

Arrow–Debreu economy. A vast literature in public finance and macroeconomics is based on the model, including studies of the national debt, social security, the incidence of taxation and bequests on the accumulation of capital, the Phillips curve, the business cycle, and the foundations of monetary theory. In the following pages I give a hint of these myriad applications only in so far as they illuminate the general theory. My main concern is with the relationship between the Samuelson model and the Arrow–Debreu model.

Samuelson's innovation was in postulating a demographic structure in which generations overlap, indefinitely into the future; up until then it had been customary to regard all agents as contemporaneous. In the simplest possible example, in which each generation lives for two periods, endowed with a perishable commodity when young and nothing when old, Samuelson noticed a great surprise. Although each agent could be made better off if he gave half his youthful birthright to his predecessor, receiving in turn half from his successor, in the marketplace there would be no trade at all. A father can benefit from his son's resources, but has nothing to offer in return.

This failure of the market stirred a long and confused controversy. Samuelson himself attributed the suboptimality to a lack of double coincidence of wants. He suggested the social contrivance of money as a solution. Abba Lerner suggested changing the definition of optimality. Others, following Samuelson's hints about the financial intermediation role of money, sought to explain the consumption loan model by the incompleteness of markets. It has only gradually become clear that the 'Samuelson suboptimality paradox' has nothing to do with the absence of markets or financial intermediation. Exactly the same equilibrium allocation would be reached if all the agents, dead and unborn, met (in spirit) before the beginning of time and traded all consumption goods, dated from all time periods, simultaneously under the usual conditions of perfect intermediation.

Over the years Samuelson's consumption loan example, infused with Arrow–Debreu methods, has been developed into a full blown general equilibrium model with many agents, multiple kinds of commodities and production. It is equally faithful to the neoclassical methodological assumptions of agent optimization, market clearing, and rational expectations as the Arrow–Debreu model. This more comprehensive version of Samuelson's original idea is known as the overlapping generations (OLG) model of general equilibrium.

Despite the methodological similarities between the OLG model and the Arrow–Debreu model, there is a profound difference in their equilibria. The OLG equilibria may be Pareto suboptimal. Money may have positive value. There are robust OLG economies with a continuum of equilibria. Indeed, the more commodities per period, the higher the dimension of multiplicity may be. Finally, the core of an OLG economy may be empty. None of this could happen in any Arrow–Debreu economy.

The puzzle is why? One looks in vain for an externality, or one of the other conventional pathologies of an Arrow–Debreu economy. It is evident that the simple fact that generations overlap cannot be an explanation, since by judicious choice of utility functions one can build that into an Arrow–Debreu model. It cannot be simply that the time horizon is infinite, as we shall see, since there are classes of infinite horizon economies whose equilibria behave very much like Arrow–Debreu equilibria. It is the combination, that generations overlap indefinitely, which is somehow crucial. In sections 4 and 5 I explain how

Note that in the Arrow–Debreu economy the number of

overlapping generations model of general equilibrium. The consumption loan model that Paul Samuelson introduced in 1958 to analyse the rate of interest, with or without the social contrivance of money, has developed into what is without doubt the most important and influential paradigm in neoclassical general equilibrium theory outside of the

commodities, and hence of time periods, is finite. One is tempted to think that if the end of the world is put far enough off into the future, that could hardly matter to behaviour today. But recalling the extreme rationality hypotheses of the Arrow-Debreu model, it should not be surprising that such a cataclysmic event, no matter how long delayed, could exercise a strong influence on behaviour. Indeed the OLG model proves that it does. One can think of other examples. Social security, based on the pay-as-you-go principle in the United States in which the young make payments directly to the old, depends crucially on people thinking that there might always be a future generation, otherwise the young will not contribute. Another similar example comes from game theory, in which cooperation depends on an infinite horizon. On the whole, it seems at least as realistic to suppose that everyone believes the world is immortal as to suppose that everyone believes in a definite date by which it will end. (In fact, it is enough that people believe, for every T , that there is positive probability the world lasts past T .)

In sections 1 and 3 I describe a simple one commodity example illustrating the four differences mentioned above between OLG and Arrow-Debreu equilibria. These are known to hold equally for economies with many commodities, as pointed out in sections 4 and 5. Section 2 discusses the possibility of equilibrium cycles in a one commodity, stationary, OLG economy.

Section 6 takes up the question of comparative statics. If there is a multiplicity of equilibria, what sense can be made of comparative statics? Section 6 summarizes the work showing that for perfectly anticipated changes, there is only one equilibrium in the multiplicity that is 'near' an original 'regular' equilibrium. For unanticipated changes, there may be a multidimensional multiplicity. But it is parameterizable. Hence by always fixing the same variables, a unique prediction can be made for changes in the equilibrium in response to perturbations. In section 7 we see how this could be used to understand some of the New Classical-Keynesian disputes about macroeconomic policy. Different theories hold different variables fixed in making predictions. Section 8 considers a neoclassical-classical controversy and section 9 summarizes some work on sunspots in the OLG model. Uncertainty in dynamic models seems likely to be very important in the future. An understanding of the one commodity model is sufficient to read all sections except 4 and 5.

The explanation of the puzzles of OLG equilibria given in section 4 is lack of market clearing 'at infinity'. By appealing to nonstandard analysis, the mathematics of infinite and infinitesimal numbers, it can be shown that there is a 'finite-like' Arrow-Debreu economy whose 'classical equilibria', those price sequences which need not clear the markets in the last period, are isomorphic to the OLG equilibria. Lack of market clearing is also used to explain the suboptimality and the positive valuation of money.

Recall the classical economists' conception of the economic process as a never ending cycle of reproduction. The state of physical commodities is always renewed. Samuelson attempted to give a completely neoclassical explanation of the rate of interest in just such a setting. It now appears that the market forces of supply and demand are not sufficient to determine the rate of interest in the OLG model. In other infinite horizon models they do. The difference, we speculate in section 5, comes from the fact that in the OLG model, no matter how large t is, there is always someone who values consumption more at date t than at date 1. In those infinite horizon models where equilibrium behaviour is like that in Arrow-Debreu, the economy is uniformly impatient.

1. INDETERMINACY AND SUBOPTIMALITY IN A SIMPLE OLG MODEL. In this section we analyse the equilibrium set of a one commodity per period, overlapping generations (OLG) economy. Although the definition of equilibrium seems firmly in the Walrasian tradition of agent optimization and market clearing, we discover three surprises. There are robust examples of OLG economies that possess an uncountable multiplicity of equilibria, that are not in the core, or even Pareto optimal. The lack of optimality (in a slightly different model, as we shall see) was pointed out by Samuelson in his seminal (1958) paper. The indeterminacy of equilibrium in the one commodity case is usually associated first with Gale (1973). In later sections we shall show that these puzzles are robust to an extension of the model to multiple commodities and agents per period, and to a nonstationary environment. We shall add still another puzzle in section 3, the positive valuation of money, which is also due to Samuelson.

A large part of this section is devoted to developing the notation and price normalization that we shall use throughout. In any Walrasian model the problem of price normalization (the 'numeraire problem') arises. Here the most convenient solution in the long run is not at first glance the most transparent.

Consider an overlapping generation (OLG) economy $E = E_{-\infty, \infty}$ in which discrete time periods t extend indefinitely into the past and into the future, $t \in \mathcal{Z}$. Corresponding to each time period there is a single, perishable consumption good x_t . Suppose furthermore that at each date t one agent is 'born' and lives for two periods, with utility

$$u^t(\dots, x_t, x_{t+1}, \dots) = a^t \log x_t + (1 - a^t) \log x_{t+1}$$

defined over all vectors

$$x = (\dots, x_{-1}, x_0, x_1, \dots) \in L = R^{\mathcal{Z}}$$

Thus we identify the set of agents A with the time periods \mathcal{Z} . Let each agent $t \in A$ have endowment

$$e^t = (\dots, 0, e_t^1, e_{t+1}^1, 0, \dots) \in L$$

which is positive only during the two periods of his life. Note that

$$\sum_{t \in A} e_t^s = e_{t-1}^{s-1} + e_t^s \quad \text{for all } s \in \mathcal{Z}.$$

An equilibrium is defined as a price vector

$$p = (\dots, p_{-1}, p_0, \dots) \in L_{++}$$

and allocation

$$\bar{x} = [x^t = (\dots, x_t^1, x_{t+1}^1, \dots), t \in A]$$

satisfying \bar{x} is feasible, i.e.

$$\sum_{t \in A} x_t^s = \sum_{t \in A} e_t^s, \quad \text{for all } s \in \mathcal{Z} \quad (1.1)$$

and

$$\sum_{s \in \mathcal{Z}} p_s e_t^s < \infty \quad \text{for all } t \in A \quad (1.2)$$

and

$$x^t \in \text{Arg Max}_{x \in L} \left\{ u^t(x) \mid \sum_{s \in \mathcal{Z}} p_s x_s \leq \sum_{s \in \mathcal{Z}} p_s e_t^s \right\} \quad (1.3)$$

The above definition of equilibrium is precisely in the Walrasian tradition, except that it allows for both an infinite number of traders and commodities. All prices are finite, and consumers treat them as parametric in calculating their budgets. The fact that the definition leads to robust examples with a continuum

of Pareto-suboptimal equilibria calls for an explanation. We shall give two of them, one at the end of this section, and one in section 4. Note that condition (1.2) becomes necessary only when we consider models in which agents can live for an infinite number of time periods.

As usual, the set of equilibrium price sequences displays a trivial dimension of multiplicity (indeterminacy), since if p is an equilibrium, so is kp for all scalars $k > 0$. We can remove this ambiguity by choosing a price normalization $q_t = p_{t+1}/p_t$, for all $t \in \mathcal{T}$. The sequence $q = (\dots, q_{-1}, q_0, \dots)$ and allocations $(x^t; t \in \mathcal{T})$ form an equilibrium if (1) above holds together with

$$x^t \in \text{Arg Max}_{x \in L} \{u^t(x) | x_t + q_t x_{t+1} \leq e_t^t + q_t e_{t+1}^t\}. \quad (1.4)$$

Notice that we have taken advantage of the finite lifetime of the agents to combine (1.2) and (1.3) into a single condition (1.4). We could have normalized prices by choosing a numeraire commodity, and setting its price equal to one, say $p_0 = 1$. The normalization we have chosen instead has three advantages as compared with this more obvious system. First, the q system is time invariant. It does not single out a special period in which a price must be 1; if we relabelled calendar time, then the corresponding relabelling of the q_t would preserve the equilibrium. In the numeraire normalization, after the calendar shift, prices would have to be renormalized to maintain $p_0 = 1$. Second, on account of the monotonicity of preferences, we know that if the preferences and endowments are uniformly bounded

$$0 < a \leq a^t \leq \bar{a} < 1, \quad 0 < e \leq e_t^t, \quad e_{t+1}^t \leq \bar{e} \leq 1$$

for all $t \in A$,

then we can specify uniform *a priori* bounds \underline{k} and \bar{k} such that any equilibrium price vector q must satisfy $\underline{k} \leq q_t \leq \bar{k}$ for all $t \in \mathcal{T}$. Thirdly, it is sometimes convenient to note that each generation's excess demand depends on its own price. We define

$$[Z_t^t(q_t), Z_{t+1}^t(q_t)] = (x_t^t - e_t^t, x_{t+1}^t - e_{t+1}^t)$$

for x^t satisfying (1.4), as the excess demand of generation t , when young and when old. We can accordingly rewrite equilibrium condition (1.1) as

$$Z_t^{t-1}(q_{t-1}) + Z_t^t(q_t) = 0 \quad \text{for all } t \in \mathcal{T}. \quad (1.5)$$

Let us now investigate the equilibria of the above economy when preferences and endowments are perfectly stationary. To be concrete, let

$$a^t = a \quad \text{for all } t \in A,$$

and let

$$e_t^t = e, \quad \text{and } e_{t+1}^t = 1 - e, \quad \text{for all } t \in A,$$

where $e > a \geq \frac{1}{2}$. Agents are born with a larger endowment when young than when old, but the aggregate endowment of the economy is constant at 1 in every time period. Furthermore, each agent regards consumption when young as at least as important as consumption when old ($a \geq \frac{1}{2}$), but on account of the skewed endowment, the marginal utility of consumption at the endowment allocation when young is lower than when old:

$$\frac{a}{e} < \frac{1-a}{1-e}.$$

If we choose

$$q_t = \bar{q} = \frac{1-ae}{1-ea} > 1$$

for all $t \in \mathcal{T}$, then we see clearly that at these prices each agent

will just consume his endowment; $q = (\dots, \bar{q}, \bar{q}, \bar{q}, \dots)$ is an equilibrium price vector, with $x^t = e^t$ for all $t \in A$. Note that if we had used the price normalization $p_0 = 1$, the equilibrium prices would be described by

$$(\dots, p_0, p_1, p_2, \dots) = (\dots, 1, \bar{q}, \bar{q}^2, \dots)$$

where $p_t \rightarrow \infty$ as $t \rightarrow \infty$.

But there are other equilibria as well. Take $q = (\dots, 1, 1, 1, \dots)$, and

$$(x_t^t, x_{t+1}^t) = (a, 1-a) \quad \text{for all } t \in A.$$

This 'golden rule' Pareto equilibrium dominates the autarkic equilibrium previously calculated, thereby raising the most important puzzle of overlapping generations economies: why is it that equilibria can fail to be Pareto optimal? We shall discuss this question at length, in section 4. For now, let us observe one more curious fact. We can define the *core* of our economy in a manner exactly analogous to the finite commodity and consumer case. We say that a feasible allocation $x = (x^t; t \in A)$ is in the core of the economy E if there is no subset of traders $A' \subset A$, and an allocation $y = (y^t; t \in A')$ for A' such that

$$\sum_{t \in A'} y^t = \sum_{t \in A'} e^t,$$

and

$$u^t(y) > u^t(x^t) \quad \text{for all } t \in A'.$$

A simple argument can be given to show that the core of this economy is empty. For example, the golden rule equilibrium allocation is Pareto optimal, but not in the core. Since $a < e$, every agent is consuming less when young than his initial endowment. Thus for any $t_0 \in A$, the coalition $A' = \{t \in A | t \geq t_0\}$ consisting of all agents born at time t_0 or later can block the golden rule allocation.

Let us continue to investigate the set of equilibria of our simple, stationary economy. One can show that for any \bar{q}_0 , with $1 < \bar{q}_0 < \bar{q}$, there is an equilibrium price sequence

$$q = (\dots, q_{-1}, q_0, q_1, \dots)$$

with $q_0 = \bar{q}_0$. In other words, there is a whole continuum of equilibria, containing a nontrivial interval of values. Incidentally, it can also be shown that for all such equilibria $q_t \rightarrow \bar{q}$ as $t \rightarrow \infty$, and $q_t \rightarrow 1$ as $t \rightarrow -\infty$. Moreover, these equilibria, together with the two steady state equilibria, constitute the entire equilibrium set.

This raises the second great puzzle of overlapping generations economies. There can be a nondegenerate continuum infinity of equilibria, while in finite commodity and agent economies there are typically only a finite number. Thus if we considered the finite truncated economy $E_{-T, T}$ consisting of those agents born between $-T$ and T , and no others, then it can easily be seen that there is only a unique equilibrium $(q_{-T}, \dots, q_T) + (\bar{q}, \dots, \bar{q})$, no matter how large T is taken. On the other hand, in the overlapping generations economy, there are a continuum of equilibria. Moreover, the differences in these equilibria are not to be seen only at the tails. In the OLG economy, as \bar{q}_0 varies from 1 to \bar{q} , the consumption of the young agent at time zero varies from a to e , and his utility from $a \log e = (1-a) \log(1-e)$ (which for e near 1 is close to $-\infty$), all the way to a $\log a + (1-a) \log(1-a)$. By pushing the 'end of the world' further into the future, one does not approximate the world which does not end. We shall take up this theme again in section 4.

It is very important to understand that the multiplicity of equilibria is not due to the stationarity of the economy. If we choose a^t near a and (e_t^t, e_{t+1}^t) near $(e, 1-e)$, we would find the

same multiplicity. One might hold the opinion that in a steady state economy, one should only pay attention to steady state equilibria, i.e. only to the autarkic and golden rule equilibria. In nonsteady state economies, there are not steady state equilibria to stand out among the continuum. One must face up to the multiplicity.

Let us reconsider how one might demonstrate the multiplicity of equilibria, even in a nonstationary economy. This will lead to a first economic explanation of indeterminacy similar to the one originally proposed by Samuelson. Suppose that in our nonstationary example we find one equilibrium $\hat{q} = (\dots, \hat{q}_{-1}, \hat{q}_0, \hat{q}_1, \dots)$ satisfying:

$$Z_t^{-1}(\hat{q}_{t-1}) + Z_t'(\hat{q}_t) = 0 \quad \text{for all } t \in \mathcal{Z}. \quad (1.6)$$

Let us look for 'nearby' equilibria.

We shall say that generation t is expectations sensitive at q_t if both $[\partial Z_t'(\hat{q}_t)/\partial q_t] \neq 0$ and $[\partial Z_{t+1}'(\hat{q}_t)/\partial q_t] \neq 0$. If the first inequality holds, then the young's behaviour at time t can be influenced by what they expect to happen at time $t+1$. Similarly, if the second inequality holds, then the behaviour of the old agent at time $t+1$ depends on the price he faced when he was young, at time t . Recalling the logarithmic preferences of our example, it is easy to calculate that the derivatives of excess demands, for any $q_t > 0$, satisfy

$$\frac{\partial Z_t'(q_t)}{\partial q_t} = a^t e_{t+1}^t \neq 0$$

and

$$\frac{\partial Z_{t+1}'(q_t)}{\partial q_t} = \frac{-(1-a^t)e_t^t}{q_t^2} \neq 0$$

Hence by applying the implicit function theorem to (1) we know that there is a nontrivial interval I_{t-1}^F containing \hat{q}_{t-1} and a function F_t with domain I_{t-1}^F such that $F_t(\hat{q}_{t-1}) = \hat{q}_t$, and more generally,

$$Z_t^{-1}(q_{t-1}) + Z_t'(F_t(q_{t-1})) = 0, \quad \text{for all } q_{t-1} \in I_{t-1}^F.$$

Similarly there is a nontrivial interval I_t^B containing \hat{q}_t , and a function B_t with domain I_t^B such that $B_t(\hat{q}_t) = \hat{q}_{t-1}$, and more generally, $Z_t^{-1}[B_t(q_t)] + Z_t'(q_t) = 0$, for all $q_t \in I_t^B$. Of course, if $F_t(q_{t-1}) = q_t \in I_t^B$, then $B_t(q_t) = q_{t-1}$.

These forward and backward functions F_t and B_t , respectively, hold the key to one understanding of indeterminacy. Choose any relative price $q_0 \in I_0^F \cap I_0^B$ between periods 0 and 1. The behaviour of the generation born at 0 is determined, including its behaviour when old at period 1. If $q_0 \neq \hat{q}_0$, and generation 1 continues to expect relative prices \hat{q}_1 between 1 and 2, then the period 1 market will not clear. However, it will clear if relative prices q_1 adjust so that $q_1 = F_1(q_0)$. Of course, changing relative prices between period 1 and 2 from \hat{q}_1 to q_1 will upset market clearing at time 2, if generation 2 continues to expect \hat{q}_2 . But if expectations change to $q_2 = F_2(q_1)$, then again the market at time 2 will clear. In general, once we have chosen $q_t \in I_t^F$, we can take $q_{t+1} = F_{t+1}(q_t)$ to clear the $(t+1)$ market. Similarly, we can work backwards. The change in q_0 will cause the period 0 market not to clear, unless the previous relative prices between period -1 and 0 were changed from \hat{q}_{-1} to $q_{-1} = B_0(q_0)$. More generally, if we have already chosen $q_t \in I_t^B$, we can set $q_{t-1} = B_t(q_t)$ and still clear the period t market.

Thus we see that it is possible that an arbitrary choice of $q_0 \in I_0^F \cap I_0^B$ could lead to an equilibrium price sequence q . What happens at time 0 is undetermined because it depends on expectations concerning period 1, and also the past. But what can rationally be expected to happen at time 1 depends on what in turn is expected to happen at time 2 etc.

There is one essential element missing in the above story. Even if $q_t \in I_t^F$, there is no guarantee that $q_{t+1} = F_{t+1}(q_t)$ is an element of I_{t+1}^F . Similarly, $q_t \in I_t^B$ does not necessarily imply that $q_{t-1} = B_t(q_t) \in I_{t-1}^B$. In our steady state example, this can easily be remedied. Since all generations are alike,

$$F_t = F_1, B_t = B_0, I_t^F = I_0^F \quad \text{and} \quad I_t^B = I_0^B \quad \text{for all } t \in \mathcal{Z}$$

One can show that the interval $(1, \bar{q}) \subset I_0^F \cap I_0^B$, and that if $q_0 \in (1, \bar{q})$, then $F_1(q_0) \in (1, \bar{q})$, and $B_0(q_0) \in (1, \bar{q})$. This establishes the indeterminacy we claimed.

In the general case, when there are several commodities and agents per period, and when the economy is nonstationary, a more elaborate argument is needed. Indeed, one wonders, given one equilibrium \hat{q} for such an economy, whether after a small perturbation to the agents there is any equilibrium at all of the perturbed economy near \hat{q} . We shall take this up in section 6.

2. ENDOGENOUS CYCLES. Let us consider another remarkable and suggestive property that one commodity, stationary OLG economies can exhibit. We shall call the equilibrium $q = (\dots, q_{-1}, q_0, q_1, \dots)$ periodic of period n if q_0, q_1, \dots, q_{n-1} are all distinct, and if for all integers t and j , $q_t = q_{t+jn}$. The possibility that a perfectly stationary economy can exhibit cyclical ups and downs, even without any exogenous shocks or uncertainty, is reminiscent of 1930s-1950s business cycle theories. In fact, it is possible to construct a robust one commodity per period economy which has equilibrium cycles of every order n ! Let us see how.

As before, let each generation t consist of one agent, with endowment $e^t = (\dots, 0, e, 1-e, 0, \dots)$ positive only in period t and $t+1$, and utility $u^t(x) = u_1(x_t) + u_2(x_{t+1})$. Again, suppose that $\hat{q} = u_2'(1-e)/u_1'(e) > 1$. It is an immediate consequence of the separability of u^t , that for

$$q_t \leq \hat{q}, \quad \frac{\partial Z_{t+1}'(q_t)}{\partial q_t} < 0.$$

From monotonicity, we know that $Z_{t+1}'(q_t) \rightarrow \infty$ as $q_t \rightarrow 0$. Hence it follows that for any $0 < q_0 < \hat{q}$, there is a unique $q_{-1} = B_0(q_0)$ with

$$Z_0^{-1}[B_0(q_0)] + Z_0'(q_0) = 0.$$

From the fact that $Z_0'(q_0) \geq -e$, for all q_0 it also follows that there is some $q \leq 1$ such that if $q_0 \in [q, \hat{q}]$, then $B_0(q_0) \in [q, \hat{q}]$.

Now consider the following theorem due to the Russian mathematician Sarkovsky.

Sarkovsky's theorem. Let $B: [q, \bar{q}] \rightarrow [q, \bar{q}]$ be a continuous function from a nontrivial closed interval into itself. Suppose that there exist a 3-cycle for B , i.e. distinct points q_0, q_1, q_2 , in $[q, \bar{q}]$ with $q_1 = B(q_0)$, $q_2 = B(q_1)$, $q_0 = B(q_2)$. Then there are cycles for B of every order n .

Grandmont (1985), following related work of Benhabib-Day and Benhabib-Nishimura, gave a robust example of a one commodity, stationary economy (u_1, u_2, e) giving rise to a 3-cycle for the function B_0 . Of course a cycle for B_0 is also a cyclical equilibrium for the economy, hence there are robust examples of economies with cycles of all orders.

Theorem (Grandmont): There exist robust examples of stationary, one-commodity OLG economies with cyclical equilibria of every order n .

This result is extremely suggestive. Note first, however, that all of the cyclical equilibria, except for the autarkic one-cycle $(\dots, \bar{q}, \bar{q}, \bar{q}, \dots)$ can be shown to be Pareto optimal (see section 4). The theory of macroeconomic business cycles is concerned with the welfare losses from cyclical fluctuations. (On the other hand, the fact that cyclical behaviour is not incompatible with

optimality is perhaps an important observation for macroeconomics) More significantly, it must be pointed out that Sarkovsky's theorem is a bit of a mathematical curiosity. No comparable general theorem is known for maps of the square, or any higher dimensional cube, into itself. The best that could be hoped for in general are robust examples of two cycles. And of course nonstationary economies, even with one commodity, will typically not have any periodic cycles. By contrast, the multiplicity and suboptimality of nonperiodic equilibria that we saw in section 1 are robust properties that are maintained in OLG economies with multiple commodities and heterogeneity across time. The main contribution of the endogenous business cycle literature is that it establishes the extremely important, suggestive principle, that very simple dynamic models can have very complicated ('chaotic') dynamic equilibrium behaviour.

In the next section we turn to another phenomenon that can generally occur in overlapping generations economies, but never in finite horizon models.

3. MONEY. Money very often has value in an overlapping generations model, but it never does in a finite horizon Arrow-Debreu model. The reason for its absence in the latter model is familiar. In the last period its marginal utility to every consumer is zero, hence so is its price. In the second to last period nobody will pay to end up holding any money, because in the last period it will be worthless. By induction it will have no value even in the first period. Evidently this logic fails in the infinite horizon setting, since there is no last period. On the other hand, there are infinite horizon models where again money can have no value. The difference between the OLG model and these other infinite horizon models will be discussed in section 5.

Strictly speaking, the overlapping generations model we have discussed so far has been modelled along the lines of Arrow-Debreu. Each agent faces only one budget constraint and equilibrium is defined as if all markets met simultaneously at the beginning of time ($-\infty$). In such a model money has no function. However, we can define another model, similar to that first considered by Samuelson, in which agents face a sequence of budget constraints and markets meet sequentially, where money does have a store of value role. Surprisingly, this model turns out to have formally the same properties as the OLG model we have so far considered. To distinguish the two models we shall refer to this latter monetary model as the Samuelson model.

Suppose that we imagine a one-good per period economy in which the markets meet sequentially, according to their dates, and not simultaneously at the beginning of time. In such a setting it is easy to see that there could be no trade, since, as Samuelson put it, there is no double coincidence of wants. The old and the young at any date t both have the same kind of commodity, so they have no mutually advantageous deal to strike. But as Samuelson pointed out, introducing a durable good called money, which affects no agent's utility, might allow for much beneficial trade. The old at date t could sell their money to the young for commodities, who in turn could sell their money when old to the next period's young. In this manner new and more efficient equilibria might be created. The 'social contrivance of money' is thus connected to both the indeterminacy of equilibrium and the Pareto suboptimality of equilibrium, at least near autarkic equilibria. The puzzle, we have said, is how to explain the positive price of money when it has no marginal utility.

A closer examination of the equilibrium conditions of Samuelson's monetary equilibrium reveals that although it

appears much more complicated, it reduces to the OLG model we have defined above, but with one difference, that the budget constraint of the generation endowed with money is increased by the value of the money. The introduction of the asset money thus 'completes the markets', in the sense of Arrow (1953), by which we mean that the equilibrium of the sequential economy can be understood as if it were an economy in which money did not appear and all the markets cleared at the beginning of time (except, as we said, that the income of several agents is increased beyond the value of their endowments). The puzzle of how money can have positive value in the Samuelson model can thus be reinterpreted in the OLG model as follows. How is it possible that we can increase the purchasing power of one agent beyond the value of his endowment, without decreasing the purchasing power of any other agent below his, and yet continue to clear all the markets? Before giving a more formal treatment of the foregoing, let me re-emphasize an important point. It has often been said that the Samuelson consumption loan model can be understood from the point of view of incomplete markets. Adding money to the model does indeed complete the markets, in the precise sense of Arrow-Debreu, but the result is the OLG model in which the puzzles remain.

Consider now a truncated example $E_{0,\infty}$ in which time begins at date $t=0$, but continues to infinity. Once again there is a new agent 'born' at each date $t \geq 0$, whose utility depends only on the two goods dated during his lifetime, and whose endowment is positive only in those same commodities. At each date $t \geq 1$ there will be two agents alive, a young one and an old one. At date 0 there is only one agent. To this truncation of our earlier model we now add one extra commodity, which we call money. Money is a perfectly durable commodity that affects no agent's utility. Agents are endowed with money (M_t, M'_{t+1}) , in addition to their commodity endowments.

A price system is defined as a sequence

$$(\pi; p) = (\pi_0, \pi_1, \pi_2, \dots; p_0, p_1, p_2, \dots)$$

of money prices π , and commodity prices p , for each $t = 0$. The budget set for any agent t is defined by

$$\{(m_t, m_{t+1}, x_t, x_{t+1}) \geq 0 \mid \pi_t m_t + p_t x_t \leq \pi_t M'_t + p_t e'_t$$

and

$$\pi_{t+1} m_{t+1} + p_{t+1} x_{t+1} \leq \pi_{t+1} M'_{t+1} + p_{t+1} e'_{t+1} + \pi_{t+1} m_t\}.$$

The budget constraint expresses the principle that in the Samuelson model agents cannot borrow at all, and cannot save, i.e. purchase more when old than the value of their old endowment, except by holding over money m_t from when they were young. Let $m'_t(\pi, p)$ and $m'_{t+1}(\pi, p)$ be the utility maximizing choices of money holdings by generation t when young and when old. As before the excess commodity demand is defined by $Z'_t(\pi, p)$ and $Z'_{t+1}(\pi, p)$.

To keep things simple, we suppose that agent 0 is endowed with $M^0_0 = M$ units of money when he is young, but all other endowments M'_t are zero. Since money is perfectly durable, total money supply in every period is equal to M . Equilibrium is defined by a price sequence (π, p) such that $m^0_0(\pi, p) = M$ and $Z^0_0(\pi, p) = 0$, and for all

$$t \geq 1, \quad m'_t(\pi, p)$$

$$+ m'_t(\pi, p) = M \quad \text{and} \quad Z'_t(\pi, p) + Z'_{t+1}(\pi, p) = 0$$

At first glance this seems a much more complicated system than before.

But elementary arguments show that in equilibrium either $\pi_t = 0$ for all t , and there is no intergenerational trade of

commodities, or $\pi_t > 0$ for all t . In the latter case $m'_{t+1}(\pi, p) = 0$ for all t , hence money market clearing is reduced to

$$m'_t(\pi, p) = M \quad \text{for all } t \geq 0$$

And for the period by period Walras's Law, if the goods market clears at date t , so must the money market. So we never have to mention money market clearing. Moreover, by taking $q_t = (\pi_t p_{t+1}) / (\pi_{t+1} p_t)$ we can write the commodity excess demands for agents $t \geq 1$ just as in section 1, by

$$[Z'_t(q_t), Z'_{t+1}(q_t)]$$

and they are the same as

$$[Z'_t(\pi, p), Z'_{t+1}(\pi, p)].$$

The only agent who behaves differently is agent 0, whose budget set must now be written

$$D^0(\mu, q_0) = \{x | x_0 + q_0 x_1 \leq e_0^0 + q_0 e_1^0 + \mu M\},$$

where

$$\mu = \frac{\pi_0}{p_0}.$$

We can then write agent 0's excess demand for goods as

$$[Z_0^0(\mu, q_0), Z_1^0(\mu, q_0)].$$

A monetary equilibrium in the OLG economy E_0 , is any (μ, q) , $\mu \geq 0$, satisfying

$$Z_0^0(\mu, q_0) = 0, \quad Z_1^0(\mu, q_0) + Z_1^1(q_1) = 0,$$

and

$$Z_i^{t+1}(q_{t-1}) + Z_i^t(q_t) = 0 \quad \text{for all } t \geq 1.$$

4. UNDERSTANDING OLG ECONOMIES AS LACK OF MARKET CLEARING AT INFINITY. In this section we point out that the suboptimality of competitive equilibria, the indeterminacy of nonstationary equilibria, the non-existence of the core, and the positive valuation of money can all occur robustly in possibly nonstationary OLG economies with multiple consumers and $L > 1$ commodities per period. We also note the important principle that the potential dimension of indeterminacy is related to L . In the two-way infinity model, it is $2L - 1$. In the one-way infinity model without money it is $L - 1$; in the one-way infinity model with money the potential dimension of indeterminacy is L .

None of these properties can occur (robustly) in a finite consumer, finite horizon, Arrow-Debreu model. In what follows we shall suggest that a proper understanding of these phenomena lies in the fact that the OLG model is isomorphic, in a precise sense, to a 'finite' model in which not all the markets are required to clear.

One of the first explanations offered to account for the differences between the Arrow-Debreu model and the Samuelson model with money centred on the finite lifetimes of the agents and the multiple budget constraints each faced. These impediments to intergenerational trade (e.g. the fact that an agent who is 'old' at time t logically cannot trade with an agent who will not be 'born' until time $t+s$) were held responsible. Indeed in a finite horizon model under uncertainty (with incomplete asset markets), if agents are confronted with a series of budget constraints, it is possible to generate robustly many dimensions of Pareto-suboptimal equilibria. But there is no uncertainty in the Samuelson model. And as we saw in the last section, without uncertainty, the presence of a single asset like money is enough to connect all the markets. Formally, as we saw, the model is identical to what we called the OLG model in which we could imagine all trade taking

place simultaneously at the beginning of time, with each agent facing a single budget constraint involving all the commodities. What prevents trade between the old and the unborn is not any defect in the market, but a lack of compatible desires and resources

Another common explanation for the surprising properties of the OLG model centres on the 'paradoxes' of infinity. In finite models, one proves the generic local uniqueness of equilibrium by counting the number of unknown prices, less one for homogeneity, and the number of market clearing conditions, less one for Walras's Law, and notes that they are equal. In the OLG model there are an infinity of prices and markets, and who is to say that one infinity is greater than another? We already saw that the backward induction argument against money fails in an infinite horizon setting, where there is no last period. Finally, it was pointed out that in many of the OLG equilibria (with unnormalized prices p) the value $\sum_t p_t (e_t^{-1} + e_t) = \sum_t p \cdot e^t$ of the social endowment is infinite. No consumer is able to afford any fraction of the social endowment, no matter how small. This suggested to some a radically different kind of budget constraint from that in the Arrow-Debreu model. Surely it is right that infinity is at the heart of the problem. But this explanation does not go far enough. In the model considered by Bewley (1972) there are an infinite number of time periods (but a finite number of consumers). In that model all equilibria are Pareto optimal, and money never has value, even though there is no last time period. And by considering a sophisticated extension of the real numbers to infinite and infinitesimal magnitudes, it is possible for consumers to imagine purchasing a positive fraction (albeit infinitesimal) of the social endowment. The problem of infinity shows that there may be a difference between the Arrow-Debreu model and the OLG model. In itself, however, it does not predict the qualitative features (like the potential dimension of indeterminacy) that characterize OLG equilibria.

Consider now a general OLG model with many consumers and commodities per period. We index utilities u^{th} by the time of birth t , and the household $h \in H$, a finite set. Household (t, h) owns initial resources e_t^{th} when young, an L -dimensional vector, and resources e_{t+1}^{th} when old, also an L -dimensional vector, and nothing else. As before utility u^{th} depends only on commodities dated either at time t or $t+1$. Given prices

$$q_t = (q_{ta}, q_{tb}) \in R_{++}^{2L},$$

consisting of all the $2L$ prices at date t and $t+1$, each household in generation t has enough information to calculate the relevant part of its budget set. Hence we can write household excess demand $[Z_t^{th}(q_t), Z_{t+1}^{th}(q_t)]$ and the aggregate excess demand of generation t as $[Z_t^t(q_t), Z_{t+1}^t(q_t)]$, where

$$Z_{t+s}^t(q_t) = \sum_{h \in H} Z_{t+s}^{th}(q_t), \quad s = 0, 1$$

Suppose that time goes from $-\infty$ to ∞ . We can write the market clearing condition for equilibrium exactly as we did in the one commodity, one consumer case, as

$$(A) \quad Z_t^{t-1}(q_{t-1}) + Z_t^t(q_t) = 0 \quad t \in \mathbb{Z}.$$

Of course we need to put restrictions on the q_t to ensure their compatibility, since q_{tb} and $q_{t+1,a}$ refer to the same period $t+1$ prices. But this is easily done.

Similarly we can define the one way infinity economy $E_{0,\infty}$, in which time begins in period 0. We retain the same market clearing conditions for $t \geq 1$, changing only the $t=0$ condition

$$(A_+) \quad Z_0^{0h}(q_0) = 0 \quad \text{and} \quad Z_t^{t-1}(q_{t-1}) + Z_t^t(q_t) = 0, \quad t \geq 1$$

Finally, let us define equilibrium in a one-way infinity model

with money, E_0^M , when agents $(0, h)$ are endowed with money M^h , in addition to their commodities, by (μ, q) , $\mu \geq 0$, satisfying

$$(A_+^M) \sum_{h \in H} Z_0^h(q_0, \mu M^h) = 0 \quad \text{and} \\ \sum_{h \in H} Z_1^h(q_0, \mu M^h) + Z_1^i(q_1) = 0,$$

and

$$Z_t^{-1}(q_{t-1}) + Z_t^i(q_t) = 0, \quad \text{for } t \geq 2.$$

With these preliminary definitions out of the way, we are ready to proceed with 'lack of market clearing at infinity'. For concreteness we shall concentrate first on $E_{0,\infty}$. Consider the truncated economy $E_{0,T}$ consisting of all the agents born between periods 0 and T . Market clearing in $E_{0,T}$ is identical to that in $E_{0,\infty}$ for $t = 0$ to $t = T$. But at $t = T + 1$, we require $Z_{T+1}^i(q_T) = 0$ in $E_{0,T}$. We have already seen in section 1 what a great deal of difference this can make: the economies $E_{0,T}$ and $E_{0,\infty}$ may be very different, for all finite T . The interesting point is that by appealing to nonstandard analysis, which makes rigorous the mathematics of infinite and infinitesimal numbers, one can easily show that the behaviour of $E_{0,T}$, for T an infinite number is very similar to the behaviour of $E_{0,T}$ for T finite. Thus the properties of the economy $E_{0,\infty}$ do not stem from infinity alone. We shall need to modify $E_{0,T}$ before it corresponds to $E_{0,\infty}$. Nevertheless, the economies $E_{0,T}$ do provide some information about $E_{0,\infty}$.

Theorem: (Cass-Balasko-Shell and Wilson): Under mild conditions, at least one equilibrium for $E_{0,\infty}$ always exists.

To see why this so, note that $E_{0,T}$ is well-defined for any finite T . From nonstandard analysis we know that the sequence $E_{0,T}$, for $T \in \mathcal{F}$ and the original economy $E_{0,\infty}$, has a unique extension to the infinite integers. Now fix T at an infinite integer. We know that $E_{0,T}$ has at least one equilibrium, since $E_{0,s}$ does for all finite s . But if T is infinite, $E_{0,T}$ includes all the finite markets $t = 1, 2, \dots$, so all those must clear at an equilibrium q^* of $E_{0,T}$. Taking the standard parts of the prices q_t^* for the finite t (and ignoring the infinite t) gives an equilibrium q for $E_{0,\infty}$.

To properly appreciate the force of this proof, we shall consider it again, when it might fail, in section 5, where we deal with infinite lived consumers.

In terms of the existence of equilibrium, $E_{0,\infty}$ (and similarly $E_{0,\infty}^M$ and E) behaves no differently from an Arrow-Debreu economy. But the indeterminacy is a different story.

Definition. A classical equilibrium for the economy $E_{0,T}$ is a price sequence $q^* = (q_0, q_1, \dots, q_T)$ that clears the markets for $0 \leq t \leq T$, but at $t = T + 1$, market clearing is replaced by

$$Z_{T+1}^i(q_T) \leq \sum_{h \in H} e_{T+1}^{T+1,h}.$$

Thus in a classical equilibrium there is lack of market clearing at the last period. The aggregate excess demand in that period, however, must be less than the endowment the young of period $T + 1$ would have had, were they part of the economy. Economies in which market clearing is not required in every market are well understood in economic theory. Note that in a classical equilibrium the agents born at time T are not rationed at $T + 1$, their full Walrasian (notional) demands are met, out of the dispossessed endowment of the young. But we do not worry about how this gift from the $T + 1$ young is obtained. The significance of our classical equilibrium for the OLG models can be summarized:

Theorem (Geanakoplos-Brown) Fix T at an infinite integer. The equilibria q for $E_{0,\infty}$ correspond exactly to the standard parts of classical equilibria q^* of $E_{0,T}$.

The Walrasian equilibria of the economy $E_{0,\infty}$, which apparently is built on the usual foundations of agent optimization and market clearing, correspond to the 'classical equilibria' of another finite-like economy $E_{0,T}$ in which the markets at $T + 1$ ('at infinity') need not clear. The existence of equilibrium in $E_{0,\infty}$ is not a problem, because marked clearing is a special case of possible non-market clearing, and $E_{0,T}$, being finite-like, always has market clearing equilibria. Thus even though the number of prices and the number of markets in $E_{0,\infty}$ are both infinite, we see that it is possible to say which is bigger, and by how much. From Walras Law we know that if all the markets but one clear, that must clear as well. Hence having L markets that do not clear provides for $L - 1$ potential dimensions of indeterminacy.

Corollary (Geanakoplos-Brown): For a generic economy $E_{0,\infty}$, there are at most $L - 1$ dimensions of indeterminacy in the equilibrium set.

It is by no means true that there must be $L - 1$ dimensions of indeterminacy in a generic economy. If we consider a classical equilibrium q^* for $E_{0,T}$, then generically we will be able to arbitrarily fix $L - 1$ prices anywhere near their q^* values, and then choose the rest of the prices to clear all the markets up through time T . But which $L - 1$ prices there are depends on which square submatrix N (of derivatives of excess demands with respect to prices) is invertible. For example, call the economy $E_{0,\infty}$ intertemporally separable if each generation t consists of a single agent whose utility for consumption at date t is separable from his utility for consumption at date $t + 1$. Then the $L - 1$ free parameters must all be chosen at date $T + 1$ (as part of q_{T+1}), i.e. way off at infinity.

Corollary (Geanakoplos-Brown-Polemarchakis): Intertemporally separable economies $E_{0,\infty}$ generically have locally unique (in the product topology) equilibria

Even when the $L - 1$ degrees of freedom may be chosen at time $t = 0$, there still may be no indeterminacy, if the matrix N has an inverse (in the nonstandard sense) with infinite norm. But when the free $L - 1$ parameters may be chosen at $t = 0$ and also the matrix N has an inverse with finite norm, then all nearby economies must also display $L - 1$ dimensions of indeterminacy.

Theorem (Kehoe-Levine and Geanakoplos-Brown). In the $E_{0,\infty}$ OLG model there are robust examples of economies with $L - 1$ dimensions of indeterminacy. In the monetary economy, $E_{0,\infty}^M$, there are robust examples of economies with L dimensions of indeterminacy.

In the monetary case, one can imagine L parallel and independent one good monetary economies of the kind studied in section 1. This is a knife-edge example of an L -commodity OLG monetary economy with L dimensions of indeterminacy, parameterized at $t = 0$. A rather simple modification of this example yields another in which the N matrix has an inverse with a finite norm. This latter example is therefore robust, which implies that there are other nearby economies with L dimensions of indeterminacy in which no utilities are separable between commodities. A similar approach works in $E_{0,\infty}$, but with a dimension less. Let us now turn our attention to the question of Pareto optimality

Definition: An allocation $\bar{x} = (x^{t,h}, 0 \leq t \leq T)$ is classically feasible for the economy $E_{0,T}$ if $\sum_{(t,h) \in A} x^{t,h} \leq \sum_{(t,h) \in A} e_x^{t,h}$, for $0 \leq s \leq T + 1$. The classically feasible allocation \bar{x} for $E_{0,T}$ is a classic Pareto optimum if there is no other classically feasible allocation \bar{y} for $E_{0,T}$ with $u^t(y^{t,h}) > u^t(x^{t,h})$ for all $(t, h) \in A$ with

$0 \leq t \leq T$, with at least one inequality representing a noninfinitesimal difference.

Theorem (Geanakoplos–Brown): The Pareto-optimal allocations \bar{x} for the OLG economy $E_{0,\infty}$ are precisely the standard parts of classical Pareto-optimal allocations \bar{x}^* for $E_{0,T}$, if T is fixed at an infinite integer.

The upshot of this theorem is that the effective social endowment includes the commodities e_{T+1}^h of the generation born at time $s = T + 1$, even though they are not part of the economy $E_{0,T}$. Since the socially available resources exceed the aggregate of private endowments, it is no longer a surprise that a Walrasian equilibrium, in which the value of aggregate spending every period must equal the value of aggregate private endowments, is not Pareto optimal.

On the other hand, this does not mean that all equilibrium are Pareto suboptimal. If the equilibrium prices $p_t \rightarrow 0$ as $t \rightarrow \infty$ (equivalently, if p_{T+1} is infinitesimal) then the value of the extra social endowment is infinitesimal, and there are no possible noninfinitesimal improvements. Let (p, \bar{x}) be an equilibrium for the OLG economy $E_{0,\infty}$. Consider the concave-convex programming problem of maximizing the utility of agent $(0, h)$, holding all other utilities of agents (t, h) with $0 \leq t \leq T$ at the levels $u^t(x^t)$ they get with \bar{x} , over all possible classically feasible allocations in $E_{0,T}$ that do not use more resources, even at time $T + 1$, than \bar{x} . Clearly \bar{x} itself is a solution to this problem. But now let us imagine raising the constraints at time $T + 1$ from

$$\sum_{h \in H} x_{T+1}^{T,h} \text{ to } \sum_{h \in H} (e_{T+1}^h + e_{T+1}^{T+1,h}).$$

What is the rate of change of the utility $u^{0,h}$? For the first infinitesimal additions to period $T + 1$ resources, the rate of change of $u^{0,h}$ is on the order of p_{T+1} . But as the increases get larger, this rate of change could drop quickly, as higher derivatives come into play (assuming that agents have strictly concave utilities). One can easily calculate that the second derivative $\partial^2 u^{0,h} / \partial e_{T+1}^2$, and all higher derivatives, depend on $\sum_{t=0}^T |1/p_t|$. Arguing this way, or even more directly, one can show:

Theorem (Cass, 1972; Benveniste–Gale, 1975; Balasko–Shell, 1980; Okuno–Zilcha, 1981): If agents have uniformly strictly concave utilities, then the equilibrium (p, \bar{x}) for an OLG economy $E_{0,\infty}$ is Pareto optimal if and only if $\sum_{t=0}^{\infty} |1/p_t| = \infty$.

Note that in this theorem it is the unnormalized prices that play the crucial role. It follows immediately from this theorem that the golden rule equilibrium $q = (. . . 1, 1, 1, . . .)$ for the simple one good, stationary economy of section 1 is Pareto optimal, since the corresponding unnormalized price sequence is also $(. . . 1, 1, 1, . . .)$. In fact, a moment's reflection shows that any periodic, nonautarkic equilibrium must also be periodic in the unnormalized prices p . Hence, as we have said, the cyclical equilibria of section 2 are all Pareto optimal.

Having explained the indeterminacy and Pareto suboptimality of equilibria for $E_{0,\infty}$ in terms of lack of market clearing at infinity, let us reexamine the monetary equilibria of OLG economies $E_{0,\infty}^M$, where $M = (M^h; h \in H)$ is the stock of aggregate money holdings by the agents $(0, h)$ at time 0.

Definition Let $z \in R^L$ be a vector of commodities for time $T + 1$. Suppose that $-\sum_{h \in H} e_{T+1}^{T,h} \leq \sum_{h \in H} M^h z \leq \sum_{h \in H} e_{T+1}^{T+1,h}$. Let the augmented economy $E_{0,T}^M(z)$ be identical to the economy $E_{0,T}$, except that the endowment of each agent $(0, h)$ is augmented by $M^h z$ units of commodities at time $T + 1$.

Theorem (Geanakoplos–Brown) The equilibria q of the monetary economy $E_{0,\infty}^M$ are precisely obtained by taking standard

parts of full market clearing equilibria q^* of the augmented economies $E_{0,T}^M(z)$ for which $q_{T,b}^* z \geq 0$, if T is fixed at an infinite integer.

The above theorem gives another view of why there are potentially L dimensions of monetary equilibria. It also explains how money can have positive value: it corresponds to the holding of extra physical commodities. In fact, this explains how the 'social contrivance of money' can lead to Pareto-improving equilibria, even in OLG economies where there is already perfect financial intermediation. The holding of money can effectively bring more commodities into the aggregate private endowment. The manifestation of the 'real money balances' is the physical commodity bundle z at date $T + 1$. Money plays more than just an intermediation role.

Before concluding this section let us consider a simple generalization. Suppose that agents live for three periods. What plays the analogous role to $E_{0,T}$? The answer is that prices need to be specified through time $T + 2$, but markets are only required to clear through time T . There are therefore $2L - 1$ potential dimensions of indeterminacy, even in the one-sided economy. In general, we must specify the price vector up until some time s , and then require market clearing only in those commodities whose excess demands are fully determined by those prices.

This reasoning has an important generalization to production. Suppose that capital invested at time t can combine with labour at time $t + 1$ to produce output at time $t + 1$, and suppose that all agents live two periods. Is there any difference between the case where labour is inelastically supplied, and the case where leisure enters the utility? In both cases the number of commodities is the same, but in the latter case the potential dimension of indeterminacy is one higher, since the supply of labour at any time might depend on further prices.

5 IMPATIENCE AND UNIFORM IMPATIENCE. We have already suggested that it is useful in understanding the OLG model to consider variations, for example in which consumers live forever. By doing so we shall also gain an important perspective on what view of consumers is needed to restore the usual properties of neoclassical equilibrium to an infinite horizon setting, a subject to which we return in section 8.

Let us now allow for consumers $t \in A$ who have endowments e^t that may be positive in all time periods, and also for arbitrary utilities u^t defined on uniformly bounded vectors $x \in L$. For ease of notation we assume one consumer per period. A minimal assumption we need about utilities u^t is continuity on finite segments i.e. fixing x_s for all $s > n$, $u^t(x)$ should be continuous in (x_1, \dots, x_n) . We also assume $\sum_{t \in A} e^t$ is uniformly bounded. In short, consumers may live forever. We shall find that in order to have Walrasian equilibria, the consumers must be impatient. In order for the equilibria to resemble Arrow–Debreu equilibria, the consumers need to be uniformly impatient.

Suppose we try to form the truncated economy $E_{0,T}$ as before, say for T finite. Since utility potentially depends on every commodity, we could not define excess demands in $E_{0,T}$ unless we knew all the prices. To make it into a finite economy, let us call $E'_{0,T}$ the version of $E_{0,T}$ in which every agent $0 \leq t \leq T$ is obliged to consume his initial endowment during periods $t > T$. Clearly $E'_{0,T}$ has an equilibrium. For this to give information about the original economy $E_{0,\infty}$, we need that consumers do not care very much about what happens to them after T , as T gets very far away. This is the notion of impatience.

For any vector x , let ${}_n x$ be the vector which is zero for $t > n$, and equal to x up until n . Thus ${}_n x$ is the initial n -segment of x . To say that agent $t \in A$ is impatient means that for any two

uniformly bounded consumption streams x and y , if $u'(x) > u'(y)$, then for all big enough n , $u'(\epsilon_n \hat{x}) > u'(y)$. Let us suppose that all consumers are impatient.

Note that the OLG agents are all impatient, since none of them cares about consumption after he dies.

The truncation argument, applied at an infinite T , still does not guarantee the existence of an equilibrium. For once we take standard parts, ignoring the infinitely dated commodities, it may turn out that the income from the sale of an agent's endowed commodities at infinite t , which he used to finance his purchase of commodities at finite t , is lost to the agent. It must also be guaranteed that the equilibria of $E_{0,T}$ give infinitesimal total value to the infinitely dated commodities. Wilson (1981) has given an example of an economy, composed entirely of impatient agents, that does not have an equilibrium precisely for this reason.

On the other hand, if there are only finitely many agents, even if they are infinitely lived, then we have:

Theorem (Bewley, 1972): Let the economy E be composed of finitely many, impatient consumers. Then there exists an equilibrium, and all equilibria are Pareto optimal.

Thus in the Bewley model there is no end to time, and no diminution in the physical size (endowment) of the economy over time. Yet unlike OLG models, equilibria are always Pareto optimal. What is the essential difference between the models?

Notice that in both the OLG model and Bewley's model, all consumers are impatient. The crucial difference is that in the OLG model, the economy is not impatient, for any date $t=n$, there is somebody who cares more about consumption at date n than at date $t=0$. In order to discuss the notion of uniform impatience, let us introduce one more notation. Let l be the bundle of goods which consists of one unit of every commodity, at all dates. If a consumer has monotonic preferences, then there is some k such that strictly more of all goods up until date k makes him better off. Thus we can re-express impatience for agent t by saying that for all uniformly bounded x , and all $\epsilon > 0$, there exists n such that

$$u'(\epsilon \cdot \hat{l} + \pi \hat{x}) > u'(x).$$

Definition: The agents $t \in A$ are said to be uniformly impatient if there is a fixed finite integer k with the following property. Let $(x^t; t \in A)$ be an allocation such that

$$x = \sum_{t \in A} x^t$$

is uniformly bounded. Let $\epsilon > 0$ be fixed. Then there are $\epsilon' > 0$ and n satisfying

$$\sum_{t \in A} \epsilon' = \epsilon$$

and

$$u'(\epsilon' \cdot \hat{l} + \pi \hat{x}) > u'(x^t) \quad \text{for all } t \in A.$$

Of course any finite set of impatient consumers is uniformly impatient.

Theorem (Geanakoplos–Brown, 1982, 1985, 1986) Let the economy E be composed of agents $t \in A$ that are uniformly impatient. Then an equilibrium exists, all equilibria are Pareto optimal and in the core, the core equivalence holds for replications of the economy, and money never has positive value.

This theorem establishes a criterion for an economy, composed possibly of infinitely many goods and consumers, to behave like an Arrow–Debreu economy. The story would be

complete if one could show, analogously to section 4, that there is an economy $E_{0,T}$ for some infinite T whose equilibria (as opposed to classical equilibria) corresponded, by taking standard parts, precisely to the equilibria of E . One could then derive information about the number of equilibria in E . Unfortunately this remains an open question.

Consider, however, the special case when there are finitely many agents, $h = 1, \dots, H$, with separable, discounted utilities of the form $u^h(x) = \sum_{t=0}^{\infty} \delta_t^h v^h(x_t)$, with $\delta_h < 1$. This is clearly an economy with uniformly impatient consumers. And here there is a result:

Theorem (Kehoe–Levine, 1985). In finite agent, separable, discounted utility economies, there are generically a finite number of equilibria.

It is extremely interesting to investigate the change in behaviour of an economy that evolves from individually impatient to uniformly impatient. Muller and Woodford (1983) consider an example with one infinitely lived agent, and infinitely many, overlapping, finite-lived agents. Wilson (1981) guarantees the existence of at least one equilibrium. They show that when the single agent's proportion of the aggregate endowment is low enough, there are a continuum of equilibria, but if it is high enough, there is no local indeterminacy.

6. COMPARATIVE STATICS FOR OLG ECONOMIES. A celebrated theorem of Debreu asserts that almost any Arrow–Debreu economy is regular, in the sense that it has a finite number of equilibria, each of which is locally unique. Small changes to the underlying structure of the economy (tastes, endowments, etc.) produce small, unique changes in each of the equilibria.

We have already seen that there are robust OLG economies with a continuum of equilibria. If attention is focused on one of them, how can one predict to which of the continuum of new equilibria the economy will move if there is a small change in the underlying structure of the economy, perhaps caused by deliberate government intervention? In what sense is any one of the new equilibria near the original one? In short, is comparative statics possible?

It is helpful at this point to recall that the OLG model is, in spirit, meant to represent a dynamic economy. Trade may occur as if all the markets cleared simultaneously at the beginning of time, but the economy is equally well described as if trade took place sequentially, under perfect foresight or rational expectations. Indeed this is surely what Samuelson envisaged when he introduced money as an asset into his model. Accordingly, when a change occurs in the underlying structure of the economy, we can interpret it as if it came announced at the beginning of time, or as if it appeared at the date in which it actually affects the economy.

We distinguish two kinds of changes to the underlying structure of an economy $E_{-\infty, \infty}$ starting from an equilibrium \bar{q} . Perfectly anticipated changes, after which we would look for a new equilibrium that cleared all the markets from the beginning of time, represent one polar case, directly analogous to the comparative statics experiments of the Arrow–Debreu economy. At the other extreme we consider perfectly unanticipated changes, say at date $t=1$. Beginning at the original economy and equilibrium $\bar{q} = (\dots, \bar{q}_{-1}, \bar{q}_0, \bar{q}_1, \dots)$, we would look, after the change from $E_{-\infty, \infty}$ to $E_{-\infty, \infty}$, at time $t=1$ (say to the endowment or preferences of the generation born at time 1), for a price sequence $q = (\dots, q_{-1}, q_0, q_1, \dots)$ in which $q_t = \bar{q}_t$ for $t \leq 0$, and $Z_1^{-1}(q_{t-1}) + Z_1^t(q_t) = 0$ for $t \geq 2$. But at date $t=1$ we would require q_1 to satisfy $Z_1^0(q_1 | \bar{q}_0) + Z_1^1(q_1) = 0$, where $Z_1^0(q_1 | \bar{q}_0)$ represents the excess demand of the old at time 1, given that when they were young they purchased commodities

on the strength of the conviction that they could surely anticipate prices $\bar{q}_{0,b}$ when they got old, only to discover prices q_{1a} instead.

To study these two kinds of comparative statics, we must describe what we mean by saying that two price sequences are nearby. Our definition is based on the view that a change at time $t = 1$ ought to have a progressively smaller impact the further away in time from $t = 1$ we move. We say that q is near \bar{q} if the difference $|q_t - \bar{q}_t|$ declines geometrically to zero, both as $t \rightarrow \infty$ and as $t \rightarrow -\infty$.

We have already noted in section 1 that the multiplicity of OLG equilibria is due to the fact that at any time t the aggregate behaviour of the young generation is influenced by their expectations of future prices, which (under the rational expectations hypothesis) depend on the next generation's expectations etc. Accordingly we restrict our attention to generations whose aggregate behaviour Z' satisfies the expectations sensitivity hypothesis:

$$\text{rank} \frac{\partial Z'_t(p_t, p_{t+1})}{\partial p_{t+1}} = \text{rank} \frac{\partial Z'_{t+1}(p_t, p_{t+1})}{\partial p_t} = L.$$

For economies composed of such generations we can apply the implicit function theorem, exactly as in section 1, around any equilibrium q to deduce the existence of the forward and backward functions F_t and B_t . We write their derivatives at \bar{q} as D_t and D_t^{-1} , respectively.

For finite Arrow-Debreu economies, Debreu gave a definition of regular equilibrium based on the derivative of excess demand at the equilibrium. He showed that comparative statics is sensible at a regular equilibrium, and then he showed that a 'generic' economy has regular equilibria. We follow the same programme.

We say that the equilibrium \bar{q} for the expectations sensitive OLG economy E is Lyapunov regular if the long-run geometric mean of the products $D_t^* D_{t-1}^* \dots D_1^* D_1$ and $D_{-1}^{-1} * D_{-2}^{-1} \dots D_{-L}^{-1} * D_{-L-1}^{-1}$ converge and if to these products we can associate at most $2L - 1$ numbers, called Lyapunov exponents. The equilibrium is also nondegenerate if in addition none of these Lyapunov exponents is equal to one.

Theorem (Geanakoplos-Brown, 1985): Let $E = E_{-\infty, \infty}$ be an expectations-sensitive economy with a regular, nondegenerate equilibrium \bar{q} . Then for all sufficiently small perturbations E of \bar{E} (including \bar{E} itself), E has a unique equilibrium q near \bar{q} .

Thus the comparative statics of perfectly anticipated changes in the structure of E , around regular, nondegenerate equilibria, is directly analogous to the Arrow-Debreu model. Note incidentally that one implication of the above theorem is that neutral policy changes, i.e. those for which \bar{q} itself remains an equilibrium cannot have any effect if they are perfectly anticipated. (A structural change is always taken to mean with respect to real magnitudes, e.g. if the government buys a certain number of capital goods.) The situation is quite different for unanticipated changes.

Theorem (Geanakoplos-Brown, 1985): Let E be an expectations-sensitive economy with a regular equilibrium \bar{q} . Then for all sufficiently small perfectly unanticipated perturbations E of \bar{E} (including \bar{E} itself), the set of unanticipated equilibria q of E near \bar{q} is either empty, or a manifold of dimension r , $0 \leq r \leq L(L - 1)$ if there is no money in the economy), independent of the perturbation.

The above theorem allows for the possibility that an unanticipated change may force the economy onto a path that diverges from the original equilibrium; the disturbance could be propagated and magnified through time. And if there are nearby equilibria, then there may be many of them. (Indeed

that is basically what was shown in section 4.) In particular, an unanticipated neutral policy change could be compatible with a continuum of different equilibrium continuations. The content of the theorem is that if there is a multiplicity of equilibrium continuations, it is parameterizable. In other words, the same r variables can be held fixed, and for any sufficiently small perturbation, there is exactly one nearby equilibrium which also leaves these r variables fixed. We shall discuss the significance of this in the next section.

This last theorem was proved first, in the special case of steady-state economies, by Kehoe-Levine, in the same excellent paper to which we have referred already several times. The theorem quoted here, together with the previous theorem on the comparative statics of perfectly anticipated policy changes, refers to economies in which the generations may be heterogeneous across time.

Let us suppose that A is a compact collection of generational characteristics, all of which obey the expectations-sensitive hypothesis. Let us suppose that each generation's characteristics are drawn at random from A , according to some Borel probability measure. If the choices are made independently across time, then the product measure describes the selection of economies. Almost any such collection will have a complex demographic structure, changing over time. The equilibrium set is then endogenously determined, and will be correspondingly complicated. It can be shown, however, that

Theorem (Geanakoplos-Brown, 1985): If the economy E is randomly selected, as described above, then with probability one, E has at least one Lyapunov regular equilibrium.

Note that the regularity theory for infinite economies stops short of Arrow-Debreu regularity. In the finite economies, with probability one all the equilibria are regular.

7. KEYNESIAN MACROECONOMICS. Keynesian macroeconomics is based in part on the fundamental idea that changes in expectations, or animal spirits, can affect equilibrium economic activity, including the level of output and employment. It asserts, moreover, that publicly announced government policy also has predictable and significant consequences for economic activity, and that therefore the government should intervene actively in the marketplace if investor optimism is not sufficient to maintain full employment.

The Keynesian view of the indeterminacy of equilibrium and the efficacy of public policy has met a long and steady resistance, culminating in the sharpest attack of all, from the so-called new classicals, who have argued that the time-honoured microeconomic methodological premises of agent optimization and market clearing, considered together with rational expectations, are logically inconsistent with animal spirits and the non-neutrality of public monetary and bond financed fiscal policy.

The foundation of the new classical paradigm is the Walrasian equilibrium model of Arrow-Debreu, in which it is typically possible to prove that all equilibria are Pareto optimal and that the equilibrium set is finite; at least locally, the hypothesis of market clearing fixes the expectations of rational investors. In that model, however, economic activity has a definite beginning and end. Our point of view is that for some purposes economic activity is better described as a process without end. In a world without a definite end, there is the possibility that what happens today is underdetermined, because it depends on what people tomorrow expect to happen the day after tomorrow, etc.

Consider the simple one good per period overlapping generations economy with money E_0^M , which we discussed in section 1. Generation 0 is endowed with money when young, and equilibrium can be described either with the unnormalized prices $p = (\bar{p}_0, \bar{p}_1, \dots)$, or with the normalized prices $(\bar{y}, \bar{q}) = (\bar{y}, \bar{q}_0, \bar{q}_1, \dots)$ where $\bar{q}_t = \bar{p}_{t+1}/\bar{p}_t$, and $\bar{y} = M/\bar{p}_0$. It is helpful to reinterpret the model as a simple production economy. Imagine that the endowment e_t^l in the first period of life is actually labour, which can be transformed into output, y_t , according to the production function, $y_t = e_t^l$. We would then think of any purchases of goods by the old generation as demand for real output to be produced by the young. The young in turn now derive utility from leisure in their youth and consumption in their old age. Notice that the quantity equation $\bar{p}_t y_t = M$, holds for this economy. (Velocity equals one.)

The indeterminacy of rational expectations equilibrium has the direct implication that optimistic expectations by themselves can cause the economy's output to expand or contract. In short the economy has an inherent volatility. The Keynesian story of animal spirits causing economic growth or decline can be told without invoking irrationality or non-market-clearing.

In fact, the indeterminacy of equilibrium expectations is especially striking when seen as a response to public (but unanticipated) policy changes. Suppose the economy is in a long-term rational expectations equilibrium \bar{p} , when at time 1 the government undertakes some expenditures, financed either by lump sum taxation on the young or by printing money. How should rational agents respond? The environment has been changed, and there is no reason for them to anticipate that $(\bar{p}_2, \bar{p}_3, \dots)$ will still occur in the future. Indeed, in models with more than one commodity (such as we will shortly consider) there may be no equilibrium (p_1, p_2, p_3, \dots) in the new environment with $p_2 = \bar{p}_2, p_3 = \bar{p}_3$, etc. There is an ambiguity in what can be rationally anticipated.

We argue that it is possible to explain the differences between Keynesian and monetarist policy predictions by the assumptions each makes about expectational responses to policy, and not by the one's supposed adherence to optimization, market clearing, and rational expectations, and the other's supposed denial of all three.

Consider now the government policy of printing a small amount of money, ΔM , to be spent on its own consumption of real output – or equivalently to be given to generation $t = 0$ (when old) to spend on its consumption. Imagine that agents are convinced that this policy is not inflationary, i.e. that \bar{p}_1 will remain the equilibrium price level during the initial period of the new equilibrium. This will give generation $t = 0$ consumption level $(M + \Delta M)/\bar{p}_1$. As long as ΔM is sufficiently small and the initial equilibrium was one of the Pareto-suboptimal equilibria described in section 1, there is indeed a new equilibrium price path p beginning with $p_1 = \bar{p}_1$. Output has risen by $\Delta M/\bar{p}_1$, and in fact this policy is Pareto improving. On the other hand, imagine that agents are convinced that the path of real interest rates

$$(q_1^{-1} - 1, q_2^{-1} - 1, \dots) = \left(\frac{p_1}{p_2} - 1, \frac{p_2}{p_3} - 1, \dots \right)$$

will remain unchanged. In this economy, price expectations are a function of p_1 . Recalling the initial period market-clearing equation, it is clear that prices rise proportionally to the growth in the money stock. The result is 'forced savings', output is unchanged and generation $t = 1$ must pay for the government's consumption. If the government's consumption gives no agent utility, the policy is Pareto worsening.

This model is only a crude approximation of the differences between Keynesian and monetarist assumptions about expectations and policy. It is quite possible to argue, for example, that holding $q_1 = \bar{p}_2/\bar{p}_1$ (the inflation rate) fixed is the natural Keynesian assumption to make. This ambiguity is unavoidable when there is only one asset into which the young can place their savings. We are thereby prevented from distinguishing between the inflation rate and the interest rate. Our model must be enriched before we can perform satisfactory policy analysis. Nevertheless the model conveys the general principle that expected price paths are not locally unique. There is consequently no natural assumption to make about how expectations are affected by policy. A sensible analysis is therefore impossible without externally given hypotheses about expectations. These can be Keynesian, monetarist, or perhaps some combination of the two.

Geanakoplos–Polemarchakis (1986) builds just such a richer model of macroeconomic equilibrium by adding commodities, including a capital good, and a neoclassical production function. With elastically supplied labour, there are two dimensions of indeterminacy. It is therefore possible to fix both the nominal wage, and the firm's expectations ('animal spirits'), and still solve for equilibrium as a function of policy perturbations to the economy. These institutional rigidities are more convincingly Keynesian, and they lead to Keynesian policy predictions. Moreover, taking advantage of the simplicity of the two-period lived agents, the analysis can be conducted entirely through the standard Keynesian (Hicksian) IS–LM diagram.

Keynesians themselves often postulate that the labour market does not clear. In Keynesian models that has at least a threefold significance, which it is perhaps important to sort out. Since labour is usually taken to be inelastically supplied, it makes it possible to conceive of (Keynesian) equilibria with different levels of output and employment. It makes the system of demand and supply underdetermined, so that endogenous variables like animal spirits (i.e. expectations) which are normally fixed by the equilibrium conditions can be volatile. It creates unemployment that is involuntary. By replacing lack of labour market clearing at time 1 with elastic labour supply and lack of market clearing 'at infinity' one no longer needs to rely on what has seemed to many an *ad hoc* assumption in order to get at least the first two *desiderata* of Keynesian analysis.

8. NEOCLASSICAL EQUILIBRIUM VS CLASSICAL EQUILIBRIUM. The Arrow–Debreu model of general equilibrium, based on agent optimization, rational expectations, and market clearing, is universally regarded as the central paradigm of the neoclassical approach to economic theory. In the Arrow–Debreu model, consumers and producers, acting on the basis of individual self-interest, combine, through the aggregate market forces of demand and supply, to determine (at least locally) the equilibrium distribution of income, relative prices, and the rate of growth of capital stocks (when there are durable goods). The resulting allocations are always Pareto optimal.

Classical economists at one time or another have rejected all of the methodological principles of the Arrow–Debreu model. They replace individual interest with class interest, ignore (marginal) utility, especially for waiting, doubt the existence of marginal product, and question whether the labour market clears. But by far the most important difference between the two schools of thought is the classical emphasis on the long-run reproduction of the means of production, in a never-ending cycle.

Thus the celebrated classical economist Sraffa writes in Appendix D to his book.

It is of course in Quesnay's *Tableau Economique* that is found the original picture of the system of production and consumption as a circular process, and it stands in striking contrast to the view presented by modern theory, of a one-way avenue that leads from 'Factors of Production' to 'Consumption Goods'.

The title of his book, *Production of Commodities by Means of Commodities*, itself suggests a world that has no definite beginning, and what is circular can have no end.

In the Arrow-Debreu model time has a definite end. As we have seen, that has strong implications. With universal agreement about when the world will end, there can be no reproduction of the capital stock. In equilibrium it will be run down to zero. Money, for example, can never have positive value. Rational expectations will fix, at each moment, and for each kind of investment, the expected rate of profit.

In the classical system, by contrast, the market does not determine the distribution of income Sraffa writes

The rate of profits, as a ratio, has a significance which is independent of any prices, and can well be 'given' before the prices are fixed. It is accordingly susceptible of being determined from outside the system of production, in particular by the money rates of interest. In the following sections the rate of profits will therefore be treated as the independent variable (Sraffa (1960, p. 33)).

Other classical writers concentrate instead on the real wage as determined outside the market forces of supply and demand, for example by the level of subsistence, or the struggle between capital and labour. Indeterminacy of equilibrium seems at least as central to classical economists as it is to Keynesians.

Like Keynesians, classicals often achieve indeterminacy in their formal models by allowing certain markets not to clear in the Walrasian sense. (Again like Keynesians, the labour market is usually among them.) Thus we have called the equilibrium in section 4 in which some of the markets were allowed not to clear a 'classical equilibrium'.

What the OLG model shows is that by incorporating the classical view of the world without definite beginning or end, it is possible to maintain all the neoclassical methodological premises and yet still leave room for the indeterminacy which is the hallmark of both classical and Keynesian economics. In particular this can be achieved while maintaining labour market clearing. The explanation for this surprising conclusion is that the OLG model is isomorphic to a finite-like model in which indeed not all the markets need to clear. But far from being the labour markets, under pressure to move toward equilibrium from the unemployed clamouring for jobs, these markets are off 'at infinity', under no pressure toward equilibrating.

We have speculated that once one has agreed to the postulate that the resources of the economy are potentially as great at any future date as they are today, then uniform impatience of consumers is the decisive factor, according to Walrasian principles, which may influence whether the market forces of supply and demand determine a locally unique, Pareto-optimal equilibrium, or leave room for extra-market forces to choose among a continuum of inefficient equilibria. In these terms, the Arrow-Debreu model supposes a short-run impatient economy, and OLG a long-run patient economy

9. SUNSPOTS. So far we have not allowed uncertainty into the OLG model. As a result we found no difference in interpreting trade sequentially, with each agent facing two budget

constraints, or 'as if' the markets all cleared simultaneously at the beginning of time, with each agent facing one budget constraint. Once uncertainty is introduced these interpretations become radically different. In either case, however, there is a vast increase in the number of commodities, and hence in the potential for indeterminacy.

If we do not permit agents to make trades conditional on moves of nature that occur before they are born, then agents will have different access to asset markets. Even in finite horizon economies, differing access to asset markets has been shown by Cass and Shell to lead to 'sunspot effects'.

A 'sunspot' is a visible move of nature which has no real effect on consumers, either on account of preferences, or endowments, or through production. In the Arrow-Debreu model it also could have no effect on equilibrium trade; this is no longer true when access to asset markets differs.

The sunspot effect is intensified when combined with the indeterminacy that can already arise in an OLG economy. Consider the simple one good, steady state OLG economy of sections 1 and 2. Suppose that there is an equilibrium two cycle $q = (\dots q_{-1}, q_0, q_1, \dots)$ with $q_{2t} = q^t$ and $q_{2t+1} = q^t$, for all $t \in \mathcal{Z}$. Now suppose that the sun is known to shine on even periods, and hide behind rain on odd periods. The above equilibrium is perfectly correlated with the sun, even though no agent's preferences or endowments are. As usual, the same prices for $t \geq 0$ support an equilibrium, given the right amount of money, in $E_{0,\infty}^M$.

More generally, suppose that the probability of rain or shine, given the previous period's weather, is given by the Markov matrix $\pi = (\pi_{SS}, \pi_{SR}, \pi_{RS}, \pi_{RR})$. A steady state equilibrium for $E_{0,\infty}^M$, given π , is an assignment of a money price for the commodity, depending only on that period's weather, such that if all agents maximize their expected utility with respect to π , then in each period the commodity market and money market clears. Azariadis (1981) essentially showed that if there is a 2-cycle of the certainty economy, then there is a continuum of steady state sunspot equilibria.

The sunspot equilibria, unlike the cyclical equilibria of section 2, are Pareto suboptimal whenever the matrix π is nondegenerate.

The combination of the dynamic effects of the infinite horizon OLG model with the burgeoning theory of incomplete markets under real uncertainty, is already on the agenda for next generation's research.

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See also ARROW-DEBREU MODEL. INTERGENERATIONAL MODELS

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