Cowles Foundation Paper 749

Journal of Mathematical Economics 19 (1990) 69–93 North-Holland

SOLVING SYSTEMS OF SIMULTANEOUS EQUATIONS IN ECONOMICS*

John GEANAKOPLOS

Cowles Foundation, Yale University, New Haven, CT 06520–2125, USA

Wayne SHAFER

University of Southern California, Los Angeles, CA 90089–0152, USA

Submitted December 1987, accepted April 1989

We show that there is a broad range of systems of simultaneous equations that arise in economics as descriptions of equilibrium that can be solved in elementary fashion via degree theory Some of these systems are not susceptible to analysis by standard Brouwer fixed point methods. Two of our applications are to general equilibrium with incomplete markets, and to non-convex production with non-competitive pricing rules

1. Introduction

Equilibrium in economic models is often described by a solution or zero to a system of simultaneous equations: $z: N \to \mathbb{R}^l$. The standard technique for proving the existence of equilibrium is to transform the system by finding a function $f: \bar{N} \to N$, where $\bar{N}$ is the closure of N, such that $f(x)=x$ if and only if $z(x)=0$. If f is continuous and $\bar{N}$ is homeomorphic to a compact, convex set, then by Brouwer's fixed point theorem, f must indeed leave some $\bar{x} \in N$ fixed.

This fixed point approach works especially well for the Walrasian system where N is S^{l-1}_{++}, the interior of the $(l-1)$-simplex, and z is a continuous function that is bounded from below, satisfying Walras Law: $z(p) \cdot p = 0$ for $p \in S^{l-1}_{++}$, and a boundary condition such as: $p_n \to p \in \partial S^{l-1}_{++}$ only if $|z(p_n)| \to \infty$. Indeed, as Uzawa (1962) has pointed out, the Gale–Nikaido–Debreu proposition that every Walrasian system has a solution is trivially equivalent to Brouwer's fixed point theorem. It is well-known that the Arrow–Debreu model of consumer general equilibrium, with strictly convex and monotonic preferences, gives rise to a Walrasian system. Furthermore, from the

*This work was supported in part by NSF grants SES 88-12051 and SES 87-09990.

Sonnenschein–Debreu–Mantel theorems on the arbitrariness of Arrow–Debreu excess demand, it follows that the existence of equilibrium for all strictly convex and monotonic Arrow–Debreu economies is also equivalent to Brouwer's fixed point theorem. Note that convexity hypotheses occur in two separate places in the Arrow–Debreu model. Applied to consumer preferences, they guarantee continuity of all the individual, and hence the aggregate, excess demand functions. Secondly, they guarantee that the set of endogenous variables (prices, *etc.*) is convex, so that Brouwer's fixed point theorem is applicable. For a history of the application of Brouwer's theorem to the Walrasian system and Arrow–Debreu economies following McKenzie (1954) and Arrow and Debreu (1954), see Debreu (1982).

The fixed point method is precisely strong enough to solve the Walrasian system, but there are other classes of economic models in which the space of endogenous variables need not be convex. For example, the excess demand z in a production economy depends on prices and the production choice y. The production choice itself may be constrained to maximize profits in some (non-convex) set Y at the prices p, or to satisfy some other criterion, such as marginal or average cost pricing, given by p. In the theory of general equilibrium with incomplete asset markets, excess demand may be thought of as depending on the prices p, and a k-dimensional subspace L of $\mathbb{R}^S$ which is contrained by p in the sense that L contains the span of the columns of a matrix $M(p)$.

More generally, let N be a subset of $S^{l-1}_{++} \times A$, where A is an auxiliary set of constraining variables. Let $z: N \to \mathbb{R}^l$ represent an economic system of simultaneous equations. Our purpose is to give a sufficient condition for there to be a solution $(\bar{p}, \bar{a}) \in N$ satisfying $z(\bar{p}, \bar{a}) = 0$. In the special case that N is the graph of a continuous function $\varphi: S^{l-1}_{++} \to A$, so that $N = \text{Graph}\,\varphi = \{(p, \varphi(p)) \mid p \in S^{l-1}_{++}\}$, the analysis reduces to the Walrasian system, for which Brouwer's theorem is applicable, provided that Walrasian-like hypotheses are made about z. However, when N has a more complicated structure, a more powerful technique than Brouwer's theorem is required. For such N, it is not even clear how to formulate the equation $z(p, a) = 0$ as a fixed point problem. We will show, nevertheless, that under Walrasian-like assumptions on z, to guarantee the existence of a solution it suffices that (1) N be an $(l-1)$-dimensional topological manifold; (2) the projection map $proj_1: N \to S^{l-1}_{++}$ given by $proj_1(p, a) = p$ is proper; and (3) $deg(proj_1)$ is non-zero. It is possible to find such sets N that are not homeomorphic to any convex body.

There are two ways in which N naturally arises which shall be of central concern to us. In the first we are given a correspondence $\Phi: S^{l-1}_{++} \to A$ and N is the graph of Φ, $N = \{(p, a) \mid a \in \Phi(p)\}$. We seek $(\bar{p}, \bar{a})$ such that $z(\bar{p}, \bar{a}) = 0$ and $\bar{a} \in \Phi(\bar{p})$. If Φ is upper hemi-continuous, then $proj_1$ is proper. Furthermore, if there is any open neighborhood $U \subset S^{l-1}_{++}$ such that, when restricted to $U, \Phi(p) = \{\varphi_1(p), \ldots, \varphi_n(p)\}$ where $\varphi_i: U \to A$ is a continuous function and n is

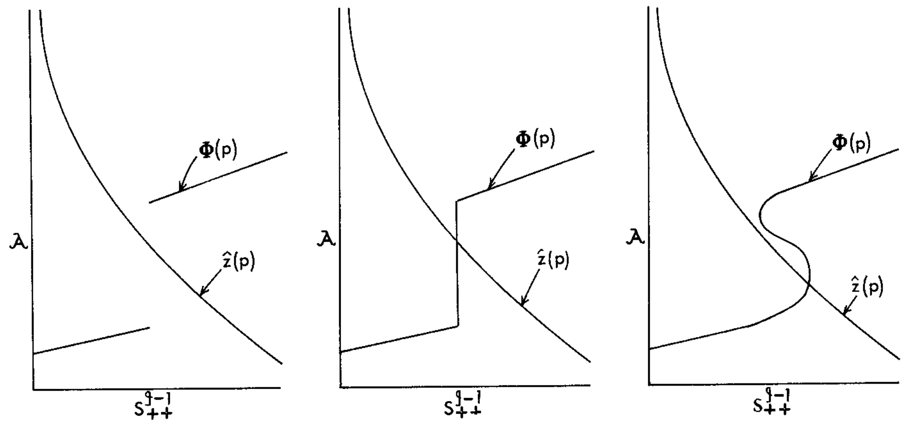

Fig. 1

odd, then $proj_1$ is of non-zero degree. To see the significance of the $(l-1)$-manifold hypothesis, consider figs. 1a, b, c.

In all three diagrams $A=\mathbb{R}_{++}$, and z has the functional form $z(p,a)=\hat{z}(p)-a\in\mathbb{R}$, so that the graph of z fits in the same diagram as the graph of Φ. Note that z is Walrasian-like. In all three diagrams Φ is upper hemi-continuous and there is an open set U on which Φ is a function. In fig. 1a Graph Φ is not a manifold and there is no solution to $z(p,a)=0$ and $a\in\Phi(p)$. In fig. 1b, Φ satisfies the further property that for each p, $\Phi(p)$ is convex valued (and N is also a topological manifold). Here there is a solution. Indeed a familiar argument along the lines of Kakutani's fixed point theorem shows that with convexity there must be a solution. In fig. 1c, $\Phi(p)$ is not convex valued. But $N=\text{Graph}\,\Phi$ is an $(l-1)$-manifold and there is a solution.

As we have said, we have in mind one application where Φ is the (aggregate) supply correspondence of the productive sector of an economy where production is chosen according to some rule, perhaps not profit maximization, where firms may not have convex technologies.

In another application to financial models of general equilibrium we think of A as the collection of all J-dimensional subspaces of some Euclidean space $\mathbb{R}^S$, $S>J$. Here $\mathbb{R}^S$ represents the state contingent money payoff space, and a subspace of $\mathbb{R}^S$ represents the set of attainable payoff vectors. It is well known that one can find a topology in which A, called the Grassmanian, is a $J(S-J)$-dimensional compact manifold. Let $v\colon S^{l-1}_{++}\to\mathbb{R}^{S\times J}$ be an arbitrary continuous function denoting the payoff of each of the J assets in each of the S states of nature. In this application we let $\Phi(p)$ be the collection of all subspaces in A that contain all the asset payoff vectors, $\Phi(p)=\{a\in A\,|\,a\supset span\langle v(p)\rangle\}$. We will show that v can always be approximated arbitrarily closely by a function $\tilde{v}\colon S^{l-1}_{++}\to\mathbb{R}^{S\times J}$, such that, letting $\tilde{\Phi}(p)=\{a\in A\,|\,a\supset span\langle\tilde{v}(p)\rangle\}$ and $\tilde{N}=\text{Graph}\,\tilde{\Phi}$, all three properties above hold. We can conclude that there must be some $(\bar{p},\bar{a})$ with $z(\bar{p},\bar{a})=0$ and $\bar{a}\in\tilde{\Phi}(\bar{p})$, and since the approximation is arbitrarily close, there must also be a solution to $z(p,a)=0$ and $a\supset span\langle v(p)\rangle$.

The second way N naturally arises is as the solution to a system of simultaneous equations. Let $f\colon S^{l-1}_{++}\times A^r\to\mathbb{R}^r$, where we have placed a superscript r on A to suggest that in this class of problems we take A to be a manifold of the same dimension as the range of f. Then $N=f^{-1}(0)$. We seek $(\bar{p},\bar{a})$ that simultaneously solves $z(\bar{p},\bar{a})=0$ and $f(\bar{p},\bar{a})=0$. Needless to say, we cannot solve both systems unless we know that $f(\bar{p},\bar{a})=0$ can be solved, which is to say that there must be a $\hat{p}\in S^{l-1}_{++}$ such that $f_{\hat{p}}(a)=f(\hat{p},\bar{a})=0$ can be solved for a. Suppose there is a $\hat{p}$ such that $deg(f_{\hat{p}},0)\neq 0$. Then we need only check that $proj_1\colon N=f^{-1}(0)\to S^{l-1}_{++}$ is proper to show that the system (z,f) has a zero. In applications it will typically be much easier to check that there is one $\hat{p}$ at which $f_{\hat{p}}$ has non-zero degree than it is to show that (z,f) has non-zero degree.

There are already two closely related alternatives to Brouwer's theorem for demonstrating the existence of a solution to the Walrasian system. In the path-following approach Smale derived from Hirsch's mathematical work, one solves $z(p)=\lambda y$ for some λ and $y\in\mathbb{R}^l$, and then follows the solutions as $\lambda\to 0$. In the homotopy approach, one finds a continuous family of functions $z_\lambda\colon N\to\mathbb{R}^l$ such that z_0 has a unique solution and $z_1=z$. One then follows the corresponding solution to $z_\lambda=0$, beginning where $z_0=0$, until a solution to $z_1=z=0$ is found. Homotopy methods were brought to the attention of economists primarily by the book of Milnor (1976). Scarf, Eaves, Balasko, and Smale, among others, have all used path-following or homotopy techniques for demonstrating existence and/or computing solutions to the Walrasian system, in lieu of applying Brouwer's theorem. Of course these same techniques have been used to prove Brouwer's theorem directly. But the important point for us is that the path-following techniques are potentially more powerful than Brouwer's fixed point theorem. In this paper we describe an economic framework more general than the Walrasian system to which Brouwer's theorem does not apply, but for which these two techniques nevertheless guarantee the existence of a solution to the resulting simultaneous equation system z. Oriented degree theory, or the related fixed point index theory and the index theory of vector fields, has been used by Dierker (1972), Balasko (1975), and Smale (1974) in the Walrasian setting, by, Mas-Colell (1977) and Kehoe (1980) in the Walrasian setting with convex production, and by Kamiya (1988) in the case with non-convex production. Duffie and Shafer (1985) used Mod 2 degree theory to demonstrate the existence of equilibrium in the incomplete markets exchange economy setting. In section 2 we provide a general framework which in principle includes these examples as special cases, and which we hope will be applicable to other similar problems. In section 3 we give some applications. In Appendix

A we give a second proof of our main theorem based on first principles (which perhaps could be used as the basis of a computational algorithm).

2. The main result

Let $S^{l-1}_{++}=\{(p_1,\dots,p_l)\in\mathbb{R}^l\,|\,p_i>0;\ \sum_{i=1}^{l}p_i=1\}$ be the interior of the $l-1$ simplex, and let $\mathscr{S}^{l-1}_{++}=\{(p_1,\dots,p_l)\in\mathbb{R}^l\,|\,p_i>0$ and $\sum_{i=1}^{l}p_i^2=1\}$. In the following pages we will nearly always use S^{l-1}_{++}, but nothing of substance would need to change if we substituted $\mathscr{S}^{l-1}_{++}$. Let A be a topological space. Let $N\subset S^{l-1}_{++}\times A$. Suppose that $z\colon S^{l-1}_{++}\times A\to\mathbb{R}^l$. What conditions on z and N suffice to guarantee the existence $(\bar{p},\bar{a})\in N$ of a solution to $z(p,a)=0$?

We begin by defining the Walrasian-like hypotheses we need for z to satisfy:

Definition. The function $z\colon S^{l-1}_{++}\times A\to\mathbb{R}^l$ is C^k-Walrasian-like (C^k means k-times continuously differentiable) on a set $N\subset S^{l-1}_{++}\times A$ if it satisfies (1)–(3) below:

(1) A is a C^k-manifold and z is C^k.
(2) Walras Law: If $(p,a)\in N$, then $p\cdot z(p,a)=0$.
(3) Boundary: If $(p^n,a^n)\in N$, and $p^n\to p\in\partial S^{l-1}_{++}$, then

$$\max_{\{i\,:\,p_i>(1/l)\}}\ \limsup_n z_i(p^n,a^n)>0.$$

The similarity to the Walrasian system z is of course unmistakable. Note that C^0 means continuous, and that continuity alone is sufficient for our existence theorems. If (1) holds and if (2) holds more generally for $(p,a)\in S^{l-1}_{++}\times A$, and if (3) can be strengthened so that $(p^n,a^n)\in S^{p-1}_{++}\times A$ and $p^n\to p\in\partial S^{l-1}_{++}$ implies that $z_i(p^n,a^n)\to\infty$ for some i with $p_i=0$, then we say that z is strongly Ck-Walrasian like. We now turn to N.

Definition. N is C^k-regular, if N is a $(l-1)$-dimensional C^k-manifold, a submanifold of $S^{l-1}_{++}\times A$ if $k\geqq 1$. N is proper if the map $proj_1\colon N\to S^{l-1}_{++}$ is proper, i.e., if $K\subset S^{l-1}_{++}$ is compact, then $\{(p,a)\in N\,|\,p\in K\}$ is compact.

Definition. (z,A,N) is C^k-admissible if z is C^k-Walrasian-like and N is C^k-regular and proper.

Remark 1. Suppose that $\Phi\colon S^{l-1}_{++}\to A$ is a correspondence, and $N=\operatorname{Graph}\Phi$. Then $proj_1$ is proper if and only if Φ is upper hemi-continuous; in particular if A is compact then $proj_1$ is proper if and only if N is closed in $S^{l-1}_{++}\times A$.

Example 1. An important kind of N arises as the solution to a system of r-simultaneous equations which depend on r auxiliary variables and the prices p. Let A be a compact, r-dimensional C^k-manifold, and let $f\colon S^{l-1}_{++}\times$

$A \to \mathbb{R}e^r$. Suppose that f is C^k, for $k \geqq 1$, and f is transverse to zero [i.e. if $f(p,a)=0$, then rank $Df(p,a)=r$], which we write $f \pitchfork 0$. Then $N=f^{-1}(0)$ is an $(l-1)$-dimensional, C^k-manifold, and $proj_1: N \to S^{l-1}_{++}$ is proper. As we shall see, in the applications of the theorem we shall present, we will also get the ontoness of $proj_1$ as well. Furthermore, in many applications a specific knowledge of f guarantees that N is proper even when A is not compact. Finally, this example can be slightly generalized to the case where there is a family of C^k-functions $f_i: T \times A_i \to \mathbb{R}^r$, where A_i is open in A and $\bigcup_{i=1}^n A_i = A$. We then require that each $f_i \pitchfork 0$, and if $a \in A_i \cap A_j$, and $f_i(p,a)=0$ for some $p \in S^{l-1}_{++}$ then $f_j(p,a)=0$.

Example 2. Let $Y \subset \mathbb{R}^l$ be a compact set with a smooth boundary ∂Y of dimension $l-1$. To each point $y \in \partial Y$ associate the outward pointing normal vector $D(y) \in \mathscr{S}^{l-1}$. Then the map $f: \mathscr{S}^{l-1}_{++} \times \partial Y \to \mathbb{R}^{l-1}$ given by $f(p,y)=[p-D(y)]_1^{l-1}$ is trivially transverse to zero, if D is a smooth function. Let $N=f^{-1}(0)$. Moreover, from the fact that $\max_{y \in \partial y} p \cdot y$ must have a solution, $proj_1(N)=\mathscr{S}^{l-1}_{++}$. When Y is convex, N corresponds to the profit maximizing choices of the firm. In the general case N corresponds to marginal cost pricing. Note that in neither case is it necessary that for all $p \in \mathscr{S}^{l-1}_{++}$, there be a unique, or even finite number of choices y in ∂Y with $(p,y) \in N$. For example, Y could display constant returns to scale over some bounded range, and then for some price there would be a continuum of profit maximizing choices.

We turn to the last property we shall impose on N.

We shall see that degree theory provides not only a powerful tool for determining existence of equilibria but also for determining index theorems and generic local uniqueness, giving a deeper insight into the structure of equilibria. For the reader unfamiliar with the details of degree theory Appendix B has been provided.

Let N and Y be C^0 manifolds of the same dimension, and suppose Y is connected and orientable, and that $0 \in Y$. Let $\hat{z}: N \to Y$, and suppose that $\hat{z}^{-1}(0)$ is compact. If N is oriented, then the oriented degree of $\hat{z}$ at 0, written $deg_\theta(\hat{z},0)$, is well defined. If N is not oriented, then the mod 2 degree of $\hat{z}$, written $deg_2(\hat{z},0)$ is well defined. If $deg_\theta(\hat{z},0) \neq 0$ or $deg_2(\hat{z},0) \neq 0$ then $\hat{z}^{-1}(0) \neq \emptyset$, and this is the basis for degree theory as an existence tool. Degree theory can also be used as a stability tool. If the degree of $\hat{z}$ is non-zero, there is a non-empty compact connected subset K of $\hat{z}^{-1}(0)$, which is stable in the sense that for any open neighborhood U of K, all sufficiently small perturbations g of $\hat{z}$ have at least one solution $x \in U \cap g^{-1}(0)$. We call such a set K a stable set of equilibria. If $\hat{z}^{-1}(0)$ is a finite set and $\hat{z}$ is a local homeomorphism in a neighborhood of each point in $\hat{z}^{-1}(0)$, then $deg_\theta(\hat{z},0)$ gives the number of points in $\hat{z}^{-1}(0)$, counted with orientation, which gives a basis for computing index formulas. In these circumstances, $deg_2(\hat{z},0)$ counts the number of points in $\hat{z}^{-1}(0) \bmod 2$, i.e., $deg_2(\hat{z},0)=1$ if $\#z^{-1}(0)$ is odd and $deg_2(\hat{z},0)=0$ if $\#z^{-1}(0)$ is even. Since S^{l-1}_{++} is connected and $proj_1: N \to S^{l-1}_{++}$ is proper, the degree of $proj_1$ is also well defined and independent of the point at which it is computed. We write $deg_\theta(proj_1)$ or $deg_2(proj_1)$ for this degree. It will turn our that for admissible (z, A, N) a sufficient condition for the existence of a zero of z is that the degree of $proj_1$ is not zero; for this reason we turn to some examples of computing the degree of $proj_1$.

Definition. N has the *global graph property* if there is a function $\varphi: S^{l-1}_{++} \to A$ with $N = \text{Graph}\,\varphi = \{(p, \varphi(p)) \mid p \in S^{l-1}_{++}\}$.

Clearly if N has the global graph property, then $deg_2(proj_1)=1$. In this case also we can orient N by the homeomorphism $\tilde{\varphi}: S^{l-1}_{++} \to N$ defined by $\tilde{\varphi}(p)=(p, \varphi(p))$, and with this orientation $deg_\theta(proj_1)=1$.

Definition. N has the *local graph property* if there is a non-empty open set $U \subset S^{l-1}_{++}$ and a function $\varphi: U \to A$ with $N \cap (U \times A) = \text{Graph}\,\varphi$.

In this case we have $proj_1^{-1}(\bar{p}) \subset U \times A$ for $\bar{p} \in U$, so that

$$deg\,(proj_1, \bar{p}) = deg\,(proj_1|_{(U \times A) \cap N}, \bar{p}).$$

But $proj_1|_{(U \times A) \cap N}$ is a local homeomorphism and since $deg\,(proj_1) = deg\,(proj_1, p)$ for any $p \in S^{l-1}_{++}$, this gives either $|deg_\theta(proj_1)| = 1$ or $deg_2(proj_1)=1$.

Definition. N satisfies the *generic graph property* if there is an open, dense subset $U \subset S^{l-1}_{++}$ and a function $\varphi: U \to A$ such that $N \cap (U \times A) = \text{Graph}\,\varphi$.

Example 3. Let A be a compact r-dimensional manifold, as in example 1 and let $f: S^{l-1}_{++} \times A \to \mathbb{R}^r$ be transverse to zero and smooth. Let $N = f^{-1}(0)$. Then by the transversality theorem, there is an open, dense subset U of S^{l-1}_{++} such that for each $p \in U$, $N \cap (\{p\} \times A)$ is a finite set. We can calculate $deg\,(proj_1) = deg\,(proj_1, p)$ for any $p \in U$. Note that $deg_2(proj_1, p)$ counts the number of elements in $N \cap (\{p\} \times A) \bmod 2$, while $deg_\theta(proj_1, p)$ counts the same elements with orientation. The generic graph property requires that U can be chosen so that $N \cap (\{p\} \times A)$ is a singleton, for all $p \in U$.

Example 4. Let $A = \mathbb{R}^r$, and suppose that $f: S^{l-1}_{++} \times A \to \mathbb{R}^r$ is transverse to 0. Furthermore, suppose that $N = f^{-1}(0)$ is proper and onto. Finally, suppose that for any fixed $p \in S^{l-1}_{++}$, $f_p: A = \mathbb{R}^r \to \mathbb{R}^r$, given by $f_p(a) = f(p, a)$, is affine in a, i.e., there exists $C = C(p) \in \mathbb{R}^r$ and $B = B(p) \in \mathbb{R}^{r \times r}$ such that $f_p(a) = C + Ba$. Then again by the transversality theorem and properness, there is an open, dense subset $U \subset S^{l-1}_{++}$ such that for all $p \in U$, $N \cap (\{p\} \times A)$ is a finite set. But from the affineness of f_p, it contains exactly one element, or else a continuum of elements, or it is empty. By the ontoness of N it cannot be empty, so the generic graph property holds for N.

Example 5. A is an oriented $(l-1)$-manifold, $F: A \to S^{l-1}_{++}$ is a proper continuous map, and $N = \{(p, a): p = F(a)\}$. Then N is proper, and in this case $F = proj_1 \circ \rho$, where $\rho: A \to N$ is the homeomorphism defined by $\rho(a) = (F(a), a)$. If we give N the oriented manifold structure induced by ρ, then $deg_\theta(\rho) = 1$, and thus

$$deg_\theta(F) = deg_\theta(proj_1)\, deg(\rho) = deg_\theta(proj_1).$$

Remark 2. The hypotheses that N is an $(l-1)$-dimensional topological manifold, that $proj_1: N \to S^{l-1}_{++}$ is proper, and that N satisfies the local graph property imply that the map $proj_j: N \to S^{l-1}_{++}$ is onto.

Remark 3. Since $proj_1: N \to S^{l-1}_{++}$ is onto, we may define the correspondence $\Phi: S^{l-1}_{++} \to A$ by $\Phi(p) \equiv \{a \in A \mid (p, a) \in N\}$. The local graph property implies that there is some open set $U \subset S^{l-1}_{++}$ such that restricted to U, the correspondence Φ is actually a function φ.

Remark 4. Similarly one might define the correspondence $Z: S^{l-1}_{++} \to \mathbb{R}^l$ by $Z(p) = \{z(p, a) \mid a \in \Phi(p)\}$. This correspondence is non-empty-valued, and upper hemi-continuous. Compared to the famous so-called Walras excess-demand correspondence, the only property that it does not satisfy is convex-valuedness.

Remark 5. Any non-empty-valued, upper hemi-continuous, convex valued correspondence can be approximated arbitrarily closely by a continuous function. Indeed that is how Kakutani's fixed point theorem is derived from Brouwer's fixed point theorem. But it is easy to see that there can be N (or equivalently Z) satisfying regularity, properness, and the local graph property that cannot be globally approximated by a continuous function on S^{l-1}_{++}. Consider for example the simple situation depicted in fig. 2.

When $l > 2$, N may also have the generic graph property, without being globally representable, even approximately, by a continuous function.

Notation. Let $H^n_b = \{x \in \mathbb{R}^{n+1} \mid \sum_i x_i = b\}$. Note that H^n_b is an n-dimensional hyperplane in $\mathbb{R}^{n+1}$, and that S^{l-1}_{++} is an open subset of H^{l-1}_1. $\hat{p} \in \mathbb{R}^l$ will always denote $\hat{p} = (1/l, 1/l, \ldots, 1/l)$. We always consider H^n_b to be the n-dimensional oriented smooth manifold with atlas $\{(H^n_b), (\psi_b)\}$, where $\psi_b: H^n_b \to \mathbb{R}^n$ is defined by

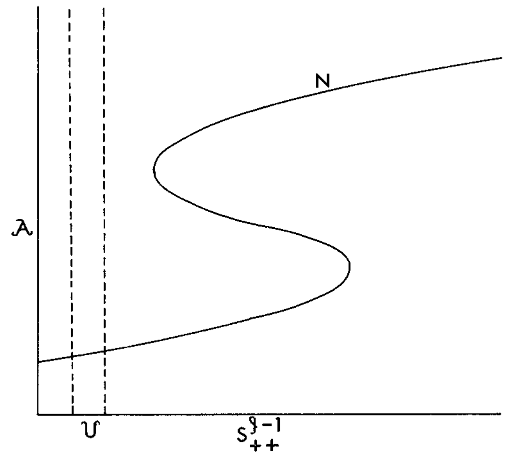

Fig 2

$$\psi_b(x_1,\dots,x_n,x_{n+1})=(x_1,\dots,x_n).$$

S^{l-1}_{++}, as an open subset of H^{l-1}_1, is always given the induced smooth, oriented manifold structure.

Given a (z,A,N) C^k admissible, $\hat{z}:N\to H^{l-1}_0$ will always denote the C^k function on N defined by $\hat{z}(p,a)=(p_1z_1(p,a),\dots,p_lz_l(p,a))$. Note that $\hat{z}^{-1}(0)=z^{-1}(0)\cap N$.

We are now ready to state our main theorem.

Theorem 1. Suppose (z,A,N) is C^0-admissible. Then:

(1) If N is oriented:

$$deg_\theta(\hat{z},0)=(-1)^{l-1}\,deg_\theta(proj_1).$$

(2) If N is not oriented or even not orientable:

$$deg_2(\hat{z},0)=deg_2(proj_1).$$

Corollary 1. Suppose (z,A,N) is C^0-admissible. If $deg_2(proj_1)\neq 0$ or $deg_\theta(proj_1)\neq 0$, then there is a stable set K of solutions to $z(p,a)=0,(p,a)\in N$.

Corollary 2. Let $z:S^{l-1}_{++}\times A^r\to\mathbb{R}^l$ be C^0-Walrasian-like, and let $f:S^{l-1}_{++}\times A^r\to\mathbb{R}^r$ be continuous. Suppose there is at least one $\tilde{p}\in S^{l-1}_{++}$ such that if $f_{\tilde{p}}(a)\equiv f(\tilde{p},a)$, then $deg(f_{\tilde{p}},0)$ is non-zero. Moreover, letting $N=f^{-1}(0)$, suppose $proj_1:N\to S^{l-1}_{++}$ is proper. Then there is a solution $(\bar{p},\bar{a})$ to the system of simultaneous equations $z(\bar{p},\bar{a})=0$ and $f(\bar{p},\bar{a})=0$.

Remark 6. If N has the global graph property, then it is homeomorphic to S^{l-1}_{++} and defining $z':S^{l-1}_{++}\to\mathbb{R}^l$ by $z'(p)=z(p,\varphi(p))$ Theorem 1 is reduced to the standard Walrasian existence problem.

Lemma 1. *Suppose (z, A, N) is C^k-admissible for some $k \geqq 0$. Then there is some open, convex set $S_\varepsilon^{l-1} \subset S_{++}^{l-1}$, with non-empty smooth boundary $\partial \bar{S}_\varepsilon^{l-1} \subset S_{++}^{l-1}$, such that if $p \in S_{++}^{l-1} \backslash S_\varepsilon^{l-1}$, and $(p, a) \in N$, then $\hat{z}(p, a) = \lambda(p - \hat{p})$ has no solution $\lambda \geqq 0$. In particular, if $h: [0,1] \times N \to H_0^{l-1}$ is defined by $h(\lambda, (p, a)) = \lambda \hat{z}(p, a) + (1 - \lambda)(\hat{p} - p)$, then h is C^k and $h^{-1}(0)$ is compact.*

Proof. Suppose $(p^n, a^n) \in N$ and $p^n \to p \in \partial S_+^{l-1}$. If $\hat{z}(p^n, a^n) = \lambda^n(p^n - \hat{p})$ for $\lambda^n \geqq 0$ and n large, then $\limsup_n z_i(p^n, a^n) = \limsup_n \lambda^n(p_i^n - \hat{p}_i) \leqq 0$ for all i with $p_i < \hat{p}_i$, contradicting the boundary condition. So all p with $\hat{z}(p, a) = \lambda(p - \hat{p})$, $\lambda \geqq 0$ and $(p, a) \in N$ for some a can be taken to lie inside some set S_ε^{l-1} satisfying the conditions of the lemma. Since $\bar{S}_\varepsilon^{l-1} \subset S_{++}^{l-1}$, $proj_1^{-1}(\bar{S}_\varepsilon^{l-1})$ is compact by the properness of $proj_1$, and $[0,1] \times proj_1^{-1}(\bar{S}_\varepsilon^{l-1}) \supset h^{-1}(0)$ is closed by continuity, it is compact. □

Remark 7. The lemma shows that the 'excess value demand' map $\hat{z}$ is homotopic to the trivial map $(p, a) \to (\hat{p} - p)$, in such a way that the zeros of the homotopy remain in a compact set. This trivial map is actually the excess value demand of a single Cobb–Douglas consumer who ignores a, has endowment of the l goods equal to $l\hat{p}$, and utility function $u(x_1, \ldots, x_l) = \Pi_i x_i$.

Proof of Theorem 1. Define a map $\phi: S_{++}^{l-1} \to H_0^{l-1}$ by $\phi(p) = \hat{p} - p$. The derivative of $\psi_0 \circ \phi \circ \psi_1^{-1}$ is just (-1) times the identity map of $\mathbb{R}^{l-1}$, so $deg_\theta(\phi, 0) = (-1)^{l-1}$. By the product rule, for the oriented case we have

$$deg_\theta(\phi \circ proj_1, 0) = deg_\theta(\phi, 0)\, deg_\theta(proj_1) = (-1)^{l-1} deg_\theta(proj_1).$$

Define $h: [0,1] \times N \to H_0^{l-1}$ by $h(t, (p, a)) = t\hat{z}(p, a) + (1 - t)\phi \circ proj_1(p, a)$. h is a homotopy between $\hat{z}$ and $\phi \circ proj_1$, and $h^{-1}(0)$ is compact by Lemma 1. Thus

$$deg_\theta(\hat{z}, 0) = deg_\theta(\phi \circ proj_1, 0) = (-1)^{l-1} deg_\theta(proj_1).$$

In the non-oriented case, the only difference is that $deg_2(\phi, 0) = 1$, so that we get $deg_2(\hat{z}, 0) = deg_2(proj_1)$. □

Degree theory can be used as a tool for demonstrating the existence of equilibrium even if z is a correspondence. Moreover, we can dispense entirely with the boundary condition if we suppose that z is also defined for $(p, a) \in S_+^{l-1} \times A$, provided that we modify the definition of equilibrium to permit goods in excess supply with a price zero. We combine both these observations in the following corollary to Theorem 1. The idea behind the corollary is that under suitable condtions on the correspondence Z there must be a sequence of continuous functions z^n satisfying the earlier boundary condition which ever more closely approximate the correspondence Z when prices are strictly positive.

Corollary 3. Suppose A is metrizable, $Z: S_{+}^{l-1} \times A \to 2^{\mathbb{R}^l}$ is a correspondence, and $N \subset S_{+}^{l-1} \times A$ such that:

(1) Z is upper hemi-continuous, convex, compact, non-empty valued.
(2) $p \cdot z = 0$ for all p such that there exists an $a \in A$ for which $(p,a) \in N$ and $z \in Z(p,a)$.
(3) N is compact.
(4) $(S_{++}^{l-1} \times A) \cap N$ is an $(l-1)$-dimensional topological manifold.

Let $proj_1$ denote the projection of $(S_{++}^{l-1} \times A) \cap N$ into S_{++}^{l-1}. Then $deg_\theta(proj_1) \neq 0$ or $deg_2(proj_1) \neq 0$ implies that there exists a $(\bar{p},\bar{a}) \in N$ and a $\bar{z} \in Z(\bar{p},\bar{a})$ such that $\bar{z} \leqq 0$.

Proof. Step 1. Suppose Z is single valued, so we can consider Z as a continuous function. For each positive integer n, define

$$z^n: S_{++}^{l-1} \times A \to \mathbb{R}^l \quad \text{by} \quad z_i^n(p,a) = Z_i(p,a) + 1/n(1/lp_i - 1), \qquad i = 1, \ldots, l.$$

Then $(z^n, A, (S_{++}^{l-1} \times A) \cap N)$ is C^0-admissible, so by Theorem 3 and Corollary 1 there exists $(p^n, a^n) \in (S_{++}^{l-1} \times A) \cap N$ such that $z^n(p^n, a^n) = 0$. By (3) we can suppose that $(p^n, a^n) \to (\bar{p}, \bar{a}) \in N$. Since we have $Z_i(p^n, a^n) = 1/n(1 - 1/lp_i^n)$ for all n, it follows that $Z_i(\bar{p},\bar{a}) = 0$ if $\bar{p}_i > 0$ and $Z_i(\bar{p},\bar{a}) \leqq 0$ if $\bar{p}_i = 0$.

Step 2. We apply Step 1 to approximate selections from Z. Let d be a metric on $S_{+}^{l-1} \times A$, and $\{q_\alpha\}_{\alpha \in I}$ a partition of unity with compact supports such that $diam(supp\, q_\alpha) < \varepsilon$ for all α. Select $(p^\alpha, a^\alpha) \in supp\, q_\alpha$ and $z^\alpha \in Z(p^\alpha, a^\alpha)$ and define a function

$$z_\varepsilon: S_{+}^{l-1} \times A \to \mathbb{R}^l \quad \text{by} \quad z_\varepsilon(p,a) = \sum_{\alpha \in I} q_\alpha(p,a)(z^\alpha - (pz^\alpha)v)$$

where $v = (1, 1, \ldots, 1) \in \mathbb{R}^l$.

Then z_ε is well defined, continuous, and satisfies $p \cdot z_\varepsilon(p,a) = 0$ for all (p,a). By Step 1, there is a $(p^\varepsilon, a^\varepsilon) \in N$ such that $z_\varepsilon(p^\varepsilon, a^\varepsilon) \leqq 0$. Let $\varepsilon \to 0$ and $(\bar{p},\bar{a})$ be a cluster point of $\{(p^\varepsilon, a^\varepsilon)\}$. Using properties (1) and (2) of Z one can readily verify that $\{z_\varepsilon(p^\varepsilon, a^\varepsilon)\}$ has a corresponding cluster point $\bar{z}$ such that $\bar{z} \leqq 0$ and $\bar{z} \in Z(\bar{p},\bar{a})$ □.

3. Applications

We now give some examples, applying Theorem 1 to the standard pure exchange model, the standard Walrasian model with production, and to the problem of pseudo-equilibrium in a model with incomplete markets, and to a model with non-convex production and general pricing rules.

Example 1. Standard pure exchange model

Take $N \equiv S^{l-1}_{++}$, and any economic model which gives rise to a C^0-Walrasian-like excess demand $z: S^{l-1}_{++} \to \mathbb{R}^l$. ($A$ can be taken to be a one point set, and hence ignored.) Thus $proj_1: S^{l-1}_{++} \to S^{l-1}_{++}$ is just the identity map, so $deg_\theta(proj_1) = 1$. If z is smooth and 0 is a regular value of $\hat{z}$, then $\hat{z}$ preserves orientation at $p \in \hat{z}^{-1}(0)$ if and only if

$$sgn\, det\, D(\psi_0 \circ \hat{z} \circ \psi_1^{-1})(x)|_{x=\psi_1(p)} = 1.$$

If we assume z is the restriction of a homogenous of degree 0 function to S^{l-1}_{++}, then a little exercise in determinates gives

$$sgn\, det\, D(\psi_0 \circ \hat{z} \circ \psi_1^{-1})(x)|_{z=\psi_1(p)} = sgn\, det[D_j z_i(p)], \quad i, j = 1, \dots, l-1.$$

Thus we get

$$(-1)^{l-1} = deg_\theta(\hat{z}, 0) = \sum_{p \in z^{-1}(0)} sgn\, det[D_j z_i(p)\}, \quad i, j = 1, \dots, l-1,$$

which is Dierker's (1972) index formula.

Example 2. Production

We consider a model (F^h, w^h, θ^h, Y) where $F^h: \mathbb{R}^l_{++} \times \mathbb{R}_+ \to \mathbb{R}^l_+$ is agent h's demand function, w^h his endowment vector, and θ^h his share of profits. Y is the production set. Assume F^h is continuous, homogenous of degree 0, $p \cdot F^h(p, m) \equiv m$, and $\lim_{p \to \partial\mathbb{R}^l_+ \setminus \{0\}} \|F^h(p, 1)\| = +\infty$. Assume $w^h \in \mathbb{R}^l_+$, $\sum w^h \in \mathbb{R}^l_{++}$, and that $\sum \theta^h = 1, \theta^h \geqq 0$. Finally, suppose Y satisfies (i) is closed and convex; (ii) $Y \cap \mathbb{R}^l_+ = \{0\}$ and (iii) $Y = Y - \mathbb{R}^l_+$. Let $A = \mathbb{R}^l$, $z: S^{l-1}_{++} \times \mathbb{R}^l \to \mathbb{R}^l$ be defined by

$$z(p, y) = \sum_h F^h(p, \max(0, pw^h + \theta^h py)) - \sum_h w^h - y,$$

and

$$N = \left\{(p, y) \in S^{l-1}_{++} \times R^l : y \in \arg\max_{y \in Y} py'\right\}.$$

We assert that $deg_\theta(\hat{z}, 0)$ is well defined and equal to $(-1)^{l-1}$.

By the usual argument the attainable production set is compact, so we can modify Y, without affecting $\hat{z}^{-1}(0)$, to satisfy: (iv) $Y=K-\mathbb{R}^l_+$, K compact, convex. From now on we assume Y satisfies (iv). It is now easy to check that z is C^0-Walrasian-like on N, and that $proj_1: N \to S^{l-1}_{++}$ is proper. We need to show that N is an $l-1$ manifold, and compute the degree of $proj_1$.

The procedure is outlined below:

Step 1. Define $N^*=\{(p,y)\in S^{l-1}_+ \times \mathbb{R}^l: y \in \arg\max p\cdot y'\}$. Clearly N is an open subset of N^*. Define $v=(1,1,\ldots,1)\in\mathbb{R}^l$, and suppose without loss of generality that $v\cdot y<1$ for all $y\in Y$. Define a map $\phi: N^* \to H^{l-1}_1$ by

$$\phi(p,y)=(1-vy)p+y.$$

Let $\pi: \mathbb{R}^l \to Y$ be the map which assigns to each $z\in\mathbb{R}^l$ the point in Y at which $(z-y)(z-y)$ is at a minimum. Then using the convexity of Y and the definition of profit maximization, one can show

$$\phi^{-1}(z)=\left(\frac{(z-\pi(z))}{v\cdot(z-\pi(z))},\pi(z)\right), \qquad z\in H^{l-1}_1.$$

[This map ϕ is a trivial modification of a map in Mas-Colell (1985, Proposition 3.4.2)]. Thus ϕ is a homeomorphism. Give N^* the $(l-1)$-dimensional oriented manifold structure determined by the homeomorphism ϕ, and N the induced manifold structure obtained as an open subset of N^*.

Step 2. We assert $deg_\theta(proj_1)=1$. This would be easy if N had the local graph property, i.e., if the 'supply function' were single valued on an open subset of prices, but this is probably not true in general under our assumptions. Instead we proceed by a homotopy argument. Let $proj_1: N^* \to S^{l-1}_+ \subset H^{l-1}_1$ be the projection on N^*. Note that for any $\bar{p}\in S^{l-1}_{++}$, $proj_1^{-1}(\bar{p})\subset N$, so that $deg_\theta(proj_1,\bar{p})=deg_\theta(proj_1|_N,\bar{p})=deg_\theta(proj_1|_N)$, for any $\bar{p}\in S^{l-1}_{++}$. We now compute $deg_\theta(proj_1,\bar{p})$. Define $h: N^*\times[0,1]\to H^{l-1}_1$ by $h(p,y,t)=t\,proj_1(p,y)+(1-t)\phi(p,y)$. Clearly h is a homotopy between $proj_1$ and, ϕ and we leave it to the reader to show that $h^{-1}(\bar{p})$ is compact for any $\bar{p}\in S^{l-1}_{++}$. Thus $deg_\theta(proj_1,\bar{p})=deg_\theta(\phi,\bar{p})=1$, so we get $deg_\theta(\hat{z},0)=(-1)^{l-1}$. Computing a useful index formula requires more structure on Y and more work; see Kehoe (1982), Mas-Colell (1985).

Example 3: Existence of equilibrium for asset economies

Let us take $A=G_J(\mathbb{R}^S)$, the smooth compact $J(S-J)$-dimensional manifold of all J-dimensional subspaces of $\mathbb{R}^S$. It is well known that A can be written as the finite union of sets $A=\bigcup A_\sigma$, where each A_σ is homeomorphic to $\mathbb{R}^{J(S-J)}$, as follows. Let

$$G=\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}=\begin{bmatrix} I \\ G_2 \end{bmatrix}$$

be an $S \times J$ matrix, where G_1 is the $J \times J$ identity matrix I and G_2 is an arbitrary $(S-J) \times J$ matrix. The columns of such a matrix G span a J-dimensional subspace of $\mathbb{R}^S$. Permuting the rows of G according to some permutation σ of S elements gives another matrix G^σ whose columns also span a J-dimensional subspace. Any $a \in A$ can be represented by G^σ for some G and some permutation σ.

Let $z: S^{l-1}_{++} \times G_J(\mathbb{R}^S) \to \mathbb{R}^l$ be C^0 Walrasian on $S^{l-1}_{++} \times G_J(\mathbb{R}^S)$. We take z to be the excess demand in an economy where trade depends on prices $p \in S^{l-1}_{++}$ and the 'potential subspace of trade' $a \in A = G_J(\mathbb{R}^S)$. We are also given an exogenous asset return matrix $v: S^{l-1}_{++} \to \mathbb{R}^{S \times J}$. Equilibrium is a $(\bar{p}, \bar{a}) \in S^{l-1}_{++} \times A$ such that $z(\bar{p}, \bar{a}) = 0$ and $\bar{a} = \text{span}\langle v(\bar{p})\rangle$. The difficulty in proving the existence of equilibrium arises because the column span of the matrix $v(p)$ need not be fully J-dimensional and can vary with p. As a result it is necessary to introduce the notion of a psuedo equilibrium $(\hat{p}, \hat{a}) \in S^{l-1}_{++} \times A$ satisfying $z(\hat{p}, \hat{a}) = 0$ and $\hat{a} \supset \text{span}\langle v(\hat{p})\rangle$. Duffie and Shafer (1985) proved that pseudo-equilibrium always exists, and that generically all pseudo-equilibria are equilibria. In papers in this volume Husseini, Lasry and Magill (1990) and Hirsch, Magill and Mas-Colell (1990) give alternative proofs of the existence of pseudo-equilibria. Here we use our method to give a brief proof of both the existence of pseudo-equilibria and the generic existence of equilibria. We state both theorems, then prove them.

Theorem (Existence of pseudo-equilibria for asset economies). *Let $z: S^{l-1}_{++} \times G_J(\mathbb{R}^S) \to \mathbb{R}^l$ be strongly C^0 Walrasian-like. Let $v: S^{l-1}_{++} \to \mathbb{R}^{S \times J}$ be continuous. Then there exists a $(\hat{p}, \hat{a}) \in S^{l-1}_{++} \times G_J(\mathbb{R}^S)$ such that $z(\hat{p}, \hat{a}) = 0$ and $\hat{a} \supset$ span $\langle v(\hat{p})\rangle$.*

Theorem (Generic existence and local uniqueness of equilibria). *Let E and M be smooth manifolds ('parameter' spaces). Let $\hat{z}. S^{l-1}_{++} \times G_J(\mathbb{R}^S) \times E \to H^{l-1}_0$ and $v. S^{l-1}_{++} \times M \to \mathbb{R}^{S \times J}$ be smooth. Suppose that $\hat{z}$ is obtained from a strongly Walrasian-like z by $\hat{z}_\iota(p, a, e) = p_\iota z_\iota(p, a, e)$. Finally suppose that for all (p, a, e, m), rank $D_e \hat{z} = l-1$ and rank $D_m v = S \times J$. Then for almost every $(e, m) \in E \times M$ (that is, except for a subset of measure 0), there are odd number of solutions $(\bar{p}, \bar{a}) \in S^{l-1}_{++} \times G_J(\mathbb{R}^S)$ to $z(\bar{p}, \bar{a}, e) = 0$ and $\bar{a} =$ span $\langle v(\bar{p}, m)\rangle$.*

Proof of both Theorems. It is convenient to begin, as in the second proposition, by assuming that $\hat{z}$ and v are smooth, and then to derive the first proposition by the approximation of continuous functions by smooth functions.

Recall that we can write $A \equiv G_J(\mathbb{R}^S) = \bigcup A_\sigma$. Let $\mathsf{D}_\sigma = S^{l-1}_{++} \times A_\sigma \times E \times M$. Let $d = (p, a, e, m)$, and if $d \in \mathsf{D}_\sigma$, we will also write $d = (p, G_2, e, m)$, where

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} I \\ G_2 \end{bmatrix}$$

and G^σ generates a. Define

$$f_\sigma \colon \mathsf{D}_\sigma \to \mathbb{R}^{J(S-J)} \quad \text{by} \quad f_\sigma(p, G_2, e, m) \equiv f_\sigma(d) = v_2^\sigma(d) - G_2 v_1^\sigma(d)$$

where $v^\sigma(d)$ is the matrix obtained from $v(d)$ by permuting the rows according to σ^{-1} and

$$v^\sigma \equiv \begin{bmatrix} v_1^\sigma(d) \\ v_2^\sigma(d) \end{bmatrix}$$

with v_1^σ a $J \times J$ matrix and v_2^σ a $(S-J) \times J$ matrix. Note that $f_\sigma(d) = f_\sigma(p, G_2, e, m) = 0$ if and only if the columns of $v^\sigma(d) = [{}^{I}_{G_2}] v_1^\sigma(d)$, that is, if and only if the columns of v^σ are spanned by the columns of $[{}^{I}_{G_2}]$. Therefore for $a \in A_\sigma$, $a \supset \operatorname{span}\langle v(d) \rangle$ if and only if $f_\sigma(p, G_2, e, m) = 0$.

For any fixed $(e, m) \in E \times M$, let

$$N_{\sigma,(e,m)} = \{(p, a) \in S^{l-1}_{++} \times A_\sigma \mid a \supset \operatorname{span}\langle v(p, m) \rangle\} \subset S^{l-1}_{++} \times A$$

and let $N_{(e,m)} = \bigcup_\sigma N_{\sigma,(e,m)}$. Suppose that the hypotheses of the second theorem above are satisfied. Then by perturbing m we can perturb v_2^σ however we want, without affecting G_2 or v_1^σ. Hence $f_\sigma \pitchfork 0$. By the Transversality Theorem, for generic (e, m), $N_{\sigma,(e,m)} = f^{-1}_{\sigma,(e,m)}(0)$ is a $(l-1)$-dimensional manifold. Next observe that if $a \in A_\sigma \cap A_{\sigma'}$, then $f_\sigma(p, a, e, m) = 0$ if and only if $f_{\sigma'}(p, a, e, m) = 0$. Hence for generic $(e, m) \in E \times M$, $N_{(e,m)}$ is an $(l-1)$-dimensional manifold.

Furthermore, since $N_{(e,m)}$ is closed in $S^{l-1}_{++} \times A$ and A is compact, $proj_1 \colon N_{(e,m)} \to S^{l-1}_{++}$ is proper. To show that $deg_2(proj_1) \neq 0$ it suffices to observe that for any $p \in S^{l-1}_{++}$, $v(p, m)$ is included in some subspace $a \in A$. Hence $proj_1$ is onto. By the Transversality Theorem, since $N_{(e,m)}$ and S^{l-1}_{++} have the same dimension $l-1$ and $proj_1$ is proper, there is an open dense set $U \subset S^{l-1}_{++}$ on which $proj_1^{-1}$ is finite valued (and non-empty valued by the ontoness of $proj_1$). But if there is more than one subspace $a \in A$ containing all the columns of $v(p, m)$, then there must be infinitely many such a. (See Example 4 in section 2). Hence $proj_1$ is one-to-one on the open set U, so the mod 2 degree of $proj_1$ is non-zero.

Thus for generic $(e, m) \in E \times M$, the hypothesis of Theorem 1 are satisfied. We conclude that if z and v are smooth, then for almost all $(e, m) \in E \times M$, we can find (p, a) with $z(p, a, e) = 0$ and $a \supset \operatorname{span}\langle v(p, m) \rangle$. But now by passing to sequences $(e^n, m^n) \to (e, m)$, a similar conclusion must hold for all (e, m).

Furthermore, for any continuous $z^*: S^{l-1}_{++} \times G_J(\mathbb{R}^S) \to \mathbb{R}^l$ and continuous $v^*: S^{l-1}_{++} \to \mathbb{R}^{S\times J}$ we can always approximate z^* and v^* by smooth z and v. Moreover, we can always take E to be a bounded open set containing the origin in H^{l-1}_0 and $M = \mathbb{R}^{S\times J}$, and define $\tilde{z}_\iota(p,a,e) = z_\iota(p,a) + (e_\iota/p_\iota)\{\min_{1\leqq j\leqq l} p_j\}$ and $\tilde{v}(p,m) = v(p) + m$ as extensions of z and v satisfying the hypotheses of the second theorem. Passing to the limit $(e^n, m^n) \to (0,0)$ yields the theorem on the universal existence of pseudo-equilibria.

To prove the generic existence of genuine equilibria, we shall show that for generic $(e,m) \in E \times M$, there is no $(\bar{p},\bar{a})$ satisfying $\hat{z}(\bar{p},\bar{a},e) = 0$, $\bar{a} \supset \operatorname{span}\langle v(\bar{p},m)\rangle$ and $\operatorname{rank} v^\sigma_1(\bar{p},m) < J$. It will follow that $v(\bar{p},m)$ has full rank, hence there is a unique $a \in A$ with $a \supset \operatorname{span}\langle v(\bar{p},m)\rangle$, hence all the pseudo-equilibria above are genuine equilibria.

Let $\mathscr{S}^{J-1}$ as before be the $(J-1)$-dimensional sphere. Extend $\hat{z}$ and f_σ to $\mathsf{D}_\sigma \times \mathscr{S}^{J-1}$ by ignoring the last coordinate. But now consider a new set of J equations $\Psi: \mathsf{D}_\sigma \times \mathscr{S}^{J-1} \to \mathbb{R}^J$ given by $\Psi_\sigma(p,a,e,m,x) = v^\sigma_1(p,m)x$. Then given (p,a,e,m), and so given $v^\sigma_1(p,m)$, there is a $x \in \mathscr{S}^J$ solving $\Psi_\sigma(p,a,e,m,x) = 0$ if and only if $v^\sigma_1(p,m)$ does not have full rank J. But now consider the system $(\hat{z}, f_\sigma, \Psi_\sigma): \mathsf{D}_\sigma \times \mathscr{S}^{J-1} \to H^{l-1}_0 \times \mathbb{R}^{J(S-J)} \times \mathbb{R}^J$. Remember that $\operatorname{rank} D_e\hat{z} = l-1$ and that $\operatorname{rank} D_m v = S \times J$. The latter assumption allows us to control v^σ_1 and v^σ_2 independently, so that we have $\operatorname{rank} D_m(f_\sigma, \Psi_\sigma) = J(S-J) + J$ as well as $\operatorname{rank} D_e\hat{z} = l-1$. Thus $(\hat{z}, f_\sigma, \Psi_\sigma) \pitchfork 0$. By the Transversality Theorem, for generic (e,m), $(\hat{z}, f_\sigma, \Psi_\sigma): S^{l-1}_{++} \times A_\sigma \times \mathscr{S}^{J-1} \to H^{l-1}_0 \times \mathbb{R}^{J(S-J)} \times \mathbb{R}^J$ is transverse to 0. But this is only possible, given the dimension of the domain and range, if $(\hat{z}, f_\sigma, \Psi_\sigma)^{-1}(0)$ is empty. Applying this argument to each σ finishes the proof.

Example 4. Increasing returns and general pricing rules

In this section we investigate a model with a non-convex production set and general pricing rules. This model is a very special case of the model in Bonnisseau and Cornet (1989); the novelty here will be the computation of $deg_\theta(\hat{z}, 0)$, which surprisingly turns out to be independent (given our method of orienting N) of the pricing rule itself. This allows the computation of an index formula in the smooth case, but we have not carried out this calculation. Kamiya (1988) has recently analyzed a much more general model using similar techniques from degree theory.

We consider an economy $E = (F, Y, f)$, with the following interpretation: Y is the production set. We assume Y contains any initial endowment of goods: for example, one could take $Y = Y' + \{w\}$, where $w \geqq 0$ is the intial endowment vector and Y' satisfies the usual assumption $Y' \cap R^l_+ = \{0\}$. $p = f(y)$ is the pricing rule. For example, $f(y)$ could be a realization of average cost pricing, a selection from the set of break-even price vectors for y. If ∂Y is smooth, f could be marginal cost pricing (see Example 2, section 2). $F(p,y)$ represents aggregate demand at prices p and production vector y. We assume any income redistribution scheme is embodied in F.

We make the following assumptions: (1) $Y \subset \mathbb{R}^l$ satisfies (i) $Y \cap \mathbb{R}^l_{++} \neq \emptyset$; (ii) $\{y \geqq w\} \cap Y$ is compact $\forall w \in \mathbb{R}^l$; and (iii) $Y = Y - \mathbb{R}^l_+$. (2) Define $Y_E = \{y \in Y : \nexists y' \in Y$ such that $y' \gg y\}$. Then $f: Y_E \to S^{l-1}_+$ is continuous and satisfies $yf(y) > 0 \, \forall y \in Y_E \cap \mathbb{R}^l_+$. (3) $F: \mathbb{R}^l_{++} \times \mathbb{R}^l_+$ is continuous and satisfies (i) $pF(p,y) = py$ for all (p,y) such that $py > 0$; (ii) If $p^n \to p \in \partial\mathbb{R}^l_+$ and $py > 0$, then $\|F(p^n, y)\| \to +\infty$.

The basic idea is as follows. Define (z, A, N) by

$$A = \mathbb{R}^l,$$

$$z(p,y) = F(p,y) - y$$

$$N = \{(p,y) \in S^{l-1}_{++} \times Y_E : p = f(y)\}.$$

A basic difficulty is that N need not be proper and z may not be C^0-Walrasian, so that Theorem 1 does not apply directly to (z, A, N). What we will do is replace (z, A, N) with a model (z, A, N^1), where N^1 is obtained from a new Y^1 and f^1, such that (z, A, N^1) satisfies the hypotheses of Theorem 3 and the changes made to Y and f have no effect on $\hat{z}^{-1}(0)$. This will allow us to show

Proposition. There is a choice of orientation of N such that

$$deg_\theta(\hat{z}, 0) = (-1)^{l-1}.$$

Proof of Proposition. For any $\varepsilon \in \mathbb{R}$, let $V(\varepsilon) = \{y \in \mathbb{R}^l : y_i \geqq \varepsilon \, \forall i\}$. For $\varepsilon > 0$, define $Y^1 = [Y \cap V(-\varepsilon)] - \mathbb{R}^l_+$. Given the assumptions on Y and f, it is possible to find a $\varepsilon > 0$ and an $f^1: Y^1_E \to S^{l-1}_+$ such that

(i) $Y^1_E \cap V(-\varepsilon) = Y_E \cap V(-\varepsilon)$,
(ii) $f^1(y) \equiv f(y) \, \forall y \in Y_E \cap V(-\varepsilon/2)$,
(iii) if $y \in Y^1_E$ and $y_i \leqq -\varepsilon$, then $f^1_i(y) = 0$,
(iv) $\exists \tau > 0$ such that $yf^1(y) \geqq \tau$, $\forall y \in Y^1_E$.

Note that if $z(p,y) = 0$ and $(p,y) \in N$, then necessarily $y \in \mathbb{R}^l_+$, so that replacing Y and f by Y^1 and f^1 can have no effect on the equilibria, nor any effect on $\hat{z}$ in a neighborhood of the equilibria.

We first show that N^1 is C^0-regular and proper. Define $\phi: Y^1_E \to H^{l-1}_1$ by

$$\phi(y) = y + \frac{(l - v \cdot y)}{l} v, \quad \text{where} \quad v = (1, 1, \ldots, 1) \in \mathbb{R}^l.$$

ϕ assigns to each $y \in Y^1_E$ the unique vector $\phi(y) \in H^{l-1}_1$ in the direction v from y. It is not difficult to check, given the properties of Y^1_E, that ϕ is a homeomorphism of Y^1_E onto H^{l-1}_1. We give Y^1_E the oriented manifold structure induced by this homeomorphism, so that $deg_\theta(\phi)=1$. $N^1=\{(p,y)\in S^{l-1}_{++}\times Y^1_E: p=f^1(y)\}$. N^1 is an open subset of $\text{Graph}^{-1}(f^1)$, and the latter can be made an oriented $(l-1)$-manifold via the homeomorphism ρ: $Y^1_E\to\text{Graph}^{-1}(f^1)$ defined by $\rho(y)=(f^1(y),y)$. Note that this makes ρ an orientation preserving homeomorphism, so that $deg_\theta(\rho)=1$. We give N^1 the oriented manifold structure induced as an open subset of $\text{Graph}^{-1}(f^1)$. If $K\subset S^{l-1}_{++}$ is compact, then $proj_1^{-1}(K)\subset K\times[V(-\varepsilon)\cap Y_E]$, so it is bounded. It is closed by continuity. Hence N^1 is proper.

It is easy to check that a is C^0-Walrasian on N^1, given the assumptions on F and the properties of f^1.

It remains to compute the degree of $proj^1: N^1\to S^{l-1}_{++}$. Note that for any $\bar{p}\in S^{l-1}_{++}$ there is by continuity an open neighborhood U of $f^{1(-1)}(\bar{p})$ such that $f^1(U)\subset S^{l-1}_{++}$. Thus on U, $f^1=proj_1\circ\rho$, so that $deg_\theta(f^1,\bar{p})=deg_\theta(proj_1,\bar{p})\,deg_\theta(\rho)$. Since $deg_\theta(\rho)=1$ and $deg_\theta(proj_1,\bar{p})=deg_\theta(proj_1)$, we need only show that $deg_\theta(f^1,\bar{p})=1$. We do this by a homotopy argument. Define h: $Y^1_E\times[0,1]\to H^{l-1}_1$ by

$$h(y,t)=tf^1(y)+(1-t)\phi(y).$$

Clearly h is a homotopy between f^1 and ϕ. We assert that for any $\bar{p}\in S^{l-1}_{++}$, $h^{-1}(\bar{p})$ is compact. $h^{-1}(\bar{p})$ is closed by continuity, so we need only show that $h^{-1}(\bar{p})$ is bounded. We have

$$(y,t)\in h^{-1}(\bar{p})\Leftrightarrow\begin{cases}tf^1(y)+(1-t)[y+((l-vy)/l)v]=\bar{p}\\ \qquad (y,t)\in Y_E\times[0,1].\end{cases}$$

We assert that $(y,t)\in h^{-1}(\bar{p})\Rightarrow y\in V(-\varepsilon-(l-1))$, so that

$$h^{-1}(\bar{p})\subset[Y^1_E\cap V(-\varepsilon-(l-1))]\times[0,1],$$

a bounded set. The details are left to the reader. Then we have $deg_\theta(f^1,\bar{p})=deg_\theta(\phi,\bar{p})\equiv deg_\theta(\phi)=1$.

Appendix A: Elementary proof

Our main Theorem 1 relied for its proof on degree theory. Here we give an elementary proof of a weaker theorem which can also be used as the basis of a computational procedure to find a zero (i.e., an equilibrium). Note that the following theorem would have sufficed for the application we gave to financial asset general equilibrium.

Theorem 1′. Let $z\cdot S^{l-1}_{++}\times A\to\mathbb{R}^l$ *and let* $N\subset S^{l-1}_{++}\times A$. *If* (z,A,N) *is*

C^1-admissible and N satisfies the generic graph property, then there exists $(\bar{p},\bar{a})\in N$ such that $z(\bar{p},\bar{a})=0$.

Proof of Theorem 1'. Step 1. Without loss of generality, we may replace $\hat{z}$ with $\bar{z}$ satisfying $\bar{z}(p,a)=\hat{p}-p$ if $p\in\partial\bar{S}_\varepsilon^{l-1}$,

Take $S_{\varepsilon/2}^{l-1}$, an open, convex body with smooth, non-empty boundary $\partial\bar{S}_{\varepsilon/2}^{l-1}$ such that $S_{++}^{l-1}\supseteqq\bar{S}_{\varepsilon/2}^{l-1}\supseteqq S_{\varepsilon/2}^{l-1}\supseteqq\bar{S}_\varepsilon^{l-1}\supseteqq S_\varepsilon^{l-1}$. Let $d:S_{++}^{l-1}\to[0,1]$ be a smooth function such that $d(p)=0$ if $p\in S_{++}^{l-1}\backslash S_{\varepsilon/2}^{l-1}$ and $d(p)=1$ if $p\in\bar{S}_\varepsilon^{l-1}$. Such a function exists by standard arguments. Then let $\bar{z}:N\to H_0^{l-1}$ be defined by $\bar{z}(p,a)=d(p)\hat{z}(p,a)+(1-d(p))(\hat{p}-p)$. Note that $\bar{z}(p,a)=0$ for $(p,a)\in N$ only if $\hat{z}(p,a)=0$. Now make the change of variables $\varepsilon\to2\varepsilon$ and the claim of Step 1 is established.

Step 2. We may assume that there is an open set $U\subset S_{++}^{l-1}$ and a function $\varphi:U\to A$ such that $(U\times A)\cap N=\operatorname{Graph}\varphi$, $U\cap\partial\bar{S}_\varepsilon^{l-1}\neq\emptyset$ and if $(p,a)\in[(U\cap\partial S_\varepsilon^{l-1})\times A]\cap N$ and $(p',a')\in(\partial S_\varepsilon^{l-1}\times A)\cap N$ and $(p',a')\neq(p,a)$ then

$$\frac{\bar{z}(p,a)}{|\bar{z}(p,a)|}\neq\frac{\bar{z}(p',a')}{|\bar{z}(p',a')|}.$$

From the generic graph property, there is an open dense set $U\subset S_{++}^{l-1}$ such that $(U\times A)\cap N=\operatorname{Graph}\varphi$, where φ is a function $\varphi:U\to A$. Since we have great freedom in choosing $\partial S_\varepsilon^{l-1}$ (anywhere in $S^{l-1}\backslash\bar{S}_{2\varepsilon}^{l-1}$) we may assume $U\cap\partial\bar{S}_\varepsilon^{l-1}\neq\emptyset$. Now take $(p,a)\in[(U\cap\partial\bar{S}_\varepsilon^{l-1})\times A]\cap N$. Then $\bar{z}(p,a)=\hat{p}-p$. If $(p',a')\in(\partial\bar{S}_\varepsilon^{l-1}\times A)$ and $\bar{z}(p',a')\equiv(\hat{p}-p')=\lambda(\hat{p}-p)$, then $p'=p$ (since the ray from $\hat{p}$ through p can only interest $\partial\bar{S}_\varepsilon^{l-1}$ in one place because we took S_ε^{l-1} convex). But if (p',a') is also in N, then by the graph property at $p=p'$, $a'=a=\varphi(p)$, and $(p,a)=(p',a')$.

Step 3. The path-following analogue of Hirsch's argument shows that there is a $(\bar{p},\bar{a})\in N$ with $\bar{z}(\bar{p},\bar{a})=0$.

Suppose there is no $(\bar{p},\bar{a})\in N$ with $\bar{z}(\bar{p},\bar{a})=0$. Then we can replace $\bar{z}:N\to H_0^{l-1}$ by $G:N\to\sum$ defined by $G(p,a)=\bar{z}(p,a)/|\bar{z}(p,a)|$, where $\sum$ is the $l-2$ sphere in H_0^{l-1}. Furthermore let us denote by $H:[(U\cap\partial\bar{S}_\varepsilon^{l-1})\times A]\cap N\to\sum$ the further restriction of G, so if (p,a) is in the domain of H, then $H(p,a)=G(p,a)$. Note that the domain of G is a smooth $(l-1)$-dimensional manifold, and the domain of H is a smooth $(l-2)$-dimensional manifold. Furthermore, since every ray from $\hat{p}$ intersects S_ε^{l-1} somewhere, the range of H contains an open set R in $\sum$.

By applying Sard's theorem twice, we deduce that there must be some element $y\in R\subset\sum$ such that both $G\pitchfork y$ and $H\pitchfork y$. Hence $G^{-1}(y)$ is a non-empty one-dimensional manifold. Furthermore, from the preceding Step

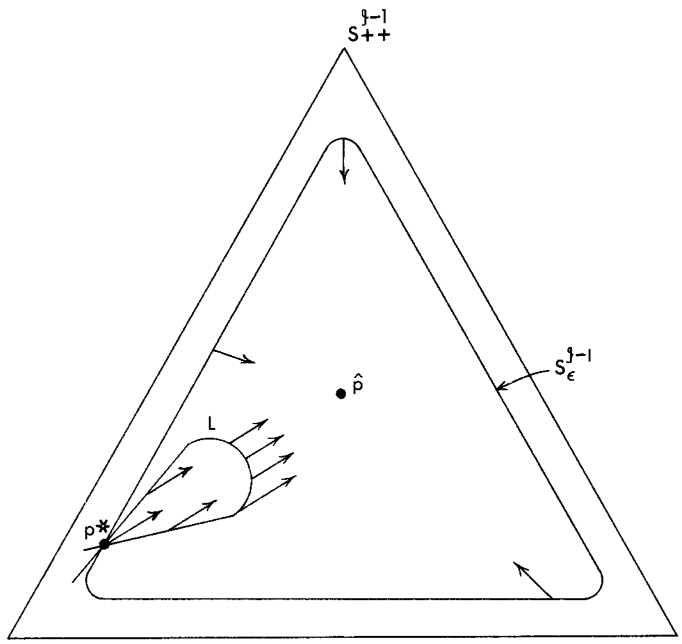

Fig. 3

2 we know that $H^{-1}(y)$ is a single point (p^*, a^*). In fact, Step 2 implies that $G^{-1}(y) \cap [(\partial\bar{S}_\varepsilon^{l-1} \times A) \cap N]$ is also the single point (p^*, a^*). Moreover, since $H \pitchfork y$, $G^{-1}(y)$ intersects domain $H = [(U \cap \partial\bar{S}_\varepsilon^{l-1}) \times A) \cap N]$ transversely. Consequently $G^{-1}(y) \cap [(S_\varepsilon^{l-1} \times A) \cap N] \neq 0$.

Now, by properness, $L \equiv G^{-1}(y) \cap cl[(S_\varepsilon^{l-1} \times A) \cap N]$ is a compact set (where the closure is taken in N). By following along the path $G^{-1}(y)$ in $cl[(S_\varepsilon^{l-1}) \times A) \cap N]$ starting from (p^*, a^*) we must come either to an end, contradicting the fact that $G^{-1}(y)$ is a one-dimensional manifold, or else come to another point of intersection of

$$G^{-1}(y) \cap \partial((\bar{S}_\varepsilon^{l-1} \times A) \cap N) = G^{-1}(y) \cap ((\partial\bar{S}_\varepsilon^{l-1} \times A) \cap N)$$

contradicting Step 2.

Remark 7. The conditions we have imposed on N are precisely those few properties of S_{++}^{l-1} used by Hirsch and Smale in proving that a smooth Walrasian-like excess value demand $\hat{z}: S_{++}^{l-1} \to H_0^{l-1}$ has a zero. The assumption that N has dimension $l-1$ guarantees that the set L of $(p,a) \in N$ such that $\hat{z}(p,a)/|\hat{z}(p,a)|$ is a given vector is typically a one-dimensional manifold. The properness assumption guarantees that $L \cap (\bar{S}_\varepsilon^{l-1} \times A) \cap N$ is compact. The generic graph property guarantees that typically L intersects $(\partial\bar{S}_\varepsilon^{l-1} \times A) \cap N$ in exactly one point. Fig. 3 illustrates the argument when

$N=S^{l-1}_{++}$. Note that since S^{l-1}_{++} is contained in the translation of H^{l-1}_0 by $\hat{p}$, we can easily imagine the domain and range of $\hat{z}$ in the same diagram.

The contradiction to the maintained hypothesis that $\hat{z}$ is never zero is seen in the above diagram in the fact that the manifold L crosses itself at p^*. This is inevitable since once L gets inside S^{l-1}_ε it cannot escape through any other boundary point $p \in \partial \bar{S}^{l-1}_\varepsilon$ because $\hat{p}-p$ points in the direction $\hat{p}-p^*$ only when $p=p^*$.

Remark 8. It is clear that the local graph property is only relevant for $p=p^*$, where L crosses the boundary of S^{l-1}_ε. In fact if there were locally an odd number of $a \in A$ with $(p^*, a) \in N$, the same proof would apply. Once L 'enters' S^{l-1}_ε through one point $(p^*, a) \in N$, it must leave through another. Pairing up points, we see there is one (p^*, a) left over through which L enters S^{l-1}_ε, but cannot leave S^{l-1}_ε without crossing itself. Now it can easily be shown from degree arguments introduced in section 4, that if N satisfies the local graph property, there is an open dense set $U \subset S^{l-1}_{++}$ such that for each $p \in U$, $proj_1^{-1}(p)$ consists of an odd number of points. In particular we can always choose $p^* \in U \cap \partial \bar{S}^{l-1}_\varepsilon$. This is one indication that by using degree theory we can greatly strengthen this theorem.

Appendix B: Degree theory

We give here a very brief review of the basic facts of degree theory. An excellent reference is Dold (1980); see also Hirsch (1976) for the case of smooth manifolds.

An n-manifold is a Hausdorff topological space X which has an open cover $\{W_\alpha\}_{\alpha \in A}$ such that for each α there exists a homeomorphism $\phi_\alpha: W_\alpha \to \mathbb{R}^n$. n is called the dimension of X, each $\{W_\alpha, \phi_\alpha\}$ is called a chart, and $\{W_\alpha, \phi_\alpha\}_{\alpha \in A}$ is called an atlas. If all the 'coordinate changes' $\phi_\alpha \circ \phi_{\bar{\alpha}}^{-1}: \phi_{\bar{\alpha}}(W_\alpha \cap W_{\bar{\alpha}}) \to \phi_{\bar{\alpha}}(W_\alpha \cap W_{\bar{\alpha}})$, $\alpha, \bar{\alpha} \in A$ are smooth (C^∞), the atlas is called smooth. If all these coordinate changes (which are homeomorphisms) preserve orientation, then the atlas is called oriented. A manifold with a countable base for the open sets is called *smoothable* if it has a smooth atlas. Such a manifold, together with a given smooth atlas, is called a *smooth manifold*. A manifold which has an oriented atlas is called *orientable*. An orientable manifold together with an oriented atlas is called an *oriented manifold*. The properties of being a manifold, dimension, smoothability and orientability are topological invariants. An open set C in a n-manifold (smooth) (oriented) X becomes an n-manifold (smooth) (oriented) in a natural way.

Given a pair of manifolds X and Y of the same dimension, a continuous map $f: X \to Y$, and a point $y \in Y$ such that $f^{-1}(y)$ is compact, degree theory can be thought of as a way of 'counting' the number of points in $f^{-1}(y)$.

Define

$C_\theta = \{(f, X, Y, y)$: (i) X and Y are oriented manifolds of the same dimension;
(ii) $f: X \to Y$ is continuous;
(iii) $y \in Y$ and $f^{-1}(y)$ is compact$\}$;

and

$C_2 = \{(f, X, Y, y)$: (i) X and Y are manifolds of the same dimension;
(ii) $f: X \to Y$ is continuous;
(iii) $y \in Y$ and $f^{-1}(y)$ is compact$\}$.

Let Z and Z_2 denote the rings of integers and integers modulo 2 respectively. We will describe properties of maps $deg_\theta: C_\theta \to Z$ and $deg_2: C_2 \to Z_2$, called respectively the 'oriented degree' and the 'mod 2 degree'. We can do this simultaneously: Let (C, R, deg) denote either $(C_\theta, Z, deg_\theta)$ or (C_2, Z_2, deg_2). We write $deg(f, y)$ instead of $deg(f, X, Y, y)$ whenever X and Y are obvious from the context. Then the basic result is [see Dold (1980)]:

Degree Theorem. There exists a map $deg: C \to R$ *satisfying:*

(*D.1*) (*normalization*): $deg(id, Y, Y, y) = 1$.

(*D.2*) (*localization*): *If* $(f, X, Y, y) \in C$, $X \supset U \supset f^{-1}(y)$, $Y \supset V \supset f(U)$, U *open in* X, V *open in* Y, *then:*

$$deg(f, X, Y, y) = deg(f|_U, U, V, y).$$

(*D.3*) (*additivity*): *If* $(f, X, Y, y) \in C$ *and* $\{G_i\}_{i=1}^m$ *is a finite partition of* X into open sets such that $(f|_{G_i}, G_i, Y, y) \in C \ \forall i$, then:

$$deg(f, X, Y, y) = \sum_{i=1}^{m} deg(f|_{G_i}, G_i, Y, y).$$

(*D.4*) (*homotopy invariance*): *If* (f, X, Y, y), $(g, X, Y, y) \in C$ *and there is a homotopy* $h. X \times [0,1] \to Y$ *between* f *and* g *such that* $h^{-1}(y)$ *is compact, then:*

$$deg(f, y) = deg(g, y).$$

(*D.5*) (*continuity*): *If* $(f, X, Y, \bar{y}) \in C$ *and* $K \subset Y$ *is a compact connected set containing* $\bar{y}$ *such that* $f^{-1}(K)$ *is compact, then:*

$$deg(f, y) = deg(f, \bar{y}), \qquad \forall y \in K.$$

(*D.6*) (*product rule*): *If* (f, X, Y, y), $(g, Y, Z, z) \in C$, Y *is connected and* f *is proper, then:*

$$deg(g \circ f, z) = deg(g, z)\, deg(f, y).$$

(*D.7*) (*non-triviality*): *If* $(f, X, Y, y) \in C$, *then*

$$deg(f,y)\neq 0 \quad \textit{implies} \quad f^{-1}(y)\neq\emptyset.$$

Remark 1. D.5 implies that if $(f,X,Y,\bar{y})\in C$, Y is connected and f is proper, then (since Y is locally compact and locally connected) $deg(f,y)=deg\,f,\bar{y})$ for all $y\in Y$. This common value is denoted $deg(f,X,Y)$ or $deg(f)$, and is called the degree of f.

Remark 2. If $(f,X,Y,y)\in C_2$ and f is a homeomorphism, then $deg_2(f,y)=1$. If $(f,X,Y,y)\in C_\theta$ and f is homeomorphism, then $deg_\theta(f,y)=1$ if f preserves orientation at $f^{-1}(y)$ and $deg_\theta(f,y)=-1$ if f reserves orientation. f preserves orientation at $f^{-1}(y)$ if there is a chart $\{W,\phi\}$ from the maximal oriented atlas on X and a chart $\{V,\psi\}$ from the maximal oriented atlas on Y such that $f^{-1}(y)\in W$, $y\in V$, and $f(W)\subset V$, so that the local homeomorphism $\psi\circ f\circ\phi^{-1}:\phi(W)\to\mathbb{R}^n$ preserves orientation at $\phi(f^{-1}(y))$. If $\psi\circ f\circ\phi^{-1}$ is smooth, then it preserves orientation at $\phi(f^{-1}(y))$ if $sgn\,det\,D\psi\circ f\circ\phi^{-1}(z)]_{z=\phi(f^{-1}(y))}=1$.

Remark 3. Suppose $(f,X,Y,y)\in C$, $f^{-1}(y)=\{x_1,\dots,x_m\}$, $m<\infty$, and there are disjoint neighborhoods W_i of x_i, $i=1,\dots,m$, such that each $f|_{W_i}$ is a local homeomorphism. Then by D.2 and D.3,

$$deg(f,X,Y,y)=\sum_{i=1}^{m} deg(f|_{W_i},W_i,f(W_i),y).$$

From Remark 2 above, in the mod 2 case $deg_2(f|_{W_i},W_i,f(W_i),y)=1$, so $deg_2(f,y)=\#f^{-1}(y)\,mod\,2$. In the oriented case, $deg_\theta(f|_{W_i},y)=1$ if f preserves orientation at x_i and -1 if f reserves orientation at x_i, so that $deg_\theta(f,y)$ counts the number of points in $f^{-1}(y)$ with orientation.

Remark 4. By Sard's theorem, the situation described in Remark 3 is 'typical' if X,Y are smooth and f is smooth [assuming $f^{-1}(y)$ is always compact]. However, there are continuous maps $g:X\to Y$ between compact, smoothable connected manifolds of the same dimension such that $g^{-1}(y)$ is an infinite set for every $y\in Y$. If $(g,X,Y,y)\in C$ and X and Y are smoothable, however, there is always a $(f,X,Y,y)\in C$ such that f and g are homotopic at y (in the sense of D.4), f is 'close' to g, and the situation described in Remark 3 holds for (f,X,Y,y). [See Hirsch (1976) for an extensive analysis of approximation by smooth maps.]

Remark 5. Combining D.4 and D.7, we see that if $(f,X,Y,y)\in C$ and $deg(f,y)\neq 0$, then for any $g:X\to Y$ homotopic to f in the sense of D.4, $g^{-1}(y)\neq\emptyset$. This is important because all sufficiently 'small' perturbations of f

are indeed homotopic to f in the sense of D.4. For example, let $f: X \to \mathbb{R}^l$, $f^{-1}(0)$ compact, and suppose X is the union of an increasing sequence of compact sets, $X = \bigcup_{n=1}^{\infty} K_n$. Suppose $g: X \to \mathbb{R}^l$, and for any sequence $x_n \in X \setminus K_n$, $|f(x_n) - g(x_n)| \leqq \frac{1}{2}|f(x_n)|$ for n large. Then g is homotopic to f in the sense of D.4, since for the homotopy given by $h(x,t) = tf(x) + (1-t)g(x)$, $h^{-1}(0)$ is compact.

Remark 6. Let $(f, X, Y, y) \in C$. We call a non-empty subset K of $f^{-1}(y)$ stable if for any open neighborhood U of K, there is a smaller open neighborhood G of K, $U \supset G \supset K$, such that $(f|_G, G, Y, y) \in C$ and $deg(f|_G, G, Y, y) \neq 0$. By Remark 5, it follows that if K is a stable set of equilibria, then for any neighborhood U of K, a sufficiently small perturbation g of f will have solutions in U.

We assert that if $deg(f, X, Y, y) \neq 0$, then there is a stable subset K of $f^{-1}(y)$ which is compact and connected. If $f^{-1}(y)$ has only a finite number of connected components, then this is a trivial consequence of D.3. Otherwise, it can be shown as follows. The collection of all compact stable subsets of $f^{-1}(y)$ is non-empty [since $f^{-1}(y)$ is one of them by D.2], and it can be partially ordered by inclusion. Any chain of compact stable sets as a lower bound; one can readily verify that the intersection of nested compact stable sets is a compact stable set. By Zorn's lemma there is a minimal compact stable set K. If K were not connected, then it could not be minimal by axiom D.3

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