Real Indeterminacy with Financial Assets*

JOHN GEANAKOPLOS

Cowles Foundation, Yale University, New Haven, Connecticut 06520

AND

ANDREU MAS-COLELL

Harvard University, Cambridge, Massachusetts 02138

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It is shown that in a two period general equilibrium securities model where assets pay in money the generic dimension of the set of equilibrium allocations, in the incomplete market situation, is $S-1$, where S is the number of assets. Hence the degree or real indeterminacy is independent of the number of assets This result requires, beyond fewer assets than states, that the number of traders be larger than the number of securities and that the asset return matrix be in general position. The generic dimension for arbitrary returns matrix is also obtained. It is argued, in addition, that the presence of real or mixed assets does not by itself lower the degree of indeterminacy. *Journal of Economic Literature* Classification Numbers. 021, 022, 023   © 1989 Academic Press, Inc

I. INTRODUCTION

It has long been known in economics that the notion of general competitive equilibrium displays a basic multiplicity, though this indeterminacy has usually been disposed of as being almost entirely nominal. An Arrow–Debreu economy, for example, typically has a continuum of equilibrium price vectors, but only a finite number of these give rise to distinct commodity allocations. The acconting relation called Walras Law implies that

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the economy-wide system of excess demands has one more endogenous price than it has independent market clearing equations. However, the homogeneity of demand, i.e., the fact that the aggregate excess demand function depends on relative prices and not on their absolute level, explains why most of the ensuing indeterminacy is nominal. Our starting point here is the observation that for economies less idealized than that of Arrow–Debreu, involving the exchange of monetarized assets, the indeterminacy caused by Walras Law is greater than one-dimensional, and because there is no corresponding increase in the homogeneity of demand, the difference manifests itself as a real indeterminacy of equilibrium.

In this paper we draw a sharp distinction between economies in which the assets promise delivery in a money (say green pieces of paper) whose exchange value can exceed its (marginal) use value, and those economies where the assets deliver in a commodity money whose exchange value is tied to its use value. In the latter situation the lock-step balance between Walras Law and the homogeneity of excess demand preserves local uniqueness of real equilibrium. But in the money case there is usually a multidimensional continuum of competitive equilibria, each representing a different commodity allocation. Often there will be Pareto comparability between equilibria.

An important, preliminary, example of real indeterminacy in a "monetary" economy occurs in the standard static Walrasian setting if we add an extra commodity, which we call money, that has no efect on any agent's utility. Let each agent be endowed with m^h green dollar bills (i.e., units of the money). As long as $\sum_h m^h > 0$, we know that the equilibrium price of money must be equal to its use value, namely zero. But if $\sum_h m^h = 0$, and we allow m^h to be negative or positive, then the price of money is not tied to zero, and in fact it is easy to show that typically there is a one-dimensional continuum of equilibria involving different commodity allocations. The same is true if $\sum_h m^h > 0$, but each agent h owes a money tax d^h to the government with $\sum_h d^h = \sum_h m^h$.[1]

The reason for this real indeterminacy with money can be expressed in two equivalent ways. Note that the excess demand for money is degenerate, i.e., at any vector of prices the demand for money will match its supply. Introducing money thus adds one more variable price, but does not add another independent market clearing condition. Equivalently, if the price of each dollar is fixed at 1, demand for real commodities is no longer homogeneous in commodity prices, yet Walras Law still applies to the commodities, so that if all the commodity markets clear but one, this last will clear as well. Note that both explanations of indeterminacy rest on the

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[1] An old argument, usually attributed to Abba Lerner, holds that govenment money taxes are a major reason government issued paper money has value.

fact that at least for some h, $m^h \neq d^h$; i.e., the economy is in midstream with some agents already debtors and others creditors.[2] Otherwise commodity demand would be homogeneous in commodity prices (or equivalently, the price of money would not affect commodity demand). From now on we shall always include money in our models, and we shall always fix the price of each dollar at 1, ignoring equilibria in which money is valueless. All prices are thus quoted in dollars, which become the units of account. When we use the terms Walras Law and homogeneity we shall mean with regard to commodity prices and demands, holding the money price fixed at 1.

Consider now an economy in which trade takes place sequentially, perhaps in different states of nature. Let trade take place in period zero ($s = 0$) and again in period 1 for each state $s = 1, ..., S$. Assume that every consumer's endowment of money in every state is zero. In each state s each agent is required to balance the value of his expenditures and sales.[3] Agents therefore face $S + 1$ butget constraints. Since Walras Law can be applied $S + 1$ times, there are $S + 1$ redundant market clearing equations and we should expect $S + 1$ dimensions of indeterminacy of equilibrium prices. However, it is clear that there are also $S + 1$ independent applications of homogeneity, since each state's commodity prices can be scaled independently without affecting demand. It is in fact easy to show that typically there are only a finite number of distinct real equilibria. Thus although there are $S + 1$ dimensions of equilibrium price vectors, differing by their absolute levels across the states, most of this indeterminacy has no effect on real consumption.

Let us next enrich the model along the lines suggested by Radner [13] and Hart [11] by allowing agents to trade in period 0 prespecified real assets as well as commodities. A real asset is a promise to deliver a vector of state contingent commodity bundles; agents can be allowed to buy or sell these claims. If every conceivable real asset were traded, the model would reduce to a special case of the Arrow–Debreu model. Whether or not the real asset market is complete, it is evident that since the assets are prefectly "indexed," there are again $S + 1$ independent operations of homogeneity, and it can generally be shown that generic, real asset

economies have a finite number of distinct real equilibria (see Geanakoplos–Polemarchakis [9]).

The first general equilibrium model involving assets occurs in Arrow [1] where assets are promises to deliver state contingent dollars. To distinguish these monetary assets from the real assets above, we shall call them financial assets or securities. Arrow concentrated on a specialized type of financial asset that promises delivery of one dollar in precisely one state s, and nothing in the other states. In his honor these have come to be called Arrow securities. Arrow proved the remarkable result that when agents are permitted to trade all S Arrow securities and spot prices are correctly anticipated then the equilibrium commodity allocations are identical to those that would arise in the Arrow–Debreu model discussed above where agents are permitted to trade all possible state contingent commodities. We can conclude that typically, in a complete Arrow securities economy almost all the indeterminacy is nominal (i.e., not real). Again Arrow's result may be looked at as a balance between the $S+1$ occurrences of Walras Law and of the homogeneity of demand, but the homogeneity is more subtle than before. As usual, demand for assets and commodities is in period zero homogeneous of degree zero relative to period zero prices. For each state $s \geq 1$, the demand for the sth Arrow security is homogeneous of degree one,[4] and the demand for all the other securities and commodities is homogeneous of degree zero, in the absolute level of prices in state s, provided that the asset price for the sth Arrow security is varied inversely with p_s. Once a state s is realized, asset promises will make some agents creditors and others debtors. According to our preliminary example, changes in the absolute level of prices p_s can have real effects. What happens is that if these price changes are anticipated, then rational agents will readjust their portfolios of Arrow securities so that in the end there are no real effects.

In this paper we consider economies with an incomplete set of arbitrary financial assets, as in Cass [4], Werner [14], and Duffie [7]. These papers all suggest that there may be real indeterminacy. In fact, Cass [3] constructs an explicit example with one financial asset and two stated in which there is a one-dimensional continuum of distinct real equilibria. In this paper we follow Cass' lead by taking up the general problem of real indeterminacy with financial assets. We find that "typically," any change in the relative rates of inflation (from 0 to s) across the states has a real effect, even if it is perfectly anticipated. This means that there are $S-1$ degrees of real indeterminacy. It is clear that there are at least two independent sources of homogeneity in demand, including the usual homogeneity in

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[4] Since the initial endowment of assets is zero for every agent, homogeneity of any degree is enough to negate one dimension of multiplicity

period zero asset and commodity prices. The second source comes from the fact that if all period one prices are doubled, and all asset prices halved, then commodity demand is unaffected and asset demands are doubled. It is not clear whether there are other sources of homogeneity, and our result implies there cannot be. The occurrences of Walras Law provide no more than $S + 1$ degrees of freedom. Of these we see that $S - 1$ are real and 2 are nominal.

More precisely, we begin by fixing the smooth preferences of each agent and the state contingent dollar payoffs of each financial asset. The payoffs of the assets can be summarized by an $S \times B$ matrix R. We say that the asset payoffs are in general position if every submatrix of R is of full rank. Clearly if the payoffs were chosen randomly, R would nearly always be in general position. Our main result is given by Theorem 1, in Section II, which essentially asserts that if there are fewer assets than states $(B < S)$, i.e., if the asset market is incomplete, and if the assets are in general position, then, provided there are at least as many agents as assets and for almost any assignement of initial commodity endowments to agents, the resulting financial asset economy has $S - 1$ dimensions of real indeterminacy.

There is something of a surprise in this result. Indeed, we had initially conjectured that the number was $S - B$ (a number consistent also with Cass' example). As it turns out the dimension of indeterminacy is independent of the number of bonds B, as long as $0 < B < S$. If $B = 0$ the model is obviously determinate. If $B \geqslant S$, one can apply Arrow's [1] logic to show that all the equilibrium commodity allocations are Arrow–Debreu equilibrium allocations, and again there is no real indeterminacy. Theorem 1 points to an intriguing discontinuity. If markets are financially complete, then the model is determinate. Let just one financial asset be missing and the model becomes highly indeterminate. Thus, in this sense, the complete markets hypothesis $(B = S)$ lacks robustness.[5] (Probably what this means is that the hypothesis has to be interpreted as $B > S$, i.e., one better have some redundancy.)

The idea behind the proof of Theorem 1 combines two essential ingredients. First, one can arbitrarily fix in advance the absolute level of commodity prices in terms of some numeraire independently across all the S states and still solve for equilibrium. The reason for this apparently puzzling phenomenon is that fixing the absolute price level in each state s is equivalent to transforming the financial assets into real assets that all deliver in the same numeraire commodity in each state s, and Geanakoplos–Polemarchakis [9] proved that numeraire asset economies

always have equilibria. Second when $0 < B < S$, changes in the relative rates of inflation change the B-dimensional span of the assets. The set of B-dimensional subspaces of $\mathbb{R}^S$ has dimension $B(S - B)$. Hence if $0 < B < S$ there are enough distinct subspaces to be filled by the $S - 1$ degrees of freedom. Finally, we show that if there are more agents than assets, then a change of subspace typically means that the old equilibrium is no longer feasible.

There is no doubt that many contracts and financial securities in the world promise state contingent delivery in money, and not in real commodities. Moreover, there is little doubt that asset markets are incomplete. Nevertheless, it is perhaps worthwhile to make three brief comments about the robustness of our results. First, recall that we have taken the financial asset payoffs as fixed exogenously. We shall not make any further effort to explain how these payoffs are determined, or why others are missing. There are obvious reasons why some contracts are denoted in money terms, not the least of which is simplicity, and we see no reason why these monetary payoffs would change to fully offset any change in the expected absolute price level across the states. If they did, then they would indeed be real assets. Or, at least, we see no reason why this would happen instantly to accommodate any unexpected shock.

Second, there is no doubt that there are a great number of real assets in the economy. One may conjecture that each independent real asset reduces by one the dimension of real indeterminacy in the economy, so that if there are A real assets, then there are only $S - 1 - A$ dimensions of real indeterminacy. But like our previous conjecture, this is incorrect. Theorem 2, in Section III, shows that as long as $A + B < S/2$, there are still $S - 1$ dimensions of real indeterminacy, independent of A or B. In summary: when markets are incomplete, the presence of financial assets creates an indeterminacy in competitive equilibrium allocations of a degree that does not depend on the absence of real assets.

Third, it is possible to give examples of financial asset payoff matrices R, that are not in general position and for which the dimension of real indeterminacy is less than $S - 1$. For example, if all the assets are Arrow securities, then there is typically no real indeterminacy. Different readers may have different views about which are the most salient financial asset payoff structures. We have accordingly, in Section III, introduced a simple formula that expresses the dimension of real indeterminacy typically associated with any financial asset payoff matrices R. We do not find that matrices R yielding no indeterminacy are more plausible than the R in general position (for which Theorem 1 yields maximal indeterminacy). In particular, our formula implies that as long as none of the rows of R is identically zero the dimension of real indeterminacy is always at least $S - B$ (see, also, Balasko–Cass [2] and Cass [5]).

II. The Model and Main Result

There are $L+1$ physical commodities $(l=0, ..., L)$ and two dates. Spot trade tomorrow will take place under any of S states $(s=1, ..., S)$. Today there is trade on current goods and on B financial assets or bonds $(b=1, ..., B)$. Bonds pay money. We express their payoff by an $S \times B$ matrix with generic entry r_{sb}. We say that R is in *general position* if every submatrix of R has full rank.

There are $H+1$ consumers $(h=0, ..., H)$. Every consumer h has a utility function u^h defined on $\mathbb{R}_{++}^{(L+1)(S+1)}$ and satisfying the standard differentiability, monotonicity, curvature, and boundary conditions needed to get a well-defined C^r differentiable excess demand function (see, for example, Mas-Colell [12, Chap. 2]). Note: The degree of differentiability r is assumed to be large enough for the subsequent transversality arguments to be justified. Every consumer also has an initial endowment vector $\omega^h \in \mathbb{R}_{++}^{(L+1)(S+1)}$. In this section, when we say that a property of economies $E = (R, u^h, \omega^h; h \in H)$ is *generic*, we mean that for any $\bar{R}$ and $\bar{u}^h$ there is an open, dense subset $\mathscr{D} \subset \mathbb{R}_{++}^{(L+1)(S+1)}$ whose complement has Lebesgue measure zero and such that the property applies to all economies $(\bar{R}, \bar{u}^h, \omega^h; h \in H)$ with endowment chosen in $\mathscr{D}$.

DEFINITION 1. An allocation $(\bar{x}, \bar{y})$ of goods and bonds is a *financial asset equilibrium* for $E = (r, u^h, \omega^h; h \in H)$ if:

(1) $\sum_h \bar{x}^h = \sum_h \omega^h$, $\sum_h \bar{y}^h = 0$, and

(ii) there is a price system $p \in \mathbb{R}_{++}^{(L+1)(S+1)}$, $q \in \mathbb{R}^B$ such that for every h, $(\bar{x}^h, \bar{y}^h)$ maximizes $u^h(x^h)$ on

$$B^h(p, q) = \left\{ (x^h, y^h): p_0 \cdot x_0^h + q \cdot y^h \leqslant p_0 \cdot \omega_0^h, \text{ and} \right.$$

$$\left. p_s \cdot x_s^h \leqslant p_s \cdot \omega_s^h + \sum_b y_b^h r_{sb}, \text{ all } s \right\}.$$

Since the budget constraints imply that $S+1$ of the market clearing conditions are redundant, there is in general some indeterminacy in the equilibrium allocations. If the indeterminacy affects only the holdings of bonds, y^h, then we call it *nominal indeterminacy*. Otherwise, we call it *real indeterminacy*. We are interested in the degree of real indeterminacy.

THEOREM 1. *If* $0 < B < S$, *R is in general position, and $H \geqslant B$, then, generically, there are $S-1$ dimensions of real indeterminacy; i.e., the set of equilibrium allocations of commodities x contains the image of a C^1, one-to-one function with domain $\mathbb{R}^{S-1}$.*

Proof of Theorem 1. The proof proceeds in four steps. The first introduces the notion of a real numeraire asset equilibrium and shows that the set of financial asset equilibrium can be parameterized as real numeraire asset equilibria, the parameter being an S-vector or prices of money in the different states. The second step gives sufficient conditions for the real numeraire asset equilibria corresponding to different parameters to be different. These conditions are of two types: (i) a full dimension requirement on the span of the vectors of individual demand for assets at equilibrium, and (ii) spanning requirements involving the return matrix and the particular parameter vector. Step 3 shows that the conditions of type (i) are generically satisfied if $H \geqslant B$. Step 4 shows that if R is in general position then the conditions of type (ii) are also satisfied for a $S - 1$ dimensional family of parameters. Combining we get the result.

Step 1. Given the prices p_{s0} of the zero commodity (or, equivalently, the price of money $\lambda_s = 1/p_{s0}$ in terms of good 0) our system of financial assets is equivalent to a system of "real numeraire assets" where each asset pays in (equivalent worth) of the zero commodity. More precisely, given a matrix $\bar{R} = (\bar{r}_{sb})$, representing the payoffs of real assets in the numeraire (commodity zero) for each state s, let us define allocation $(\bar{x}, \bar{y})$ of goods and real assets to be a *real numeraire asset equilibrium* if (i) and (ii) of Definition 1 are satisfied, but with respect to the budget set

$$\bar{B}^h(p, q) = \left\{ (x, y) : p_0 \cdot x_0 + q \cdot y \leqslant p_0 \cdot \omega_0^h \text{ and} \right.$$

$$\left. p_s \cdot x_s \leqslant p_s \cdot \omega_s^h + p_{s0} \cdot \sum_b y_b^h \bar{r}_{sb} \text{ for all } s \right\}.$$

It is easy to see that $(\bar{x}, \bar{y})$ is a financial asset equilibrium, with asset return R, if and only if $(\bar{x}, \bar{y})$ is a real numeraire asset equilibrium with $p_{s0} = 1$ for all $s \in S$ and asset return matrix $\bar{R} = AR$, where A is some diagonal $S \times S$ matrix having $\lambda_s > 0$ for all s.

Step 2. For any $S \times B$ matrix A, let us denote by $sp[A]$ the linear subspace of $\mathbb{R}^s$ spanned by the B columns of A.

LEMMA 1. *Let* (x, y) *and* $(\hat{x}, \hat{y})$ *be real numeraire asset equilibria for, respectively,* $E = (AR, u^h, \omega^h; h \in H)$ *and* $\hat{E} = (\hat{A}R, u^h, \omega^h; h \in H)$. *Suppose that* $\dim sp[y^1, ..., y^H] = \dim sp[\hat{y}^1, ..., \hat{y}^H] = B$ *and* $sp[AR] \neq sp[\hat{A}R]$. *Then* $x \neq \hat{x}$.

Proof. Consider the vectors $\{ARy^h : h \in H\}$ and $\{\hat{A}R\hat{y}^h : h \in H\}$. By hypothesis there is some h such that $\hat{A}R\hat{y}^h \neq ARy^h$. Suppose that $x^h = \hat{x}^h$. From the smoothness and boundary conditions on u^h, we must have that

the goods equilibrium prices p and $\hat{p}$ are equal. But by Walras Law, which holds state by state, that implies $\hat{A}R\hat{y}^h = ARy^h$. Contradiction. $\blacksquare$

Step 3. Let M be the set of diagonal positive matrices. We will now establish a fairly intuitive fact. Namely that if $H \geq B$ then generically at equiblirium the vectors of individual assets demands span $\mathbb{R}^B$. More precisely, we show that generically there is an open nonempty subset $V \subset M$ and a C^1 parameterization of allocations $x(A)$, $y(A)$, $A \in V$, such that, first, $(x(A), y(A))$ is a real numeraire asset equilibrium with return matrix AR, and, second, $y(A)$ satisfies the full dimension condition of Lemma 1 (and, therefore, if A, $A' \in V$ and $sp[AR] \neq sp[A'R]$ then $x(A) \neq x(A')$).

The proof uses standard transversality techniques. We will not repeat here the most familiar arguments.

Let $f(p, q, A, \omega)$ be the excess demand function from $P = \mathbb{R}^{L(S+1)}_{++} \times \mathbb{R}^B \times \mathbb{R}^S_{++} \times \mathbb{R}^{(L+1)(S+1)(H+1)}_{++}$ to $\mathbb{R}^{L(S+1)} \times \mathbb{R}^B$. Of course this function is not defined for all $q \in \mathbb{R}^B$ but only for those asset prices which satisfy a "nonarbitrage" condition.

LEMMA 2. *f is a C' function on the (nonempty) interior of its domain of definition. Moreover, $f(p, q, A, \omega) = 0$ if and only if p, q are real numeraire asset equilibrium prices for $E = (AR, u^h, \omega^h; h \in H)$. Also, $f(p, q, A, \omega) = 0$ implies that rank $\partial_{\omega_0} f(p, q, A, \omega) = L(S+1) + B$.*

Proof. See Geanakoplos and Polemarchakis [9]. $\blacksquare$

Define now $g: P \times J \to \mathbb{R}^{L(S+1)} \times \mathbb{R}^B \times \mathbb{R}^B$, where J is the $B - 1$ sphere, by

$$g(p, q, A, \omega, z) = \left(f(p, q, A, \omega), \sum_{h=1}^{B} z_h y_1^h, ..., \sum_{h=1}^{B} z_h y_B^h \right),$$

where y_b^h is the hth consumer demand for bond b at (p, q, A, ω).

LEMMA 3. *If $g(p, q, A, \omega, z) = 0$ then   rank $\partial_\omega g(p, q, A, \omega, z) = L(S+1) + 2B$.*

Proof. Let $(p, q, A, \omega, \bar{z}) \in g^{-1}(0)$. Because $\bar{z} \in J$ we know that $\bar{z}_h \neq 0$ for some h. Given Lemma 2 it suffices to show that for any $b = 1, ..., B$ there is some perturbation Δ^h and Δ^0 of the endowments of consumers h and 0 that leaves f and $y_{h' b'}$ unaffected for all $(h', b') \neq (h, b)$ but does change $y_{h,b}$. Let Δ^h be given by a decrease in ω_{00}^h of q_b and an increase in ω_{s0}^h of $\lambda_s R_{sb}$ for all $s = 1, ..., S$. Let Δ^0 be given by an increase in ω_{00}^h of q_b and a decrease in ω_{s0}^h of $\lambda_s R_{sb}$. Then consumer h decreases his demand $y_{h,b}$ by one unit and aggregate f is unaffected. $\blacksquare$

By the Transversality Theorem (se, e.g., Mas-Colell [12, Subsection 1.I]) for a.e. ω the sets $f_\omega^{-1}(0)$ and $g_\omega^{-1}(0)$ are C^r- manifolds of respective dimensions S and $S-1$. By Sard's theorem (see reference above) the projection of $f_\omega^{-1}(0)$ on M has a regular value $\hat{A}$. From Geanakoplos and Polemarchakis [9] we know that $(\hat{p}, \hat{q}, \hat{A}, \omega) \in f^{-1}(0)$ for some $\hat{p}, \hat{q}$. Hence the regular value $\hat{A}$ is actually in the range of the projection. Therefore, from the Implicit Function Theorem, there are open sets $P'_\omega \subset P_\omega = \{(p, q, A): (p, q, A, \omega) \in P\}$, $V' \subset M$, and a C^1 function $\xi: V' \to P'_\omega$ such that $(p, q, A, \omega) \in f^{-1}(0) \cap (P_\omega \times \{\omega\})$ if and only if $\xi(A) = (p, q, A, \omega)$. Let $\bar{P}'_\omega \subset P_\omega$ be the closure of P'_ω. Then the projection of $g_\omega^{-1}(0) \cap (\bar{P}'_\omega \times J)$ on M is compact and so we can find a nonempty open set $V \subset V'$ which is disjoint from this projection. But this means that if $A \in V$ then the $\{y^h\}_{h=1}^B$ corresponding to $\xi(A)$ satisfy the spanning condition of Lemma 1. We have thus obtained the desired parameterization of equilibria.

Step 4. We now complete the proof by exploiting the hupotheses not yet used, namely that $B < S$ and R is in general position. We will see that this implies that $sp[AR] \neq sp[A'R]$ unless $A = \alpha A'$ for some $\alpha > 0$. Therefore, using Lemmata 1–3, the subset of M where $\lambda_1 = 1$ provides our $S-1$ parameterization. We begin by a preliminary lemma. We say that a collection of subspaces $L_1, ..., L_K \subset \mathbb{R}^B$ is linearly independent if $\sum_k y_k = 0$, $y_k \in L_k$ implies $y_k = 0$ for all h.

LEMMA 4. *Let R be an $S \times B$ matrix with nonzero rows and A a diagonal invertible matrix. If $sp[AR] \subset sp[R]$ then there are linearly independent subspaces $L_1, ..., L_K \subset \mathbb{R}^B$, such that, first, every row of R is contained in some subspace and, second, two rows belong to the same subspace if and only if the corresponding entries of A are equal.*

Proof. The hypothesis $sp[AR] \subset sp[R]$ is equivalent to the following: there is a $B \times B$ matrix Y such that $AR = RY$. This means that every row of R is an eigenvector of Y with the corresponding element of A being the eigenvalue. Given a linear transformation Y to each of its distinct real eigenvalues $\lambda_1, ..., \lambda_K$ we can associate the linear subspace $L_1, ..., L_K \subset \mathbb{R}^B$ where each L_k is spanned by the eigenvectors corresponding to λ_k. The collection $\{L_k\}$ is linearly independent (see, e.g., Halmos [10, p. 113]). ∎

In our case we should have $K = 1$. Otherwise, because $B < S$ we would have a subspace $L \subset \mathbb{R}^B$ with dim $L < B$ but containing a number of rows of R larger than dim L. This contradicts the general position of R.

Summarizing, in our case $sp[AR] \subset sp[R]$ implies $A = \alpha I$, $\alpha > 0$. Let now $sp[AR] = sp[A'R]$. Then $sp[A'^{-1}AR] = sp[R]$. Hence $A = \alpha A'$ for some $\alpha > 0$, as we wanted to prove.

Remark 1. Observe that Theorem 1 and its corollary hold for any

smooth utilities and asset matrix R, but only when there are more agents than assets, and only for a generic choice of endowments. There are good reasons for this, aside from the technical requirements of the transversality theorem. For example, if the endowment assignement were Pareto optimal, then there would be a unique (no trade) equilibrium allocation, no matter what the asset structure. Of course, if there is only one agent, then all endowment assignments are Pareto optimal, and Theorem 1 could not possibly apply. But if there is more than one agent, then generically an endowment assignement is not Pareto optimal.

Remark 2. Suppose that $B < H + 1 \leqslant S - 1$. In that case there are at least as many dimensions of indeterminacy as there are individual types. One would expect very often to find Pareto comparable financial equilibria.

Remark 3. If the assets were Arrow securities (i.e., every asset pays one dollar in a state of nature and nothing otherwise) then the model is generically determinate (see Geanakoplos and Polemarchakis [9]) Theorem 1 does not apply because R fails to be in general position when $B < S$. See Section III for more on this.

Remark 4. The conclusion of the theorem implies that the set of equilibrium real allocations x contains a nonempty $S - 1$ topological (i.e., C^0) manifold. The conclusion can be strengthened to C^1 manifold (one only needs to show that the derivative of the parameterization has full rank everywhere). Because nothing of economic substance is involved we skip the extra technical work.

Remark 5. The conclusion of the theorem can be sharpened when $H \geqslant SB$. In this case the entire set of equilibrium real allocations can be expressed as the differentiable one-to-one image of an $S - 1$ C^1 manifold (the observation parallel to Remark 4 also applies here). For the proof one considers the function $g: P \times J \to \mathbb{R}^{L(S+1)} \times \mathbb{R}^B \times \mathbb{R}^{SB}$, where J is the $S(B - 1)$ sphere, defined by

$$g(p, q, \Lambda, \omega, z^1, ..., z^s)$$

$$= \left(f(p, q, \Lambda, \omega), \sum_{h=1}^{B} z_h^1 y_1^h, ..., \right.$$

$$\left. \sum_{h=1}^{B} z_h^1 y_B^h, ..., \sum_{h=1}^{B} z_h^S y_1^{(S-1)B+h}, ..., \sum_{h=1}^{B} z_h^S y_B^{(S-1)B+h} \right).$$

Exactly as in the proof of Theorem 1, one shows that 0 is a regular value of g, hence for a generic ω, 0 is a regular value of g_ω. But this is impossible unless $g_\omega^{-1}(0) = \varnothing$ because the range of g_ω has greater dimension than its domain. If ω is generic for f and g we have then that $f_\omega^{-1}(0)$ is an S

manifold. This yields that $E = \{(p, q, \Lambda, \omega) \in f_\omega^{-1}(0): \lambda_1 = 1\}$ is an $S - 1$ manifold. It is easily seen (use Lemma 1) that the real allocations corresponding to *any* two points in E are necessarily distinct.

Remark 6. At the risk of repeating ourselves (see the Introduction), it may be useful to devote a few words to understanding the failure of determinacy in Theorem 1 in the light of the conventional theory of regular economies (see, e.g., Mas-Colell [12, Sects. 8.2 and 8.3]). Formally, our general framework falls within the scope of the theory because money can be viewed as a physical commodity as any other and, similarly, its price is just one more relative price. The reason that the conclusions of the theory (i.e., generically the economy is determinate) do not apply is that technically our universe of admissible economies is degenerate. As long as consumers do not derive direct utility from money and the total endowment of the latter is kept equal to zero the excess demand for money remains null. In fact, the decisive factor is that the total endowment of money be zero (even if money is directly valued at equilibrium its consumption must be zero, i.e., consumers must be at the boundary of their consumption sets and, therefore, their demands for money may be locally insensitive to its relative price). If money aggregate endowments become positive the model is determinate (with unvalued money its price can only be zero).

But even if the general theory does not apply one could ask: How can the presence of an unvalued money commodity available in zero aggregate amount affect the equilibrium prices and allocations of the remaining commodities (we, after all, would not care if the indeterminacy fell entirely on the relative price of money)? The answer should be obvious: the relative price of the money commodity may have real (wealth) effects if consumers arrive to the market with nonzero entitlements of money (aggregating to zero). This can happen even in a one period two commodity world. What the incomplete markets contribute to the story is an endogenous reason (trade at time 0) for the nonzero individual endowments of money in the markets of period one.

Remark 7. Financial assets in our model yield payoffs in what might be called "inside money." The aggregate endowment of each asset, and the aggregate payoff in each state, is zero. This is, of course, of central importance to the indeterminacy that we find in financial assets markets since in any equilibrium for a finite horizon model outside money cannot be positively priced. However, in an infinite horizon model, like the overlapping generations model, it is possible to have nontrivial outside money. One could easily introduce uncertainty and financial assets that have nonzero aggregate supply into an overlapping generations economy. Indeed, what is called money in that model is the archetypical financial asset.

Remark 8. We have considered an economy with only two time periods. This is more general than it may appear at first sight. We could imagine an economy with many time periods, as in Debreu [6], where time and uncertainty resolve themselves as in a tree. If all assets are traded once and for all at data zero then the tree model can be regarded as a special case of our two period model with as many states of nature, as there are nodes in the tree (less one for date 0). The number of states of nature, hence the degree of indeterminacy, can grow geometrically with the length of the time horizon.

Remark 9. The possibility of combining Remarks 7 and 8 is intriguing. One is irresistibly lead to conjecture that in an overlapping generations economy with repeated moves of nature and incomplete *financial* markets there will be an infinity of dimensions of indeterminacy!

Remark 10. Although our theorem only holds for a generic set of endowments, once can guess that there are economies where across states the endowments and von Neumann–Morgenstern utilities are identical, and yet if markets are incomplete, the presence of financial assets creates $S-1$ dimensions of real indeterminacy, i.e., $S-1$ dimension of "sunspot" equilibria.

III. Refinements

In this section we present two refinements of Theorem 1. In the first (done in collaboration with J. Moore) we derive the general formula for the degree of indeterminacy when no restrictions whatsoever are imposed on the return matrix. In the second we discuss our problem when there are both financial and real assets.

For any $S \times B$ matrix R let $K(R)$ be the maximal number of linearly independent subspaces of $\mathbb{R}^B$ which satisfy the property that every subspace contains some nonzero row of R and that every row is contained in some subspace. Let $Z(R)$ be the number of rows which are identically zero. Then Theorem 1 can be strengthtened to:

THEOREM 1'. *If* $H \geqslant B > 0$ *then for any asset matrix* R *there are, generically,* $S - K(R) - Z(R)$ *dimensions of real indeterminacy.*

Proof. We give the proof for the case $Z(R) = 0$. Accounting for the more general case is a trivial matter.

Denote $K = K(R)$. By hypothesis we can assume that the rows of R have been renumbered so that

$$R = \begin{bmatrix} R_1 \\ \vdots \\ R_K \end{bmatrix},$$

where the rows of the matrices $\{R_1, ..., R_K\}$ span linearly independent subspaces of $\mathbb{R}^B$.

Let M_R be the collection of positive diagonal matrices having the properties that for every k the diagonal entries of the k-block are all equal (and denoted λ_k).

LEMMA 5. *We have* $sp[\Lambda R] \subset sp[R]$ *if and only if* $\Lambda \in M_R$.

Proof. (i) Necessity. By Lemma 4 if two rows of some R_k are associated with different diagonal entries of Λ then it is possible to split the rows of R_k so that they generate two linearly independent subspaces. But this contradicts the maximality property of K. Hence $\Lambda \in M_R$.

(ii) Sufficiency. Let $\Lambda \in M_R$. Then $qR = 0 \Leftrightarrow q_k R_k = 0$ for all $k \Leftrightarrow \lambda_k q_k R_k = 0$ for all $k \Leftrightarrow q\Lambda R = 0$. The first and last implication follows from the linear independence of the subspaces generated by the rows of the different R_k (or $\lambda_k R_k$). Hence $sp[\Lambda R] = sp[R]$. ∎

Because $\dim M = S$ and $\dim M_R = K$ Lemma 5 implies that there are precisely $S - K$ directions of perturbations Λ of the identity for which $sp[\Lambda R] \neq sp[R]$. Hence, noting that Steps 1 to 3 of the proof of Theorem 1 never use the hypothesis "$B < S$ and R is in general position," we have proved the more general Theorem 1'. ∎

When R general position, Theorem 1 sharply distinguishes between the complete asset markets case ($B \geqslant S$), and the incomplete asset markets case ($B < S$) for which there are $S - 1$ dimensions of real indeterminacy. The general picture, however, is given by Theorem 1'.

Notice first that if the asset markets are complete, i.e., if all the rows of R are linearly independent, then $Z(R) = 0$ and $K(R) = S$, because the one-dimensional subspaces spanned by each row separately are linearly independent. Hence the dimension of indeterminacy is $S - S - 0 = 0$, as it should be. Furthermore, if R consists of B distinct Arrow securities, then $K(R) = B$, $Z(R) = S - B$, and again the dimension of indeterminacy is $S - (S - B) - B = 0$, as it should be. If R is in general position and $0 < B < S$, then $K(R) = 1$ and $Z(R) = 0$, as we argued in the last two paragraphs of the proof of Theorem 1.

There is another special case which is of interest, and which indicates that the dimension of indeterminacy in practice is probably considerably less than $S-1$. Suppose that S can be partitioned into disjoint subsets $S = S_1 \cup \cdots \cup S_K$ and that for each $i \leqslant K$, there is an asset in R which pays out one dollar if $s \in S_i$ and nothing otherwise. Then clearly $K(R) \geqslant K$, and so the dimension of indeterminacy is at most $S - K$. Observe, however, that the formula can also be made to yield a lower bound. As long as every state can be reached by at least one asset (i.e., $Z(R) = 0$) the dimension of indeterminacy must be at least $S - B \leqslant S - K(R)$.

Our second refinement takes as starting point the fact that in actuality there are both nominal and real assets. It may seem reasonable to conjecture that the larger is the proportion of real assets, the smaller is the indeterminacy associated with financial assets. However, we now show that the dimension of real indeterminacy is robust to the introduction of real securities, as long as markets are sufficiently incomplete. We will not make here an effort to get the best possible result.

In order to avoid the difficulties with existence that are known to plague models with real assets which yield vector-valued payoffs (see Hart [11]), we shall confine our attention to real numeraire assets, i.e., real assets that, for each state $s \in S$, pay only in commodity 0.

Let R be the $S \times B$ matrix representing the B financial assets and let $\bar{R}$ be the $S \times A$ matrix representing the real numeraire assets. Thus r_{sb} is the number of dollars paid by financial asset b in state s, and $\bar{r}_{sa}$ is the amount of good 0 paid by real asset a in state s.

The definition of an equilibrium is now a triple $(x, y, \bar{y})$ satisfying (i) and (ii) of Definition 1 with the budget set $B^h(p, q, \bar{q})$ defined as

$$\left\{ (x, y, \bar{y}): p_0 \cdot x_0 + q \cdot y + \bar{q} \cdot \bar{y} \leqslant p_0 \cdot \omega_0^h \text{ and} \right.$$

$$\left. p_s \cdot x_s \leqslant p_s \cdot \omega_s^h + \sum_{b=1}^{B} y_b^h r_{sb} + p_{s0} \sum_{a=1}^{A} y_a^h \bar{r}_{sa} \text{ for all } s \right\}.$$

THEOREM 2. *Suppose that $B \geqslant 2$, $S > 2(A + B)$, and $H > A + B$. Then for a generic choice of matrices R and $\bar{R}$, there is a generic set of endowments such that each of the corresponding economies has $S - 1$ dimensions of real indeterminacy (in the sense of Theorem 1).*

Proof By following the logic of the proof of Theorem 1, it suffices to show that for a generic choice of matrices R and $\bar{R}$, there is no diagonal matrix $\Lambda \neq \sigma I$ with $\Lambda R \subset sp[R, \bar{R}]$.

Let $\mathscr{D}$ be the set of $S \times (B + A)$ matrices $W = (R, \bar{R})$ which have rank $B + A$, satisfy $r_{sb} \neq 0$ for $b \in \{1, 2\}$ and $s \in S$, and have $r_{s1}/r_{s'1} \neq r_{s2}/r_{s'2}$ for all $s \neq s' \in S$. Clearly, $\mathscr{D}$ is an open, dense subset of all $S \times (B + A)$ matrices. It

has a complement of null measure. We shall show that there is a generic subset $\mathcal{D}' \subset \mathcal{D}$ of matrices $(R, \bar{R})$ for which only diagonal matrices Λ that are multiples of the identity satisfy $\Lambda R \subset sp[R, \bar{R}]$.

Suppose in particular that $\Lambda R^1 \in sp[R, \bar{R}]$ and $\Lambda R^2 \in sp[R, \bar{R}]$, where R^1, R^2 are, respectively, the first and second column of R. Since we can rewrite

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_S \end{bmatrix} \begin{bmatrix} r_{11} \\ \vdots \\ r_{S1} \end{bmatrix} = \begin{bmatrix} 1/r_{11} & & 0 \\ & \ddots & \\ 0 & & 1/r_{S1} \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_S \end{bmatrix}$$

we must have two $(A + B)$-dimensional vectors z and $\hat{z}$ with

$$\begin{bmatrix} 1/r_{11} & & 0 \\ & \ddots & \\ 0 & & 1/r_{s1} \end{bmatrix} [R, \bar{R}] z = \begin{bmatrix} 1/r_{12} & & 0 \\ & \ddots & \\ 0 & & 1/r_{S2} \end{bmatrix} [R, \bar{R}] \hat{z}.$$

We know, of course, We know, or course, that $z = (\lambda, 0, ..., 0)$, $\hat{z} = (0, \lambda, 0, ..., 0)$ is always a solution for any λ. We show that for a generic choice of R and $\bar{R}$, there is no other choice of z and $\hat{z}$ that constitutes a solution.

Note first that if $S > 2(A + B)$, then there are more equations to satisfy than there are unknowns. It suffices to show, therefore, that eliminating from the domain the previous special configuration of z and $\hat{z}$, the above system of equations has zero as a regular value. That is, it suffices that given any R, $\bar{R}$ and solution $z \neq (\lambda, 0, ..., 0)$, $\hat{z} \neq (0, \lambda, 0, ..., 0)$, we can perturb any equation s by changing the R, $\bar{R}$, z, $\hat{z}$ in such a manner that the remaining equations are not disturbed. A routine application of the Transversality Theorem would then finish the proof.

Suppose that for a solution R, $\bar{R}$, z, $\hat{z}$, there is some k, $3 \leqslant k \leqslant A + B$ with either $z_k \neq 0$ or $\hat{z}_k \neq 0$ (or both). It follows that $z_k/r_{s1} \neq \hat{z}_k/r_{s2}$ for at least $S - 1$ of the S states. For any such state s, a small perturbation of w_{sk} (if $k \leqslant B$, $w_{sk} = r_{sk}$; if $B < k \leqslant A + B$, $w_{sk} = \bar{r}_{s, A+B-k}$) will change the sth equality without disturbing the rest. For the remaining state s_0, one can change z_1. That will affect every equality, including s_0; but this is clearly a perturbation with an effect which is independent of the other $S - 1$ perturbations.

Suppose alternatively that $z_k = \hat{z}_k = 0$ for all $k \geqslant 3$. Then we are back to exactly the framework of Step 4 in the proof of Theorem 1, with only two financial assets, and we know that there are no solutions to the system of equations except for $z = (\lambda, 0, ..., 0)$, $\hat{z} = (0, \lambda, ..., 0)$, which we have excluded from the domain.

Remark 11.   More generally, we could, and should, also consider mixed assets which pay both in real commodities and in money. Once again there will be natural sufficient conditions guaranteeing that the dimension of indeterminacy is $S - 1$. For example, suppose that for each asset the states can be divided into those in which the asset pays in units of account and those in which the asset pays in numeraire commodities. Loans with collateral are of this type: there is a specified financial payment and a real collateral payoff in case of default (which here should be thought of as an exogenous event). One could also think of form-issued debt in similar terms. Let A be the total number of mixed assets. Suppose that for every $s \in S$ there are two assets and a collection $F(s) \subset S$ of at least $2A + 1$ states (including s) on which the two assets both pay in money. Then the proof of Theorem 2 does easily yield that there are $S - 1$ dimensions of real indeterminacy.

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