Arrow–Debreu model of general equilibrium. I INTRODUCTION; II THE MODEL; III EQUILIBRIUM; IV PARETO OPTIMALITY; V WHAT THE MODEL DOESN'T EXPLAIN.

I. INTRODUCTION

It is not easy to separate the significance and influence of the Arrow–Debreu model of general equilibrium from that of mathematical economics itself. In an extraordinary series of papers (Arrow, 1951; Debreu, 1951; Arrow–Debreu, 1954), two of the oldest and most important questions of neoclassical economics, the viability and efficiency of the market system, were shown to be susceptible to analysis in a model completely faithful to the neoclassical methodological premises of individual rationality, market clearing, and rational expectations, through arguments at least as elegant as any in economic theory, using the two techniques (convexity and fixed point theory) that are still, after thirty years, the most important mathematical devices in mathematical economics. Fifteen years after its birth (e.g. Arrow, 1969), the model was still being reinterpreted to yield fresh economic insights, and twenty years later the same model was still capable of yielding new and fundamental mathematical properties (e.g. Debreu, 1970, 1974). When we consider that the same two men who derived the most fundamental properties of the model (along with McKenzie, 1954) also provided the most significant economic interpretations, it is no wonder that its invention has helped earn for each of its creators, in different years, the Nobel Prize for economics.

In the next few pages I shall try to summarize the primitive mathematical concepts, and their economic interpretations, that define the model. I give a hint of the arguments used to establish the model's conclusions. Finally, on the theory that a model is equally well described by what it cannot explain, I list several phenomena that the model is not equipped to handle.

II. THE MODEL

Commodities and Arrow–Debreu Commodities (A.1) Let there be L commodities, $l=1,\dots,L$. The amount of a commodity is described by a real number. A list of quantities of all commodities is given by a vector in $\mathbb{R}^L$.

The notion of commodity is the fundamental primitive concept in economic theory. Each commodity is assumed to have an objective, quantifiable, and universally agreed upon (i.e. measurable) description. Of course, in reality this description is somewhat ambiguous (should two apples of different sizes be considered two units of the same commodity, or two different commodities?) but the essential quantitative aspect of commodity cannot be doubted. Production and consumption are defined in terms of transformations of commodities that they cause. Conversely, the set of commodities is the minimum collection of objects necessary to describe production and consumption. Other objects, such as financial assets, may be traded, but they are not commodities.

General equilibrium theory is concerned with the allocation of commodities (between nations, or individuals, across time, or under uncertainty etc.). The Arrow–Debreu model studies those allocations which can be achieved through the exchange of commodities at one moment in time.

It is easy to see that it is often important to the agents in an economy to have precise physical descriptions of commodities, as for example when placing an order for a particular grade of steel or oil. The less crude the categorization of commodities becomes, the more scope there is for agents to trade, and the greater is the set of imaginable allocations. Two agents may each have apples and oranges. There is no point in exchanging one man's fruit for the other man's fruit, but both might be made better off if one could exchange his apples for the other's oranges. Of course there need not be any end to the distinctions which in principle could be drawn between commodities, but presumably finer details become less and less important. When the descriptions are so precise that further refinements cannot yield imaginable allocations which increase the satisfaction of the agents in the economy, then the commodities are called Arrow–Debreu commodities.

A field is better allocated to one productive use than another depending upon how much rain has fallen on it; but it is also better allocated depending on how much rain has fallen on other fields. This illustrates the apparently paradoxical usefulness of including in the description of an Arrow–Debreu commodity characteristics of the world, for example the commodity's geographic location, its temporal location (Hicks, 1939), its state of nature (Arrow, 1953; Debreu, 1959; Radner, 1968), and perhaps even the name of its final consumer (Arrow, 1969), which at first glance do not seem intrinsically connected with the object itself (but which are in principle observable).

Hicks, perhaps anticipated by Fisher and Hayek, was the first to suggest an elaborate notion of commodity; this idea has been developed by others, especially Arrow in connection with uncertainty. Hicks was also the first to understand apparently complicated transactions, perhaps involving the exchange of paper assets or other noncommodities, over many time periods, in terms of commodity trade at one moment in time. Thus saving, or the lending of money, might be thought of as the purchase today of a particular future dated commodity. The second welfare theorem, which we shall shortly discuss, shows that an 'optimal' series of transactions can always be so regarded. By making the distinction between the same physical object depending, for example, on the state of nature, the general equilibrium theory of the supply and demand of commodities at one moment in time can incorporate the analysis of the optimal allocation of risk (a concept which appears far removed from the mundane qualities of fresh fruit) with exactly the same apparatus used to analyse the exchange of apples and oranges. Classifying physical objects according to their location likewise allows transportation costs to be handled in the same framework. Distinguishing commodities by who ultimately consumes them could allow general equilibrium analysis to systematically include externalities and public goods as special cases, though this has not been much pursued.

In reality, it is very rare to find a market for a pure Arrow–Debreu commodity. The more finely the commodities are described, the less likely are the commodity markets to have many buyers and sellers (i.e. to be competitive). More commonly, many groups of Arrow–Debreu commodities are traded together, in unbreakable bundles, at many moments in time, in 'second best' transactions. Nevertheless, this understanding of the limitations of real world markets, based on the concept of the Arrow–Debreu commodity, is one of the most powerful analytical tools of systematic accounting available to the general equilibrium theorist. Similarly, the model of Arrow–Debreu, with its idealization of a separate market for each Arrow–Debreu commodity, all simultaneously meeting, is the benchmark against which the real economy can be measured.

Consumers. (A.1) Let there be H consumers, $h = 1, \ldots, H$. □

Each consumer h can imagine consumption plans $x \in \mathbb{R}^L$ lying in some consumption set X^h. (A.2) X^h is a closed subset in $\mathbb{R}^L$ which is bounded from below. □

Each consumer h also has well defined preferences $\succcurlyeq h$ over every pair $(x, y) \in X^h \times X^h$, where $x \succcurlyeq y$ means x is at least as desirable as y. Typically it is assumed that (A.3) $\succcurlyeq$ is a complete, transitive, continuous ordering. □

Notice that in general equilibrium consumers make choices between entire consumption plans, not betweeen individual commodities. A single commodity has significance to the consumer only in relation to the other commodities he has consumed, or plans to consume. Together with transitivity and completeness, this hypothesis about consumer preferences embodies the neoclassical ideal of rational choice.

Rationality has not always been a primitive hypothesis in neoclassical economics. It was customary (e.g. for Bentham, Jevons, Menger, Walras) to regard satisfaction, or utility, as a measurable primitive; rational choice, when it was thought to occur at all, was the consequence of the maximization of utility. And since utility was often thought to be instantaneously produced, sequential consumer choice on the basis of sequential instantaneous utility maximization was sometimes explicitly discussed as irrational (see e.g. Böhm-Bawerk on saving and the reasons why the rate of interest is always positive)

Once utility is taken to be a function not of instantaneous consumption, but of the entire consumption plan, then rational choice is equivalent to utility maximization. Debreu (1951) proved that any preference ordering $\succcurlyeq_h$ defined on $X^h \times X^h$ satisfies (A.1)–(A.3) if and only if there is a utility function u^h: $X^h \to \mathbb{R}$ such that $x \succcurlyeq_h y$ exactly when $u^h(x) \geqslant u^h(y)$.

Under the influence of Pareto (1909), Hicks (1939) and Samuelson (1947), neoclassical economics has come to take rationality as primitive, and utility maximization as a logical consequence. This has had a profound effect on welfare economics, and perhaps on the scope of economic theory as well. In the first place, if utility is not directly measurable, then it can only be deduced from observable choices, as in the proof of Debreu. But at best this will give an 'ordinal' utility, since if f: $\mathbb{R} \to \mathbb{R}$ is any strictly increasing function, then u^h represents $\succcurlyeq_h$ if and only if $v^h \equiv f \circ u^h$ represents $\succcurlyeq_h$. Hence there can be no meaning to interpersonal utility comparisons; the Benthamite sum $\Sigma_{h=1}^{H} u^h$ is very different from the Benthamite sum $\Sigma_{h=1}^{H} f^h \circ u^h$. In the second place, the ideal of rational choice or preference, freed from the need for measurement, is much more easily extended to domains not directly connected to the market and commodities such as political candidates or platforms, or 'social states'. The elaboration of the nature of the primitive concepts of commodity and rational choice, developed as the basis of the theory of market equilibrium, prepared the way for the methodological principles of neoclassical economics (rational choice and equilibrium) to be applied to questions far beyond those of the market.

Although the rationality principle is in some respects a weakening of the hypothesis of measurable utility and instantaneous utility maximization, when coupled with the notion of consumption plan it is also a strengthening of this hypothesis, and a very strong assumption indeed. For example there is not room in this theory for the Freudian split psyche (or self-deception), or for Odysseus-like changes of heart. Perhaps more importantly, a consumers's preferences (for example how thrifty he is) do not change according to the role he plays in the process of production (e.g. on whether he is a capitalist or landowner), nor do they change depending on other consumers' preferences, or the supply of commodities. As an instance of this last case, note that it follows from the rationality hypothesis that the surge in the microcomputer industry influenced consumer choice between typewriters and word processors only through availability (via the price), and not through any learning effect. (Consumers can 'learn' in the Arrow–Debreu model, e.g. their marginal rates of substitution can depend on the state of nature, but the rate at which they learn is independent of production or consumption – it depends on the exogenous realization of the state. We shall come back to this when we consider information.) If for no other reason, the burden of calculation and attention which rational choice over consumption plans imposes on the individual is so large that one expects rationality to give way to some kind of bounded rationality in some future general equilibrium models

Two more assumptions on preferences made in the model of Arrow–Debreu are nonsatiation and convexity:

(A.4) For each $x \in X^h$, there is a $y \in X^h$ with $y \succ_h x$, i.e. such that $y \succcurlyeq_h x$ and not $x \succcurlyeq_h y$. □

(A.5) X^h is a convex set, and $\succcurlyeq_h$ is convex, i.e., if $y \succ_h x$ and $0 < t \leqslant 1$, then $[ty + (1-t)x] \succ_h x$. □

The nonsatiation hypothesis seems entirely in accordance with human nature. The convexity hypothesis implies that commodities are infinitely divisible, and that mixtures are at least as good as extremes. When commodities are distinguished very finely according to dates, so that they must be thought of as flows, then the convexity hypothesis is untenable. In a standard example, a man may be indifferent between drinking a glass of gin or of scotch at a particular moment, but he would be much worse off if he had to drink a glass of half gin–half scotch. On the other hand, if the commodities were not so finely dated, then they would be more analogous to stocks, and a consumer might well be better off with a litre of gin and a litre of scotch, than two litres of either one. In any case, as we shall remark later, if every agent is small relative to the market (i.e. if there are many agents) then the nonconvexities in preferences are relatively unimportant.

Each agent h is also characterized by a vector of initial endowments

(A.6) $e^h \in X^h \subset \mathbb{R}^L$ for all $h = 1, \ldots, h$ □

The endowment vector e^h represents the claims that the consumer has on all commodities, not necessarily commodities in his physical possession. The fact that $e^h \in X^h$ means that the consumer can ensure his own survival even if he is deprived of all opportunity to trade. This is a somewhat strange hypothesis for the modern world, in which individuals often have labour but few other endowments, e.g. land. Doubtless the hypothesis could be relaxed; in any case, survival is not an issue that is addressed in the Arrow–Debreu model.

Each individual h is also endowed with an ownership share of each of the firms $j = 1, \ldots, J$

(A.7) For all $h = 1, \ldots, H$, $j = 1, \ldots, J$, $d_{hj} \geqslant 0$, and for all $j = 1, \ldots, J$, $\Sigma_{h=1}^{H} d_{hj} = 1$. □

Firms. (A.8) Let there be J firms, $j = 1, \ldots, J$. □

The firm in Arrow–Debreu is characterized by its initial distribution of owners, and by its technological capacity

$Y_j \subset \mathbb{R}^L$ to transform commodities. Any production plan $y \in \mathbb{R}^L$, where negative components of y refer to inputs and positive components denote outputs, is feasible for firm j if $y \in Y_j$. A customary assumption made in the Arrow–Debreu model is free disposal: if $l = 1, \ldots, L$ is any commodity, and v_l is the unit vector in $\mathbb{R}^L$, with one in the lth coordinate and zero elsewhere, then

(A.9) For all $l = 1, \ldots, L$ and $k > 0$, $-kv_l \in Y_j$, for some $j = 1, \ldots, J$. □

Although it is strange, when thinking of nuclear waste etc., to think that any commodity can be disposed without cost (i.e. without the use of any other inputs), as we shall remark later, this assumption can be relaxed, if negative prices are introduced (or if weak monotonicity is assumed).

The empirically most vulnerable assumption to the Arrow–Debreu model, and one crucial to its logic, is.

(A.10) For each j, Y_j is a closed, convex set containing 0. □

This convexity assumption rules out indivisibilities in production (e.g. half a tunnel), increasing returns to scale, gains from specialization, etc. As with consumption, if the indivisibilities of production are small relative to the size of the whole economy, then the conclusions we shall shortly present are not much affected. But when they are large, or when there are significant increasing returns to scale, the model of competitive equilibrium that we are about to examine is simply not applicable. Nevertheless, convexity is consistent with the traditionally important cases of decreasing and constant returns to scale in production

We conclude by presenting three final assumptions used in the Arrow–Debreu model.

(A.11) Let $e = \Sigma_{h=1}^{H} e^h$,
let $F = \{y \in \mathbb{R}^L | y = \Sigma_{j=1}^{J} y_j,\ y_j \in Y^j, j = 1, \ldots, J\}$,
let $\bar{F} = \{y \in F | y + e \geqslant 0\}$, and
let $K = \{(y_1, \ldots, y_J) \in Y_1 \times \cdots \times Y_J | \equiv y\Sigma_{j=1}^{J} y^j \in \bar{F}\}$.
Then $\bar{F} \cap \mathbb{R}_{++}^L \neq \phi$, and K is compact □

Assumption (A.11) requires that the level of productive activity that is possible even if the productive sector appropriates all the resources of the consuming sector is bounded (as well as closed)

Notice that these assumptions are consistent with firms owning initial resources, as well as individuals. In the original Arrow–Debreu model (1954), the firms were prohibited from owning initial resources (they were assigned to the firm owners. with complete markets there is little difference, but with incomplete markets the earlier assumption is restrictive)

(A.12) The economy is irreducible □

We shall not elaborate this assumption here. It means that for any two agents h and h', the endowment e^h of agent h is positive in some commodity l, which (taking into account the possibilities of production) agent h' could use to make himself strictly better off. It certainly seems reasonable that each agent's labour power could be used to make another agent better off.

Lastly, we assume that

(A.13) The commodities are not distinguished according to which firm produces them, or who consumes them □

Assumption (A.13) is made simply for the purposes of interpretation. When put together with the definition of competitive equilibrium, it implies that there are no externalities to production or consumption, no public goods, etc. Mathematically, however, (A.13) has no content. In other words, if we dropped assumption (A.13), the Arrow–Debreu notion of competitive equilibrium would still make sense (even in the presence of externalities and public goods) and it would still have the optimality properties we shall elaborate in Section III, but it would require an entirely different interpretation. Consumers, for example, would be charged different prices for the same physical commodities (same, that is, according, to date, location and state of nature). In more technical language, a Lindahl equilibrium is a special case of an (A 1)–(A.12) Arrow–Debreu equilibrium, with the commodity space suitably expanded and interpreted. Thus each physical unit of a public good is replaced by H goods, one unit for the public good indexed by which agent consumes it. Also the physical technology set describing the production of the public good is replaced by a different set in the Arrow–Debreu model, lying in a higher dimensional space, where the output of the same amount of H goods. In an Arrow–Debreu equilibrium, consumers will likely pay different prices for these H goods, i.e. for what in reality represents the same physical public good. Hence the differential pay principle for the optimal provision of public goods elucidated by Samuelson, which appeared to point to a qualitative difference between the analytical apparatus needed to describe optimality in public goods and private goods economies, is thus shown to be explicable by exactly the same apparatus used for private goods economies, simply by multiplying the number of commodities. The same device can also be used for analysing the optimal provision of goods when there are externalities, provided that negative prices are allowed Assumption (A 13) thus seriously limits the normative conclusions that can be drawn from the model. From a descriptive point of view, however, rationality and the price taking behaviour which equilibrium implies, make (A.13) necessary

III. Equilibrium

Price is the final primitive concept in the Arrow–Debreu model Like commodity it is quantifiable and directly measurable. As Debreu has remarked, the fundamental role which mathematics plays in economics is partly owing to the quantifiable nature of these two primitive concepts, and to the rich mathematical relationship of dual vector spaces, into which it is natural to classify the collections of price values and commodity quantities Properly speaking, price is only sensible (and measurable) as a relationship between two commodities, i e. as relative price. Hence there should be $L^2 - L$ relative prices in the Arrow–Debreu model. But the definition of Arrow–Debreu equilibrium immediately implies that it suffices to give $L - 1$ of these ratios, and all the rest are determined

For mathematical convenience (namely to treat prices and quantities as dual vectors), one price is specified for each unit quantity of each commodity The relative price of two commodities can be obtained by taking the ratio of the Arrow–Debreu prices of these commodities.

I shall proceed by specifying the definition of Arrow–Debreu equilibrium, and then I make a number of remarks emphasizing some of the salient characteristics of the definition. The longest remark concerns the differences between the historical development of general equilibrium, up until the time of Hicks and Samuelson and the particular Arrow–Debreu model of general equilibrium.

An Arrow–Debreu economy E is an array $E = \{L, H, J(X^h, e^h, \succcurlyeq_h), (Y^j), (d^{hj}), h = 1, \ldots, H, j = 1, \ldots, J\}$ satisfying assumptions (A.1)–(A.13). An Arrow–Debreu equilibrium is an array $[(\bar{p}_l), (\bar{x}_l^h), (\bar{y}_l^j), l = 1, \ldots, L, h = 1, \ldots, H, j = 1, \ldots, J]$ satisfying:

For all $j = 1, \ldots, J, \bar{y}_j \in$

$$\arg\max \left\{ \sum_{l=1}^{L} \bar{p}_l y_l | (y = y_1, \ldots, y_L) \in Y^j \right\} \qquad (1)$$

For all $h = 1, \ldots, H, \bar{x}^h \in B^h(\bar{p})$, where $B^h(\bar{p})$

$$\equiv \left\{ x \in X^h \mid \sum_{l=1}^{L} \bar{p}_l x_l \leqslant \sum_{l=1}^{L} \bar{p}_l e_l^h + \sum_{j=1}^{J} d^{hj} \sum_{l=1}^{L} \bar{p}_l \bar{y}_l^j \right\} \quad (2)$$

and if $x \in B^h(\bar{p})$, then not $x \succ_h \bar{x}^h$,

$$\text{For all} \quad l = 1, \ldots, L, \quad \sum_{h=1}^{H} \bar{x}_l^h = \sum_{h=1}^{H} e_l^h + \sum_{j=1}^{J} \bar{y}_l^j. \quad (3)$$

The most striking feature of general equilibrium is the juxtaposition of the great diversity in goals and resources it allows, together with the supreme coordination it requires. Every desire of each consumer, no matter how whimsical, is met precisely by the voluntary supply of some producer. And this is true for all markets and consumers simultaneously.

There is a symmetry to the general equilibrium model, in the way that all agents enter the model individually motivated by self-interest (not as members of distinct classes motivated by class interests), and simultaneously, so that no agent acts prior to any other on a given market (e.g. by setting prices). If workers' subsistence were not assumed, for example, that would break the symmetry; workers income could have to be guaranteed first, otherwise demand would (discontinuously) collapse. As it is, at the aggregate level, supply and demand equally and simultaneously determine price; in equilibrium, both the consumers' marginal rates of substitution and the producers' marginal rates of transformation are equal to relative prices (assuming differentiability and interiority). There are gains to trade both through exchange and through production This point of view represents a significant break with the classical tradition of Ricardo and Marx. We shall come to the main difference between the classical and neoclassical approaches shortly. Another difference is that here need not be fixed coefficients of production in the Arrow–Debreu model – the sets Y are much more general. Also in an Arrow–Debreu equilibrium, there is no reason for here to be a uniform rate of profit. There is none the less one aspect of the model which these authors would have greatly approved, namely the shares d^{hj} which allow the owners of firms to collect profits even though they have contributed nothing to production.

Notice that in general equilibrium each agent need only concern himself with his own goals (preferences or profits) and the prices. The implicit assumption that every agent 'knows' all the prices is highly non-trivial. It means that at each date each agent is capable of forecasting perfectly all future prices until the end of time. It is in this sense that the Arrow–Debreu model depends on 'rational expectations'. Each agent must also be informed of the 'price q_j of each firm j, where $q_j = \Sigma_{l=1}^{L} \bar{p}_l \bar{y}_l^j$. (Firms that produce under constant returns to scale must also discover the level of production, which cannot be deduced from the prices alone.) Assuming that the 'man on the spot' (Hayek's expression) knows much better than anyone else what he wants, or best how his changing environment is suited to producing his product, decentralized decision making would seem to be highly desirable, if it is not incompatible with coordination. Indeed, harmony through diversity is one of the sacred doctrines of the liberal tradition.

The greatest triumph of the Arrow–Debreu model was to lay out explicitly the conditions (roughly (A.1)–(A.13)) under which if is possible to claim that a properly chosen price system must always exist that, like the invisible hand, can guide diverse and independent agents to make mutually compatible choices. The idea of general equilibrium had gradually developed since the time of Adam Smith, mostly through the pioneering work of Walras (1874), Von Neumann (1937), Wald (1932), Hicks (1939) and Samuelson (1947). By the late 1940s the definition of equilibrium, including ownership shares in the firms, was well-established. But it was Arrow–Debreu (1954) that spelled out precise microeconomic assumptions at the level of the individual agents that could be used to show the model was consistent.

The axiomatic and rigorous approach that characterized the formulation of general equilibrium by Arrow–Debreu has been enormously influential. It is now taken for granted that a model is not properly defined unless it has been proved to be logically consistent. Much of the clamour for 'microeconomic foundations to macroeconomics', for example, is a desire to see an axiomatic clarity similar to that of the Arrow–Debreu model applied to other areas of economics. Of course, there were other earlier economic models that were similarly axiomatic and rigorous; one thinks especially of Von Neumann–Morgenstern's *Theory of Games* (1944). But game theory was, at the time, on the periphery of economics. Competitive equilibrium is at its heart.

The central mathematical techniques, convexity theory (separating hyperplane theorem) and Brouwer's (Kakutani's) fixed point theorem, used in Arrow-Debreu are, thirty years later, still the most important tools used in mathematical economics. Both elements had played a (hidden) role in Von Neumann's work. Convexity had been prominent in the work of Koopmans (1951) on activity analysis, in the work of Kuhn and Tucker (1951) on optimization, and in the papers of Arrow (1951) and Debreu (1951) on optimality Fixed point theorems had been used by Von Neumann (1937), by Nash (1950) and especially by McKenzie (1954), who one month earlier than Arrow–Debreu had published a proof of general equilibrium using Kakutani's theorem, albeit in a model where the primitive assumptions were made on demand functions, rather than preferences. McKenzie (1959) also made an early contribution to the notion of an irreducible economy (assumption (A.9)).

The first fruit of the more precise formulation of equilibrium that began to emerge in the early 1950s was the transparent demonstration of the first and second welfare theorems that Arrow and Debreu simultaneously gave in 1951. Particularly noteworthy is the proof that every equilibrium is Pareto optimal. So simple and illuminating is this demonstration that it is no exaggeration to call it the most frequently imitated argument in all of neoclassical economic theory.

Among the confusions that were cleared away by the careful axiomatic treatment of equilibrium was the reliance of the discussions by Hicks and Samuelson on interior solutions and differentiability When discussing the optimal allocation of housing, for example, it is evident that most agents will consume nothing of most houses, but this does not affect the Pareto optimality of a free (and complete) market allocation of housing. Similarly, it is not necessary to either the existence of Arrow–Debreu equilibrium, nor to the first and second welfare theorems, that preferences or production sets be either differentiable or strictly convex. In particular, it is possible to incorporate the 'neoclassical production function' with constant returns to scale with variable inputs, the classical fixed coefficients methods of production, and the strictly concave production functions of the Hicks–Samuelson vintage, all in the same framework.

This is not to say that differentiability has no role to play in the Arrow–Debreu model. In his seminal paper (1970), Debreu resurrected the role of differentiability by showing, via the methods of transversality theory (a branch of differential topology) that almost every differentiable economy is regular, in the sense that small perturbations to the economic data (e.g. the endowments) make small changes in all the equilibrium prices Before Debreu, comparative statics could be handled only under specialized hypotheses, for example, the invertibility of excess demand at all prices, etc. We shall give a fuller discussion of the three crucial mathematical results of the Arrow–Debreu model – existence, optimality and local uniqueness – in the next section.

Observe finally, that although the commodities may include physical goods dated over many time periods, there is only one budget constraint in an Arrow–Debreu equilibrium. The income that could be obtained from the sale of an endowed commodity, dated from the last period, is available already in the first period.

IV. PARETO OPTIMALITY

The first theorem of welfare economics states that any Arrow–Debreu equilibrium allocation $\bar{x} = (\bar{x}^h)$, $h = 1, \ldots, H$ is Pareto optimal in the sense that if $[(x^h), (y^j)]$ satisfies $y^j \in Y^j, \Sigma_{h=1}^H x^h = \Sigma_{j=1}^J y_h^j + e$, then it cannot be the case that $x^h \succ_h \bar{x}^h$ for all h. The second theorem of welfare analysis states the converse, namely that any Pareto optimal allocation for an Arrow–Debreu economy E is a competitive equilibrium allocation for an Arrow–Debreu economy $\hat{E}$ obtained from E by rearranging the initial endowments of commodities and ownership shares.

The first welfare theorem expresses the efficiency of the ideal market system, although it makes no claim as to the justice of the initial distribution of resources. The second welfare theorem implies that any income redistribution is best effected through a lump sum transfer, rather than through manipulating the market, e.g. through rent control, etc.

The connection between competitive equilibrium and Pareto optimality has been perceived for a long time, but until 1951 there was a general confusion between the necessity and sufficiency part of the arguments. The old proof of Pareto optimality (see Lange, 1942) assumed differentiable utilities of production sets, and a strictly positive allocation $\bar{x}$. It noted the first order conditions to the problem of maximizing the ith consumer's utility, subject to maintaining all the others at least as high as they got under $\bar{x}$, and feasibility, are satisfied at $\bar{x}$, if and only if $\bar{x}$ is a competitive equilibrium allocation for a 'rearranged' economy $\hat{E}$. This first order, or infinitesimal, proof of equivalence between competitive equilibrium and Pareto optimality could have been made global by postulating in addition that preferences and production sets are convex.

The Arrow and Debreu (1954) proofs of the equivalence between competitive equilibrium and Pareto optimality, under global changes, do not require differentiability, nor do they require that all agents consume a strictly positive amount of every good. In fact the proof of the first welfare theorem, that each competitive equilibrium is Pareto optimal, does not even use convexity.

The only requirement is local nonsatiation, so that every agent spends all his income in equilibrium. If (x, y) Pareto dominates the equilibrium allocation $(\bar{p}, \bar{x}, \bar{y})$, then for all $h, \bar{p} \cdot x^h < \bar{p} \cdot \bar{x}^h$. Since profit maximization implies that for all $j, \bar{p} \cdot \bar{y}^j \geqslant \bar{p} \cdot y^j$, it follows that $\bar{p} \cdot (\Sigma_h x^h - \Sigma_j y^j) > \bar{p} \cdot (\Sigma_h \bar{x}^h - \Sigma_j \bar{y}^j)$, contradicting feasibility.

The proof of the second welfare theorem, on the other hand, does require convexity of the preferences and production sets (though not their differentiability, nor the interiority of the candidate allocation $\bar{x}$). Essentially it depends on Minkowski's theorem, which asserts that between any two disjoint convex sets in $\mathbb{R}^L$ there must be a separating hyperplane.

In this connection let us mention one more remarkable mathematical property of the Arrow–Debreu model. Let us suppose that all production takes place under constant returns to scale: if $y \in Y^j$, then so is λy, for $\lambda \geqslant 0$. We say that a feasible allocation $\bar{x}$ for the economy E is in the core if there is no coalition of consumers $S \subset \{1, \ldots, H\}$ such that using only their initial endowments of resources, as well as access to all the production technologies, they cannot achieve an allocation for themselves which they all prefer to $\bar{x}$. The core is meant to reflect those allocations which could be maintained when bargaining (the formation of coalitions) is costless. In a status quo core allocation, any labour union or cartel of owners that threatens to withhold its goods from the market knows that another coalition could form and by withholding its goods, prevent some members of the original coalition from being better off than they were under the status quo. It is easy to see that any competitive equilibrium is in the core. Debreu–Scarf (1963), building on earlier work of Scarf, showed by using the separating hyperplane theorem, that if agents are small relative to the market, in the sense they made precise through the notion of replication, then the core consists only of competitive allocations. Such a theorem can also be proved even if there are small nonconvexities in preferences (see Aumann (1964) for a different formulation of the small agent).

EXISTENCE OF EQUILIBRIUM. Suppose that agents' preferences and firms' production sets are strictly convex, and that agents strictly prefer more of any commodity to less (strict monotonicity) and that they all have strictly positive endowments. Let Δ be the set of L-price vectors, all non-negative, summing to one. Let $f^h(p)$ be the commodity bundle most preferred by agent h, given the strictly positive prices $p \in \Delta_{++}$. Similarly let $g^j(p)$ be the profit maximizing choice of firm j, given prices $p \in \Delta_{++}$. Finally, let $f(p) = \Sigma_{h=1}^H f^h(p) - \Sigma_{j=1}^J g^j(p) - e$. It is easy to show that f is a continuous function at all $p \in \Delta_{++}$. A price $\bar{p} \in \Delta_{++}$ is an Arrow–Debreu equilibrium price if and only if $f(\bar{p}) = 0$.

In general there is no reason to expect a continuous function to have a zero. Thus Wald could prove only with great difficulty in a special case that an equilibrium necessarily exists. Now observe that the function must satisfy Walras' Law, $p \cdot f(p) = 0$, for all p. So f is not arbitrary.

Consider the convex, compact set Δ_ϵ of prices $p \in \Delta$ with $p_l \geqq \epsilon > 0$, for all l. Consider also the continuous function ϕ: $\Delta_\epsilon \to \Delta_\epsilon$ mapping p to the closest point $\hat{p}$ in Δ_ϵ to $f(p) + p$. By Brouwer's fixed point theorem, there must be some $\bar{p}$ with $\phi(\bar{p}) = \bar{p}$. From strict monotonicity, it follows that $\bar{p}$ cannot be on the boundary of Δ_ϵ, if ϵ is chosen sufficiently small. From Walras' Law it follows that if $\bar{p}$ is in the interior of Δ_ϵ, then $f(\bar{p}) = 0$. The demonstration of the existence of equilibrium by Arrow and Debreu, as modified later by Debreu (1959), followed a similar logic.

Note the essential role of convexity in two parts of the above proof. It was used with respect to agents' characteristics to guarantee that their optimizing behaviour is continuous. And it was also used to ensure that the space Δ_ϵ has the fixed point property. Smale (1976) has given a path-following proof (related to Scarf's (1973) algorithm) that on closer inspection does not require convexity of the price space. (Dierker (1974) and Balasko (1986) have given homotopy proofs.) This is not only of computational importance. It appears that there may be economic problems, dealing with general equilibrium with incomplete markets, in which the price space is intrinsically nonconvex, and in which the existence of equilibrium can only be proved using path-following methods (see Duffie-Shaffer, 1985).

To weaken the assumption of strict convexity, in the above proof, one can replace Brouwer's fixed point theorem with Kakutani's. An important conceptual point arises in connection with strict montonicity. If that is dropped, and the production sets do not have free disposal, then in order to guarantee the existence of equilibrium, the definition must be revised to require either $f_l(\bar{p}) = 0$, or $f_l(\bar{p}) < 0$ and $\bar{p}_l = 0$. There may be free goods, like air, in excess supply. One cannot drop monotonicity and free disposal without allowing for negative prices.

Finally, it can be shown that if there are small nonconvexities in either preference or production, and if all the agents are small relative to the market (either in the replication sense of Debreu–Scarf, or the measure zero sense of Aumann), then there will be prices at which the markets nearly clear. On the other hand, increasing returns to scale over a broad range is definitely incompatible with equilibrium.

LOCAL UNIQUENESS AND COMPARATIVE STATICS. Another property of the excess demand function $f(p)$ is that it is homogeneous of degree zero. So instead of taking $p \in \Delta$, let us fix $p_1 = 1$. Similarly, let $F(p)$ be the $L - 1$ vector of excess demands for goods $l = 2, \dots, L$ If $F(p) = 0$, then by Walras' Law, $f(p) = 0$.

Suppose furthermore that agent characteristics are smooth. Then $F(P)$ is a differentiable function. If $D_pF(\bar{p})$ has full rank at an equilibrium $\bar{p}$, then $\bar{p}$ is locally unique. Moreover, the equilibrium $\bar{p}$ will move continuously, given continuous, small changes in the agents' characteristics, such as their endowments e. If $D_pF(\bar{p})$ has full rank at all equilibria $\bar{p}$, then there are only a finite number of equilibria. Debreu (1970) called an economy E regular if $D_pF(\bar{p})$ has full rank at all equilibrium $\bar{p}$ of E.

The problem of trying to give sufficient conditions on preferences etc. to guarantee that $D_p\ F$ has full rank in equilibrium has proved intractable (except for restrictive, special cases). But Debreu (1970) solved the problem in classic style, appealing to the transversality theorem of differential topology (or Sard's theorem), to show that if one were content with regularity for 'almost all' economies, then the problem is simple He proved that for almost all economies, D_pF has full rank at every equilibrium. Hence, in almost all economies comparative statics (the change in equilibrium, given exogenous changes to the economy) is well defined.

Observe that excess demand F depends on the agents' characteristics, including their endowments, so we could write $F(e, p)$. Now the transversality theorem says that (given some technical conditions) if $D_eF(e, \bar{p})$ has full rank at all equilibria $\bar{p}$ for the economy $E(e)$ with endowments e, for all e, then for 'almost all' e, $D_pF(e, \bar{p})$ has full rank at all equilibrium $\bar{p}$ of $E(e)$. But it is easy to show that $D_eF(e, p)$ always has full rank. Along similar lines, Debreu proved the 'generic regularity' of equilibrium.

There is one unfortunate side to this comparative statics story. One would like to show not only that comparative statics are well defined, but also that they have a definite form. In a concave programming problem, for example, a small increase in an input results in a decrease in that input's shadow price, and an increase in output approximately equal to the size of the input increase multiplied by its original shadow price. Given the strong rationality hypothesis of the Arrow–Debreu model, one would hope for some sort of analogous result. Following a conjecture of Sonnenschein, Debreu proved in 1974 that given any function $f(p)$ on Δ_ϵ satisfying Walras' Law, he could find an Arrow–Debreu economy such that $f(p)$ is its aggregate excess demand on Δ_ϵ. Thus assumptions (A.1)–(A.13) do not permit any *a priori* predictions about the changes that must occur in equilibrium given exogenous changes to the economy. An increase in the aggregate endowment of a particular good, for example, might cause its equilibrium price to rise. The possibility of such pathologies is disappointing. It means that to make even qualitative predictions, the economist needs detailed data on the excess demands F.

V. WHAT THE MODEL DOESN'T EXPLAIN

We have already discussed the implications of the notion of Arrow–Debreu commodities and the second welfare theorem for insurance, namely that since every Pareto optimal allocation is supportable as an Arrow–Debreu equilibrium, every optimal allocation of risk bearing can be accomplished by the production and trade of Arrow–Debreu commodities, i.e. without recourse to additional kinds of insurance markets specializing in risks. Every Arrow–Debreu commodity is as much a diversifier in location, or time, or physical quality as it is for risk. This leads to a great simplification and economy of analysis. But it also means, that from the positive point of view, the Arrow–Debreu economy cannot directly provide an analysis of insurance markets (except as a benchmark case) In this section I shall try to point out a few of the other phenomena which recede into the background in the Arrow–Debreu model, but which would emerge if the assumption of a finite, but complete set of Arrow–Debreu commodities, and consumers was dropped.

There are four currently active lines of research which attempt to come to grips in a general equilibrium framework with some of these phenomena, while preserving the fundamental neoclassical Arrow–Debreu principles of agent optimization, market clearing, and rational expectations, that I think are particularly worthy of attention. They are the theory of general equilibrium with incomplete asset markets which can be traced back to Arrow's (1953) seminal paper on securities; overlapping generations economies, whose study was initiated by Samuelson (1958) in his classic consumption loan model; the Cournot theory of market exchange with few traders, first adapted to general equilibrium by Shapley–Shubik (1977), and the model of rational expectations equilibrium, pioneered by Lucas (1972)

Let us note first of all that in Arrow–Debreu equilibrium ther is no trade in shares of firms. A stock certificate is not an Arrow–Debreu commodity, for its possession entitles the owner to additional commodities which he need not obtain through exchange. Note also that in Arrow–Debreu equilibrium, the hypothesis that all prices will remain the same, no matter how an individual firm changes its production plan, guarantees that firm owners unanimously agree on the firm objective, to maximize profit. If there were a market for firm shares, there would not be any trade anyway, since ownership of the firm and the income necessary to purchase it would be perfect substitutes. In an incomplete markets equilibrium, different sources of revenue are not necessarily perfect substitutes. There could be active trade on the stock market. Of course, such a model would have to specify the firm objectives, since one would not expect unanimity. The theory of stock market equilibrium is still in its infancy, although some important work has already been done. (See Dreze, 1974, and Grossman–Hart, 1979.)

Bankruptcy is not allowed in an Arrow–Debreu equilibrium. That follows from the fact that all agents must meet their budget constraints. In a game theoretic formulation of equilibrium (such as I shall discuss shortly), it is achieved by imposing an infinite bankruptcy penalty. Since every Arrow–Debreu equilibrium is Pareto optimal, there would be no benefit in reducing the bankruptcy penalty to the point where someone might choose to go bankrupt. But with incomplete markets, such a policy might be Pareto improving, even allowing for the deadweight loss of imposing the penalties.

Money does not appear in the Arrow–Debreu model. Of course, all of the reasons for its real life existence: transactions demand, precautionary demand, store of value, unit of account, etc. are already taken care of in the Arrow–Debreu model. One could imagine money in the model: at data zero every agent could borrow money from the central bank. At every date afterwards he would be required to finance his purchases out of his stock of money, adding to that stock from his sales. At the last date he would be required to return to the bank exactly what he borrowed (or else face an infinite bankruptcy penalty). In such a model the Arrow–Debreu prices would appear as money prices. The absolute level of money prices and the aggregate amount of borrowing would not be determined, but the allocations of commodities would be the same as in Arrow–Debreu. There is no point in making the role of money explicit in the Arrow–Debreu model, since it has no effect on the real allocations. However, if one considers the same model with incomplete asset markets, the presence of explicitly financial securities can be of great significance to the real allocations

In the Arrow–Debreu model, all trade takes place at the beginning of time. If markets were reopened at later dates for the same Arrow–Debreu commodities, then no additional trade would take place anyway. At the other extreme, one might consider a model in which at every date and state of nature only those Arrow–Debreu commodities could be traded which were indexed by the corresponding (date, state) pair. An intermediate case would also permit the trade of some (but not all) differently indexed Arrow–Debreu commodities. Now the Arrow–Debreu proofs of the existence and Pareto optimality of equilibrium do not apply to such an incomplete markets economy, as Hart (1975) first pointed out. We have already noted the existence problem. As for efficiency, the Pareto optimality of Arrow–Debreu equilibria might suggest the presumption that, though there might be a loss to eliminating markets, trade on the remaining markets would be as efficient as possible. In fact, it can be shown (generically) that equilibrium trades do not make efficient use of the existing markets.

The Arrow–Debreu model of general equilibrium is relentlessly neoclassical; in fact it has become the paradigm of the neoclassical approach. This stems in part from its individualistic hypothesis, and its celebrated conclusions about the potential efficacy of unencumbered markets. (Although Arrow, for example, has always maintained that a proper understanding of Arrow–Debreu commodities is also useful to showing how inefficient is the limited real world market system.) But still more telling is the fact that the assumption of a finite number of commodities (and hence of dates) forces upon the model the interpretation of the economic process as a one-way activity of converting given primary resources into final consumption goods. If there is universal agreement about when the world will end, there can be no question about the reproduction of the capital stock. In equilibrium it will be run down to zero. Similarly when the world has a definite beginning, so that the first market transaction takes place after the ownership of all resources and techniques of production, and the preferences of all individuals have been determined, one cannot study the evolution of the social norms of consumption in terms of the historical development of the relations of production. One certainly cannot speak about the production of all commodities by commodities (Sraffa, 1960) (since at date zero there must be commodities which have not been produced by commodities, i.e. by physical objects which are traded).

It seems natural to suppose that as L becomes very large, so that the end of the world is put off until the distant future, that this event cannot be of much significance to behaviour now. But let us not forget the rationality imposed on the agents. Far off as the end of the world might be, it is perfectly taken into account. Thus, for example, social security (funded as it is in the US by taxes on the young) could not exist if rational agents agreed on a final stopping time to transactions.

Consider a model satisfying all the assumptions (A.1)–(A.13), except that L and H are allowed to be infinite, such as the overlapping generations model. It can be shown that there is a robust collection of economies which have a continuum of equilibria, most of which are Pareto suboptimal, which differ enormously in time 0 behaviour. Thus in a model where time does not have a definite end, the optimality and comparative statics properties of equilibria are radically different. (For example, there may be a continuum of equilibria, indexed by the level of period 0 real wages (inversely related to the rate of profit) or the level of output or employment. The interested reader can consult the entry on OVERLAPPING GENERATIONS MODELS. A systematic study of economies where only L is allowed to be infinite was begun by Bewley (1972). Such economies tend to have properties similar to those of Arrow–Debreu.)

There is no place in the Arrow–Debreu model for asymmetric information. The second welfare theorem, for example, relies on lump sum redistributions, i.e. redistributions that occur in advance of the market interactions. But if agents cannot be distinguished except through their market behaviour, then the redistribution must be a function of market behaviour Rational agents, anticipating this, will distort their behaviour and the optimality of the redistribution will be lost.

Similarly, in the definition of equilibrium no agent takes into account what other agents know, for example about the state of nature. Thus it is quite possible in an Arrow–Debreu equilibrium for some ignorant agents to exchange valuable commodities for commodities indexed by states that other agents know will not occur. This problem received enormous attention in the finance literature, and some claim (see Grossman 1981) that it has been solved by extending the Arrow–Debreu definition of equilibrium to a 'rational expectations equilibrium' (Lucas, 1972; see also Radner, 1979). But this definition is itself suspect; in particular, it may not be implementable.

Even if rational expectations equilibrium (REE) were accepted as a viable notion of equilibrium, it could not come to grips with the most fundamental problems of asymmetric information. For like Arrow–Debreu equilibrium, in REE all trade is conducted anonymously through the market at given prices. Implicit in this definition is the assumption of large numbers of traders on both sides of every market. But what has come to be called the incentive problem in economics revolves around individual or firm specific uncertainty, i.e. trade in commodities indexed by the names of the traders, which by definition involves few traders.

This brings us to another major riddle: how are agents supposed to get to equilibrium in the Arrow–Debreu model?

The pioneers of general equilibrium never imagined that the economy was necessarily in equilibrium; Walras, for example, proposed an explicit tâtonnement procedure which he conjectured converged to equilibrium. But that idea is flawed in two respects: in general, it can be shown not to converge, and more importantly, it is an imaginary process in which no exchange is permitted until equilibrium is reached. This illustrates a grave shortcoming of any equilibrium theory, namely that it cannot begin to specify outcomes out of equilibrium. The major crisis of labour market clearing in the 1930s, and again recently, argues strongly that there are limits to the applicability of equilibrium analysis.

One is led naturally to consider market games, in which the outcomes are well-specified even when agents do not make their equilibrium moves The most famous market game is Cournot's duopoly model, which has been extended to general equilibrium by Shapley–Shubik (1977). When there are a large number of agents of each type, the Nash equilibria of the Shapley–Shubik game give nearly identical allocations to the competitive allocations of Arrow–Debreu. This justifies (to first approximation) the price taking behaviour of the Arrow–Debreu agents. But note that the informational requirements of Nash equilibrium are at least twice that of Arrow–Debreu competitive equilibrium (each agent must know the aggregates of bids and offers on each market). It is also extremely interesting that trade takes place in the Shapley–Shubik game even if there is only one trader on each side of the market. Hence many problems in asymmetric information which have no place in the Arrow–Debreu model, because they involve too fine a specification of the commodities to be consistent with price taking, might be sensible in a market game context. Finally, it can be shown that REE is not consistent with the Shapley–Shubik game, or indeed with any continuous game.

We have indicated some of the ways in which it is possible to extend general equilibrium analysis to phenomena outside the scope of the Arrow–Debreu model, while at the same time preserving the neoclassical methodological premises of agent optimization, rational expectations, and equilibrium. It is important to note that these variations have extended the definition of equilibrium as well; this is most obvious in the case of market games, where Nash equilibrium replaces competitive equilibrium. All of the models have retained, on the other hand, more or less the same notion of rationality, sometimes at the cost of increasing the demands on the rationality of expectations. A great challenge for future general equilibrium models is how to formulate a sensible notion of bounded rationality, without destroying the possibility of drawing normative conclusions.

JOHN GEANAKOPLOS

See also EXISTENCE OF GENERAL EQUILIBRIUM, GENERAL EQUILIBRIUM, INTERTEMPORAL EQUILIBRIUM AND EFFICIENCY, OVERLAPPING GENERATIONS MODEL, STABILITY; UNIQUENESS OF EQUILIBRIUM

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