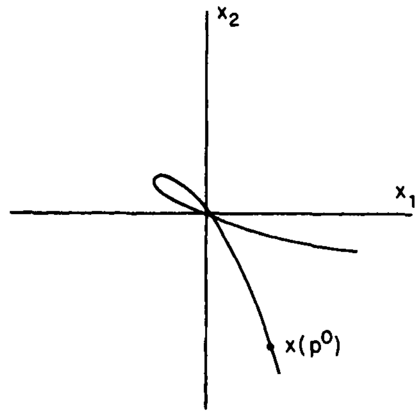

FIGURE 2a

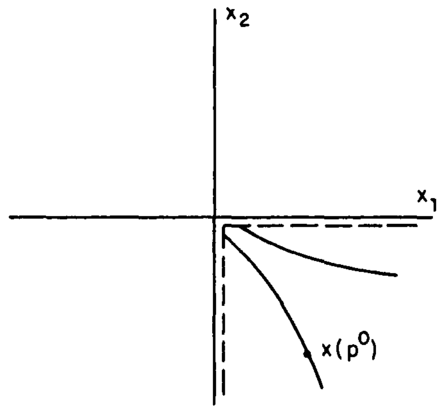

FIGURE 2b

compact and $p'e^{k_i} > \alpha > 0$ everywhere on S_ε^{l-1}, $i = 1, \ldots, l-1$. Thus by choosing y^{k_i} in an open neighborhood around e^{k_i} it will still be the case that $p'y^{k_i} > \frac{1}{2}\alpha > 0$ everywhere on S_ε^{l-1}, $i = 1, \ldots, l-1$. Hence $p'y^{k_i} > 0$ everywhere on $\mathscr{R}_\varepsilon^l$. Furthermore, the y^{k_i}'s can be chosen such that everywhere on $\mathscr{R}_\varepsilon^{l-1}$, $p \notin [y^{k_1}, \ldots, y^{k_{l-1}}]$. Consequently, defining $\bar{y}^{k_i} = [I - pp'/\|p\|^2]y^{k_i}$, $i = 1, \ldots, l-1$, we see that $T(p) = [\bar{y}^{k_1}, \ldots, \bar{y}^{k_{l-1}}]$ and $x(p^0) = \Sigma_{i=1}^{l-1}\xi_i(p^0)\bar{y}^{k_i}$ with $\xi_i(p^0) > 0$, $i = 1, \ldots, l-1$ For any $x(p)$ let $x(p) = \Sigma_{i=1}^{l-1}\xi_i(p)\bar{y}^{k_i}$. As before, $\xi_i(p)$ is continuous in p. Proceeding as in the earlier part of the proof we define

$$x^i(p) = b_i(p)[I - pp'/\|p\|^2]y^{k_i} = b_i(p)\bar{y}^{k_i}, \quad p \in \mathscr{R}_\varepsilon^l, \quad i = 1, \ldots, l-1,$$

where $b_i(p) = \xi(p) + \max\{0, \delta - \xi_i(p)\}$. Hence $p'y^{k_i} > 0$ for all $p \in \mathscr{R}_\varepsilon^l$ (but not necessarily on $\mathring{\mathscr{R}}_+^l$), and $b_i(p)$ is strictly positive and homogeneous on $\mathscr{R}_\varepsilon^l$.

Moreover, on $H=\{p\in\mathscr{R}^l_\varepsilon | p^t y^{k_i}>\delta,\ i=1,\ldots,l-1\}$ we have that

$$\sum_{i=1}^{l-1} x^i(p)=\sum_{i=1}^{l-1} b_i(p)\bar{y}^{k_i}=\sum_{i=1}^{l-1}\xi_i(p)\bar{y}^{k_i}=x(p).$$

Consequently, as long as $x(p)$ lies bounded in the interior of the positive quadrant defined by $\bar{y}^{k_1},\ldots,\bar{y}^{k_{l-1}}$ in $T(p)$, it can be derived as the aggregate excess demand of $l-1$ rational agents. *Q.E.D.*

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REFERENCES

[1] DEBREU, G.: "Definite and Semidefinite Quadratic Forms," *Econometrica*, 20 (1952), 295–300.
[2] ———: "Smooth Preferences," *Econometrica*, 40 (1972), 603–615.
[3] ———: "Excess Demand Functions," *Journal of Mathematical Economics*, 1 (1974), 15–23.
[4] DIEWERT, W. E.: "Generalized Slutsky Conditions for Aggregate Consumer Demand Functions," *Journal of Economic Theory*, 15 (1977), 353–362.
[5] GEANAKOPLOS, J. G.: "Utility Functions for Debreu's Excess Demands," mimeo, Harvard University, 1978.
[6] MCFADDEN, D., A. MAS-COLLEL, R. MANTEL, AND M. RICHTER: "A Characterization of Community Excess Demand Functions," *Journal of Economic Theory*, 7 (1974), 361–374.
[7] MANTEL, R.: "On the Characterization of Aggregate Excess Demand," *Journal of Economic Theory*, 7 (1974), 348–353.
[8] ———: "Homothetic Preferences and Community Excess Demand Functions," *Journal of Economic Theory*, 12 (1976), 197–201.
[9] ———: "Implications of Microeconomic Theory for Community Excess Demand Functions," Cowles Foundation Discussion Paper No. 451, Yale University, 1977.
[10] POLEMARCHAKIS, H.: "Is There an Income Effect," mimeo, Harvard University, 1977.
[11] SONNENSCHEIN, H.: "Market Excess Demand Functions," *Econometrica*, 40 (1972), 549–563.
[12] ———: "Do Walras' Identity and Continuity Characterize the Class of Community Excess Demand Functions?", *Journal of Economic Theory*, 6 (1973), 345–354
[13] ———: "The Utility Hypothesis and Market Demand Theory," *Western Economic Journal*, 11 (1973), 404–410.