Sankhyā : The Indian Journal of Statistics

Asymptotic Behavior of a Stochastic Discount Rate

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Abstract

The mean discount rate for a simple stochastic model behaves asymptotically roughly like $1/\sqrt{n}$ in contrast to the usual geometric discounting in a deterministic model.

AMS (2000) *subject classifications.* 60F99, 60J05, 60G42.
Keywords and phrases. discount rate, random walk.

1. Introduction

A stochastic model for interest rates was studied in Farmer and Geanakoplos [1]. In their model, the current rate, say r_n, is multiplied by a volatility factor $e^{vX_{n+1}}$ so that the rate in the next period is $r_{n+1} = r_n e^{vX_{n+1}}$. Here v is a given scale parameter and $\{X_n\}_n$ is a sequence of independent random variables, that equal plus or minus 1 with probability 1/2 each. If the initial rate is r_0, then the rate in period n is

$$r_n = r_0 e^{vS_n}$$

where $S_n = X_1 + \cdots + X_n$ is a simple, symmetric random walk on the integers. The value of future consumption in period n should accordingly be discounted by the amount

$$D_n = e^{-r_0} e^{-r_1} \cdots e^{-r_n} = e^{-\sum_{i=0}^{n} r_i} = e^{-r_0 \sum_{i=0}^{n} e^{vS_i}},$$

where $S_0 = 0$. Farmer and Geanakoplos [1] show that the mean discount factor $M_n = E(D_n)$ behaves roughly like a constant times $1/\sqrt{n}$ as $n \to \infty$. Thus future consumption or wealth is discounted much less in this stochastic model than in the usual geometric one.

The proof in [1] used ingenious path counting methods, and does not seem to carry over to more general distributions of the increments. In this note, we use classical Markovian tools to generalize the result in [1] to the class of random walks S_n on the real line that have bounded increments with mean zero.

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2. The Model and Main Result

Let $X_1, X_2, \ldots$ be independent, identically distributed random variables that are bounded with mean zero and a positive variance. Consider the random walk

$$S_n = X_1 + \cdots X_n,\ n = 1, 2, \ldots$$

and define the discount factor

$$D_n = e^{-\sum_{i=1}^{n} e^{S_i}}.$$

Let $M_n = E(D_n)$ be the mean discount factor.

THEOREM 2.1. *Under the assumptions above, we have that*

$$M_n = \frac{1}{n^{\frac{1}{2}+o(1)}} \tag{2.1}$$

where $o(1)$ denotes a function of n that converges to 0 as $n \to \infty$.

Our methods do not give optimal rates, so we do not characterize the behavior of the $o(1)$.

The next two sections are devoted to the proof of Theorem 2.1. For simplicity, we assume from now on that the random variables X_n are bounded by 1 in absolute value. Absolute constants that appear in the proof may depend on the distribution of the increments.

3. The Lower Bound for M_n

The key to the proof that $M_n \geq 1/(n^{\frac{1}{2}+o(1)})$ is the following lemma. Let ϵ be a small positive number, say $\epsilon < 1/5$.

LEMMA 3.1. *There is a positive integer $n_0 = n_0(\epsilon)$ such that*

$$P[S_i \leq -(i-n_0)^{\epsilon},\ i = n_0+1, \ldots, n_0+n] \geq \frac{1}{n^{\frac{1}{2}+o(1)}}. \tag{3.1}$$

Before proving the lemma, we first observe that the lower bound in (2.1) will follow from it. Let

$$D'_n = e^{-\sum_{i=n_0+1}^{n_0+n} e^{S_i}},\ \text{and}\ M'_n = E(D'_n).$$

Then, if the lower bound holds with M'_n in place of M_n, then it also holds in its original form. This is because

$$|\sum_{i=1}^{n_0+n} e^{S_i} - \sum_{i=n_0+1}^{n_0+n} e^{S_i}| \leq \sum_{i=1}^{n_0} e^{|S_i|} \leq e + e^2 + \cdots e^{n_0} = c_0.$$

So
$$e^{-c_0} M'_n \leq M_n \leq e^{c_0} M'_n.$$
Now expressions of the form $1/(n^{\frac{1}{2}+o(1)})$ remain of the same form when multiplied by a constant. Hence, it suffices to prove the bound for M'_n.

Assume now that (3.1) holds. Then, for $n \geq n_0$,
$$M'_n \geq e^{-\sum_{i=n_0+1}^{n_0+n} e^{-(i-n_0)^\epsilon}} \cdot P[S_i \leq -(i-n_0)^\epsilon,\ i = n_0+1, \ldots, n_0+n]$$
and
$$e^{-\sum_{i=n_0+1}^{n_0+n} e^{-(i-n_0)^\epsilon}} \geq e^{-\sum_{i=1}^{\infty} e^{-i^\epsilon}} > 0.$$
So, for the lower bound, it suffices to prove the lemma.

The proof will be in several steps and it will be convenient to prove
$$P[S_i \geq (i-n_0)^\epsilon,\ i = n_0+1, \ldots, n_0+n] \geq \frac{1}{n^{\frac{1}{2}+o(1)}} \tag{3.2}$$
rather than (3.1). The two inequalities are equivalent since $\{-S_n\}$ satisfies the same assumptions as $\{S_n\}$.

Fix a large positive integer K to be specified below and set $\delta = 5\epsilon$. So $\delta < 1$. The idea of the proof of Lemma 1 is to get bounds on the behavior of the random walk as it travels through the sequence of intervals $I_i = [K^{i\delta}, K^{i+1}), i \geq 1$, starting, in each case, from a point in the interior near K^i.

Notice that, since the random walk is recurrent with increments bounded by 1, there is a positive integer n_0 such that
$$p = P[S_{n_0} \in [K, K+1)] > 0.$$
This is the same n_0 that appears in the statement of Lemma 1.

Let P^x denote the distribution of the random walk starting from the real number x (so that $P = P^0$). It follows from the Markov property that
$$P^0[S_i \geq (i-n_0)^\epsilon,\ i = n_0+1, \ldots, n_0+n] \geq p \cdot P^K[S_i \geq i^\epsilon,\ i = 1, \ldots, n].$$
Thus it suffices to prove that
$$P^K[S_i \geq i^\epsilon,\ i = 1, \ldots, n] \geq \frac{1}{n^{\frac{1}{2}+o(1)}}. \tag{3.3}$$

The first step in the proof gives a lower bound on the probability that the random walk exits each of the intervals I_i at the upper boundary. For y real, let
$$\tau_y^+ = \min\{n : S_n \geq y\} \text{ and } \tau_y^- = \min\{n : S_n \leq y\}.$$

Step 1. *For $i = 1, 2, \ldots$, it holds uniformly for $x \in [K^i, K^i + 1]$ and uniformly in i that*
$$P^x[\tau^+_{K^{i+1}} < \tau^-_{K^{i\delta}}] \geq \frac{1}{K}(1 + o(1)). \tag{3.4}$$
(Here $o(1) \to 0$ as $i \to \infty$.)

Proof. Let τ be the exit time for the interval $(K^{i\delta}, K^{i+1})$; that is,

$$\tau = \min\{\tau^-_{K^{i\delta}}, \tau^+_{K^{i+1}}\}.$$

By the optional sampling theorem, $E^x(S_\tau) = x$. Also, since the increments of $\{S_n\}$ are bounded by 1, we have

$$K^i - 1 \le x = E^x(S_\tau) \le \bar{p}(K^{i+1} + 1) + (1 - \bar{p})K^{i\delta},$$

where $\bar{p} = P^x[\tau^+_{K^{i+1}} < \tau^-_{K^{i\delta}}]$. Hence,

$$\bar{p} \ge \frac{K^i - K^{i\delta} - 1}{K^{i+1} - K^{i\delta} + 1}.$$

Inequality (3.4) now follows. □

Notice that when the random walk exits the interval $I_i = [K^{i\delta}, K^{i+1}]$ to the right, it is then at a position in $[K^{i+1}, K^{i+1} + 1)$ so that Step 1 can then be applied to the next interval I_{i+1}.

The next step gives an upper bound for the time the process spends in the interval $(0, K^{i+1})$. The integer K will also be specified in this step.

Step 2. *There exist K_0 and a positive constant c_1 such that, for all $K \ge K_0$, $i \ge 1$ and all $x \in (0, K^{i+1})$, we have*

$$P^x[\tau^+_{K^{i+1}} \wedge \tau^-_0 > K^{2(i+1)+i\delta}] \le e^{-c_1 K^{i\delta}}. \tag{3.5}$$

(Here $a \wedge b$ denotes the minimum of a and b.)

Proof. Recall that the X_i have mean 0 and finite, positive variance σ^2. Let Z be a standard normal random variable and let $a = P[Z > 2/\sigma]$. Then, by the central limit theorem,

$$P[\frac{S_n}{\sqrt{n}\sigma} \ge \frac{1}{\sigma}] > a > 0,$$

for n sufficiently large. Choose the integer $K_0 > 2^{1/\delta}$ so that this inequality holds for all $n \ge K_0$. Now let $K \ge K_0$ and

$$\tau = \tau^+_{K^{i+1}} \wedge \tau^-_0$$

be the exit time from the interval $(0, K^{i+1})$. Then, for x in the interval,

$$P^x[\tau \le K^{2(i+1)}] \ge P[S_{K^{2(i+1)}} > K^{i+1}] = P[\frac{S_{K^{2(i+1)}}}{K^{i+1}\sigma} > \frac{1}{\sigma}] \ge a.$$

It now follows from the strong Markov property that, with $\lfloor K^\delta \rfloor$ denoting the largest integer that is less than or equal to K^δ,

$$P^x[\tau > K^{2(i+1)+i\delta}] \le (1 - a)^{\lfloor K^{i\delta} \rfloor} \le e^{-c_1 K^{i\delta}},$$

where $c_1 = -\log(1 - a)/2$. □

Now we combine Steps 1 and 2.

Step 3. *Under the same assumptions as in Step 1, it holds uniformly for* $x \in [K^i, K^i+1]$ *and uniformly in* i *that*

$$P^x[\tau^+_{K^{i+1}} < \tau^-_{K^{i\delta}} \wedge (K^{2(i+1)+i\delta})] \geq \frac{1}{K}(1+o(1)). \tag{3.6}$$

Also, if $S_l = x$ *and* $S_l, S_{l+1}, \ldots, S_{\tau_{K^{i+1}}}$ *is a path satisfying the condition of the event above, then* $S_j \geq j^\epsilon$ *for* $j = l, l+1, \ldots, \tau^+_{K^{i+1}} - 1$.

Proof. Inequality (3.6) follows from Steps 1 and 2. For the second assertion, recall that $\delta = 5\epsilon$. Now $j < K^{2(i+1)+i\delta} < K^{5i}$. So $j^\epsilon = j^{\delta/5} < K^{i\delta} < S_j$. □

Next we need an upper bound on the time for the random walk to reach a positive boundary.

Step 4. *If* n *is a large positive integer and* $\tilde{n} = \sqrt{n}\log n$, *then*

$$P[\tau^+_{\tilde{n}} < n] \leq e^{-\frac{1}{4}(\log n)^2}. \tag{3.7}$$

Proof. Since the increments of $\{S_n\}$ are bounded by 1, it folows from Azuma's inequality that for $i = 1, 2, \ldots, n$

$$P[S_i > \tilde{n}] \leq e^{-\frac{\tilde{n}^2}{2i}} \leq e^{-\frac{\tilde{n}^2}{2n}} = e^{-\frac{1}{2}(\log n)^2}.$$

Thus

$$P[\tau^+_{\tilde{n}} < n] \leq \sum_{i=1}^{n} P[S_i > \tilde{n}] \leq n e^{-\frac{1}{2}(\log n)^2} \leq e^{-\frac{1}{4}(\log n)^2},$$

where the final inequality holds for $n > e^4$. □

Suppose now that K^m is the largest power of K smaller than $\tilde{n}$. Then, by (3.7), there is an exponentially small chance that the random walk will traverse more than m of the intervals I_i by time n. Now use Step 3 and the strong Markov property to see that

$$P^K[S_i > i^\epsilon, i = 1, \ldots, n] \geq (\frac{1}{K})^{\log_K \tilde{n}}(1+o(1))^{\log_K \tilde{n}}.$$

It is straightforward to verify that the quantity above on the right is bounded below by $1/(n^{1/2+o(1)})$. This completes the proof of (3.3) and of Lemma 1.

4. The Upper Bound for M_n

The argument that $M_n \leq 1/(n^{\frac{1}{2}+o(1)})$ is based on certain properties of the excursions of the random walk $\{S_n\}$. Recall that the increments of the walk are bounded in absolute value by 1. An *excursion* is here defined to be any path $S_n, S_{n+1}, \ldots, S_{n+k-1}$ such that S_n and S_{n+k} belong to the interval $(-1, 1)$ but the intermediate positions $S_{n+1}, \ldots, S_{n+k-1}$ do not. The *length* $L = k$ of such an excursion is measured in units of time.

Let ϵ be a small positive number. The following lemma is probably known.

LEMMA 4.1. *Let L be the length of an excursion of the random walk $\{S_n\}$. Then there is a constant c_ϵ such that, for $x > 0$ sufficiently large,*

$$P[L > x] \leq \frac{c_\epsilon}{x^{\frac{1}{2}-\epsilon}}.$$

Proof. Assume, without loss of generality, that the position of the random walk at the beginning of the excursion L is in the interval $[0, 1)$. Let $A_x = [\tau^+_{x^{1/2-\epsilon}} < \tau^-_0]$ be the event that the walk exceeds $x^{1/2-\epsilon}$ before reaching a negative value. An argument using the optional sampling theorem similar to that for Step 1 in the previous section shows that there is a constant c_2 such that, for x large,

$$P(A_x) \leq \frac{c_2}{x^{\frac{1}{2}-\epsilon}}.$$

Next let $\tau = \min\{n \geq 1 : S_n \notin (0, x^{\frac{1}{2}-\epsilon})\}$ and let $B_x = [\tau > n]$. Then an argument similar to that for Step 2 in the previous section shows there is a constant c_3 such that, for x large,

$$P(B_x) \leq e^{-c_3 x^{2\epsilon}}.$$

Now, if L exceeds x, then either the walk reaches $x^{1/2-\epsilon}$ before it reaches a negative value or it stays in the interval $(0, x^{1/2-\epsilon})$ for an amount of time at least as large as x. So

$$P[L > x] \leq P(A_x) + P(B_x) \leq \frac{c_2}{x^{\frac{1}{2}-\epsilon}} + e^{-c_3 x^{2\epsilon}}.$$

The assertion of the lemma follows easily. □

To complete the proof for the upper bound, consider, for large n, the event E_n that there are less than $e \log n$ excursions that begin by time n. Then, if E_n occurs, the length of at least one of these excursions must exceed $n/(e \log n)$. So if $m = \lfloor e \log n \rfloor$ is the greatest integer less than or equal to $e \log n$ and $L_1, \ldots, L_m$ are the lengths of the first m excursions, it follows from Lemma 4.1 that,

$$\begin{aligned} P(E_n) \leq \sum_{i=1}^{m} P[L_i > n/(e \log n)] \leq & \frac{c_\epsilon m}{(\frac{n}{e \log n})^{1/2-\epsilon}} \\ \leq & \frac{c_\epsilon e \log n}{(\frac{n}{e \log n})^{1/2-\epsilon}} = \frac{c_\epsilon{}'(\log n)^{3/2-\epsilon}}{n^{1/2-\epsilon}}. \end{aligned}$$

On the complement E_n^c, there are at least $e \log n$ excursions by time n and therefore at least $e \log n$ visits to the interval $(-1, 1)$. Hence, on E_n^c

$$\sum_{i=1}^{n} e^{S_i} \geq e^{-1} e \log n = \log n$$

and

$$D_n \leq e^{-\log n} = n^{-1}.$$

Now D_n is everywhere less than or equal to 1. So we have

$$M_n = E(D_n) = \int_{E_n} D_n \, dP + \int_{E_n^c} D_n \, dP$$
$$\leq P(E_n) + n^{-1} \leq \frac{c_\epsilon{}'(\log n)^{3/2-\epsilon}}{n^{1/2-\epsilon}} + n^{-1}.$$

The last expression is easily seen to be bounded above by $1/n^{1/2-2\epsilon}$ for n sufficiently large. Since ϵ is arbitrary, the upper bound in (2.1) follows and the proof of Theorem 2.1 is complete.

References

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