A STRATEGIC MARKET GAME WITH ACTIVE BANKRUPTCY

BY

JOHN GEANAKOPLOS, IOANNIS KARATZAS
MARTIN SHUBIK
AND
WILLIAM SUDDERTH

COWLES FOUNDATION PAPER NO. 1008

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281
2001

ELSEVIER Journal of Mathematical Economics 34 (2000) 359–396

JOURNAL OF Mathematical ECONOMICS

www.elsevier.com/locate/jmateco

A strategic market game with active bankruptcy

J. Geanakoplos [a,1], I. Karatzas [b,2], M. Shubik [a,*], W. Sudderth [c,3]

[a] *Yale University, Box 208281, New Haven, CT 06520-8281, USA*

[b] *Columbia University, 619 Mathematics, New York, NY 10027, USA*

[c] *University of Minnesota, 206 Church Street S.E., Minneapolis, MN 55455, USA*

Received 28 June 1998; received in revised form 14 December 1999; accepted 28 February 2000

Abstract

We construct stationary Markov equilibria for an economy with fiat money, one nondurable commodity, countably many time periods, and a continuum of agents. The total production of *commodity* remains constant, but individual agents' endowments fluctuate in a random fashion from period to period. In order to hedge against these random fluctuations, agents find it useful to hold *fiat money*, which they can borrow or deposit at appropriate rates of interest; such activity may take place either at a *central bank* (which fixes interest rates judiciously) or through a *money-market* (in which interest rates are determined endogenously).

We carry out an *equilibrium analysis*, based on a careful study of Dynamic Programming equations and on properties of the *invariant measures* for associated optimally controlled Markov chains. This analysis yields the stationary distribution of wealth across agents, as well as the stationary *price* (for the commodity) and *interest rates* (for the borrowing and lending of fiat money).

A distinctive feature of our analysis is the incorporation of *bankruptcy*, both as a real possibility in an individual agent's optimization problem, and as a determinant of interest rates through appropriate *balance equations.* These allow a central bank (a money-market) to announce (to determine endogenously) interest rates in a way that conserves the total money-supply and controls inflation.

* Corresponding author. Tel.: +1-203-432-3704; fax: +1-203-432-6167.

E-mail addresses: john.geanakoplos@yale.edu (J. Geanakoplos), ik@math.columbia.edu (I. Karatzas), martin.shubik@yale.edu (M. Shubik), bill@stat.umn.edu (W. Sudderth).

[1] Tel.: +1-203-432-3397.

[2] Tel.: +1-212-854-3177.

[3] Tel.: +1-612-625-2801.

PII: S0304-4068(00)00038-0

General results are provided for the existence of such stationary equilibria, and several explicitly solvable examples are treated in detail.

Keywords: Bankruptcy; Strategic market game; Markov equilibrium

1. Introduction

There are some relatively mundane aspects of economic activity, which can easily be ignored in an equilibrium theory concerned with the existence of prices, but not with the mechanisms which bring them into being. These are: (1) the presence of fiat money and the nature of the conservation laws governing its supply in the markets and in the banking system; (2) the existence of the "float", or a transactions need for money; (3) the need for default, bankruptcy and reorganization rules, if lending is permitted; and (4) the nature of interest rates as parameters or control variables or as endogenous variables.[4] A process-model requires that these aspects be explained and analyzed.

As in two previous papers [KSS1] (1994) and [KSS2] (1997), we study here an *infinite-horizon strategic market game with a continuum of agents.* The game models a simple economy with one nondurable good (or "perishable commodity"), which is produced in the same quantity in every period. The commodity endowments of individual agents are random, and fluctuate from period to period. At the start of each period, agents must decide how much of their current monetary wealth to spend on consumption of the commodity ("cash-in-advance" model). In [KSS1] (1994), the only choice was between spending, and hoarding cash for the future. In [KSS2] (1997), agents were able to borrow or lend money before spending, but were not allowed to borrow more than they could pay back from their earnings in the next period. Since bankruptcy is a prominent feature of real economies, we introduce here a more general model where bankruptcy can and does occur.

The main focus of this paper is on a model with a *Central Bank*, which makes *loans* and accepts *deposits.* The bank sets two interest rates, one for borrowers and one for depositors. Some unfortunate borrowers may not receive sufficient income to pay back their debts. To avoid inflation, the bank must set the interest rate for borrowers sufficiently high, so that it will receive enough money from high-in-

[4] A fifth key phenomenon is the *velocity of money*, which is defined as the volume of transactions per unit of money per period of time. We delay our study of velocity and avoid the new difficulties it poses, by considering discrete-time models where velocity is constrained to be between 0 and 1. The relationship between the discrete- and continuous-time models is of importance. But we suspect that the transactions need for money cannot be adequately modeled by continuous-time models alone, without superimposing some discrete-time aspects of economic life on the continuous-time structure.

come borrowers to offset the bad debts of the bankrupt, and also be able to pay back depositors at a (possibly) different rate of interest. We assume that the bank seeks not to make profit, but only to control inflation in the economy.

The rules of the game must, of course, specify the terms of bankruptcy. Almost every conceivable rule seems to have occurred in history, but we have chosen for our model what appears to be the simplest rule that can be analyzed mathematically. Namely, the bankrupt receive a nonmonetary "punishment" in units of utility, but are then forgiven their debts and allowed to continue to play.

An interesting alternative to the model with a Central Bank is one with a *money-market*. In this model, agents offer fiat money for lending, or bid I.O.U. notes for loans, and thereby determine interest rates endogenously. Such a model is studied in [KSS2] (1997). For the sake of clarity and brevity, we shall concentrate here on the model with a central bank and limit ourselves to a few remarks on the model with a money-market.

The game with a central bank, as we define it in Section 3, is a full-process model with completely specified dynamics. Indeed, the game can be simulated for a finite number of players as was done for the model of [KSS1] (1994) by Miller and Shubik (1994). It would be very interesting to obtain theorems about the limiting behavior of the model when it begins out of equilibrium, and we do obtain some information (Lemma 7.5) about the extent to which the bank is able to influence prices by its control of interest rates. However, we concentrate on the existence and structure of a special type of equilibrium, called "stationary Markov equilibrium" (Theorems 5.1, 7.1 and 7.2, and Examples 6.1–6.3), in which the price of the good and the distribution of wealth among the agents remain constant.

In ongoing work, we propose to extend these results to a cyclical or stochastic overall supply of goods per period, to examine the critical role played by a Central Bank in controlling the money-supply, and to study the limitations in the Central Bank's ability to control inflation — as a function of the control variables it can utilize and of the frequency of its interventions.

We have limited our investigation to the trading of a single commodity. It appears that the existence results may be extended to the case of several commodities; but in that context, uniqueness will certainly not hold. Even with one commodity, if the system is started *away from the equilibrium distribution,* we have been unable to establish general conditions for convergence to equilibrium. We leave all these issues as open problems for further research.

2. Summary

In the next section, we provide a careful definition of the model under study, and also of the notion of "stationary Markov equilibrium". The key to our construction of such an equilibrium is a detailed study in Section 4 of the one-person, dynamic programming problem faced by a single player when the many-person model is in equilibrium (with constant price and interest rates). We are then able to show in Section 5 that equilibrium occurs for given price, interest rates, and wealth distribution, if two conditions hold: (i) the wealth distribution corresponds to the aggregate of the invariant measures for the Markov chains associated with the wealth processes of individual agents, and (ii) the bank balances its books by earning from borrowers exactly what it owes its depositors. After a collection of illustrative examples in Section 6, fairly general existence theorems are proved in Section 7 for the case of homogeneous agents. Section 8 offers a brief discussion of the model with a money-market (instead of a central bank).

3. The model

Time in the economy is discrete and runs $n = 1, 2, \ldots$. Uncertainty is captured by a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on which all the random variables of our model will be defined. There is a continuum of agents $\alpha \in I \triangleq [0, 1]$, distributed according to a *nonatomic* probability measure φ on the collection $\mathscr{B}(I)$ of Borel subsets of I. We regard such a nonatomic distribution of agents as a fairly reasonable approximation of an anonymous mass-market, in which the actions of a single agent do not have an appreciable effect on prices.

On each day, or time-period, $n = 1, 2, \ldots$, each agent $\alpha \in I$ receives a random endowment $Y_n^\alpha(w) = Y_n(\alpha, w)$ in units of a single perishable commodity. The endowments $Y_1^\alpha, Y_2^\alpha, \ldots$ for a given agent α are assumed to be nonnegative, integrable, and independent, with common distribution λ^α. We also assume that the variables $Y_n(\alpha, w)$ are jointly measurable in (α, w), so that the total endowment or ''production'':

$$Q_n(w) \triangleq \int Y_n(\alpha, w)\, \varphi(\mathrm{d}\alpha) > 0$$

is a well-defined, positive and finite random variable, for every n.

There is a loan-market and a commodity-market in each time-period $t = n$. For the loan-market, the bank sets two interest rates, namely, $r_{1n}(w) = 1 + \rho_{1n}(w)$ to be paid by borrowers and $r_{2n}(w) = 1 + \rho_{2n}(w)$ to be paid to depositors. In the commodity market, agents bid money for consumption of the commodity, thereby determining its price $p_n(w)$ endogenously as will be explained below. The interest rates are assumed to satisfy:

$$1 \le r_{2n}(w) \le r_{1n}(w) \quad \text{and} \quad r_{2n}(w) < \frac{1}{\beta}, \tag{3.1}$$

for all $n \in \mathbb{N}$, $w \in \Omega$, where $\beta \in (0, 1)$ is a fixed *discount factor*.

Each agent $\alpha \in I$ has a *utility function* u^α: $\mathbb{R} \to \mathbb{R}$, which is assumed to be increasing and concave with $u^\alpha(0) = 0$. For $x < 0$, $u^\alpha(x)$ is negative and measures the ''disutility'' for agent α of going bankrupt by an amount x; for $x > 0$,

$u^{\alpha}(x)$ is positive and measures the "utility" derived from the consumption of x units of the commodity.

At the beginning of day $t=n$, the price of the commodity is $p_{n-1}(w)$ (from the day before) and the total amount of money held in the bank is $M_{n-1}(w)$. An agent $\alpha \in I$ enters the day with wealth $S_{n-1}^{\alpha}(w)$. If $S_{n-1}^{\alpha}(w) < 0$, then agent α has an unpaid debt from the previous day, is assessed a nonmonetary punishment of $u^{\alpha}(S_{n-1}^{\alpha}(w)/p_{n-1}(w))$, is then forgiven the debt, and plays the game from the wealth position 0. If $S_{n-1}^{\alpha}(w) \geq 0$, then agent α has fiat money on hand and plays from position $S_{n-1}^{\alpha}(w)$. In both cases, an agent α (possibly after having been punished and forgiven) will play from the wealth position $(S_{n-1}^{\alpha}(w))^{+} = \max\{S_{n-1}^{\alpha}(w), 0\}$.

Agent α also begins day n with information $\mathscr{F}_{n-1}^{\alpha} \subset \mathscr{F}$, a σ-algebra of events that measures past prices p_k, past total endowments Q_k and interest-rates r_{1k}, r_{2k}, as well as past individual wealth-levels, endowments, and actions S_0^{α}, S_k^{α}, Y_k^{α}, b_k^{α}, for $k=1, \ldots, n-1$. (It may, or may not, measure the corresponding quantities for other agents.) Based on this information, agent α bids an amount:

$$b_n^{\alpha}(w) \in \left[0, \quad (S_{n-1}^{\alpha}(w))^{+} + k^{\alpha}\right], \tag{3.2}$$

of fiat money for the commodity on day n. The constant $k^{\alpha} \geq 0$ is an upper bound on allowable loans. It is assumed that the mapping $(\alpha, w) \mapsto b_n^{\alpha}(w)$ is $\mathscr{B}(I) \otimes \mathscr{F}_{n-1}$-measurable, where:

$$\mathscr{F}_{n-1} \triangleq \bigvee_{\alpha} \mathscr{F}_{n-1}^{\alpha}$$

is the smallest σ-algebra containing $\mathscr{F}_{n-1}^{\alpha}$ for all $\alpha \in I$. Consequently, the *total bid*:

$$B_n(w) \triangleq \int b_n^{\alpha}(w) \varphi(\mathrm{d}\alpha) > 0 \tag{3.3}$$

is a well-defined random variable, assumed to be strictly positive.

After the price $p_n(w)$ for day $t=n$ has been formed, according to the rule of Eq. (3.12) below, each agent α receives his bid's worth $x_n^{\alpha}(w) \triangleq b_n^{\alpha}(w)/p_n(w)$ of the commodity, consumes it in the same period (the "perishable" nature of the commodity), and thereby receives $u^{\alpha}(x_n^{\alpha}(w))$ in utility. The total utility that agent α receives during the period is thus:

$$\xi_n^{\alpha}(w) \triangleq \begin{Bmatrix} u^{\alpha}(x_n^{\alpha}(w)), & \text{if } S_{n-1}^{\alpha}(w) \geq 0 \\ u^{\alpha}(x_n^{\alpha}(w)) + u^{\alpha}(S_{n-1}^{\alpha}(w)/p_{n-1}(w)), & \text{if } S_{n-1}^{\alpha}(w) < 0 \end{Bmatrix}. \tag{3.4}$$

The total payoff for agent α during the entire duration of the game is the discounted sum $\sum_{n=1}^{\infty} \beta^{n-1} \xi_n^{\alpha}(w)$.

3.1. Strategies

A *strategy* π^α for an agent α specifies the bids b_n^α as random variables that satisfy Eq. (3.2) and are $\mathscr{F}_{n-1}^\alpha$-measurable (thus, also $\mathscr{F}_{n-1}$-measurable), for every $n \in \mathbb{N}$. A collection $\Pi = \{\pi_\alpha,\ \alpha \in I\}$ of strategies for all the agents is *admissible* if, for every $n \in \mathbb{N}$, the mapping $(\alpha,\ w) \rightarrowtail b_n^\alpha(w)$ is $\mathscr{G}_{n-1} \equiv \mathscr{B}(I) \otimes \mathscr{F}_{n-1}$-measurable. We shall always assume that the collection of strategies played by the agents is admissible. Consequently, the macro-variable $B_n(w)$ of Eq. (3.3), which represents the total bid in period n, is well-defined and $\mathscr{F}_{n-1}$-measurable.

3.2. Dynamics

In order to explain the dynamics of the model, we concentrate again on day $t = n$. After the bids for this day have been made and the price $p_n(w)$ has been formed, according to the rule of Eq. (3.12) below, the agents' endowments $Y_n^\alpha(w)$ are revealed and each agent α receives his endowment's worth $p_n(w)Y_n^\alpha(w)$ in fiat money, according to the day's price. Now there are three possible situations for agent α on day n:

(i) *Agent α is a depositor*: this means that α's bid $b_n^\alpha(w)$ is strictly less than his wealth $(S_{n-1}^\alpha(w))^+ = S_{n-1}^\alpha(w)$ and he deposits (or lends) the difference:

$$l_n^\alpha(w) \triangleq S_{n-1}^\alpha(w) - b_n^\alpha(w) = \left(S_{n-1}^\alpha(w)\right)^+ - b_n^\alpha(w). \tag{3.5}$$

(We set $l_n^\alpha(w)$ equal to 0 if $b_n^\alpha(w) \geq (S_{n-1}^\alpha(w))^+$.) At the end of the day, α gets back his deposit with interest, as well as his endowment's worth in fiat money and, thus, moves to the new wealth level:

$$S_n^\alpha(w) \triangleq r_{2n}(w) l_n^\alpha(w) + p_n(w) Y_n^\alpha(w) > 0. \tag{3.6}$$

(ii) *Agent α is a borrower*: this means that α's bid $b_n^\alpha(w)$ exceeds his wealth $(S_{n-1}^\alpha(w))^+$, so he must borrow the difference:

$$d_n^\alpha(w) \triangleq b_n^\alpha(w) - \left(S_{n-1}^\alpha(w)\right)^+. \tag{3.7}$$

(We set $d_n^\alpha(w)$ equal to 0 if $b_n^\alpha(w) \leq (S_{n-1}^\alpha(w))^+$.) At the end of the day, α owes the bank $r_{1n}(w)d_n^\alpha(w)$, and his new wealth position is:

$$S_n^\alpha(w) \triangleq p_n(w) Y_n^\alpha(w) - r_{1n}(w) d_n^\alpha(w),$$

a quantity which may be negative. Agent α is then *required* to pay back, from his endowment $p_n(w)Y_n^\alpha(w)$, as much of his debt $r_{1n}(w)d_n^\alpha(w)$ as he can. Thus, agent α pays back the amount:

$$h_n^\alpha(w) \triangleq \min\{r_{1n}(w) d_n^\alpha(w), p_n(w) Y_n^\alpha(w)\} \tag{3.8}$$

and his cash holdings at the end of the period are $(S_n^\alpha(w))^+ = p_n(w)Y_n^\alpha(w) - h_n^\alpha(w)$.

(iii) *Agent α neither borrows nor lends*: in this case, the agent bids his entire cash-holdings $b_n^\alpha(w) = (S_{n-1}^\alpha(w))^+$ and ends the day with exactly his endowment's worth in fiat money:

$$S_n^\alpha(w) = p_n(w) Y_n^\alpha(w) \geq 0.$$

Using the notation of Eqs. (3.5)–(3.8), we can write a single formula for α's wealth position at the end of the period:

$$S_n^\alpha(w) = p_n(w) Y_n^\alpha(w) + r_{2n}(w) l_n^\alpha(w) - r_{1n}(w) d_n^\alpha(w), \tag{3.9}$$

and another formula for α's cash-holdings:

$$(S_n^\alpha(w))^+ = p_n(w) Y_n^\alpha(w) + r_{2n}(w) l_n^\alpha(w) - h_n^\alpha(w). \tag{3.10}$$

The wealth position $S_n^\alpha(w)$ may be negative, but the amount $(S_n^\alpha(w))^+$ of holdings in cash is, of course, nonnegative.

3.3. The conservation of money

Let $M_n(w)$ be the total quantity of fiat money held by the bank, and:

$$\tilde{M}_n(w) \triangleq \int (S_n^\alpha(w))^+ \varphi(\mathrm{d}\alpha) \tag{3.11}$$

be the total amount of fiat money held by the agents, at the end of the time-period $t = n$. Thus, the total wealth in fiat money in the economy is $W_n(w) \triangleq M_n(w) + \tilde{M}_n(w)$. Consider the simple rule:

$$p_n(w) = \frac{B_n(w)}{Q_n(w)}, \tag{3.12}$$

which forms the commodity price as the ratio of the total bid to total production. This rule turns out to be both necessary and sufficient for the conservation of money. For the rest of the paper, we assume that Eq. (3.12) holds and, thus, money is conserved.[5]

Lemma 3.1. *The total quantity $W_n(w) = M_n(w) + \tilde{M}_n(w)$ of fiat money in the economy is the same for all n and w, if and only if Eq. (3.12) holds.*

[5] The combination of the assumption of a fixed time-grid for payments, together with a conservation law for fiat money, apparently give us a *quantity theory of money*, as the velocity of money is constrained to be less than or equal to one. Superficially, we have: Price × Quantity = Money × Velocity ($PQ = MV$), but this artifact of our model is for simplicity. Velocity can, and is, varied both by strategic delays in payments and by the creation and use of different financial instruments which are close money-substitutes.

Proof. Use Eqs. (3.5)–(3.11) to see that $l_n^\alpha(w) - d_n^\alpha(w) = ((S_{n-1}^\alpha(w))^+ - b_n^\alpha(w))^+ - (b_n^\alpha(w) - (S_{n-1}^\alpha(w))^+)^+ = (S_{n-1}^\alpha(w))^+ - b_n^\alpha(w)$, and check that Eq. (3.12) implies then:

$$\begin{aligned} M_n(w) - M_{n-1}(w) &= \int [l_n^\alpha(w) - d_n^\alpha(w) + h_n^\alpha(w) \\ &\quad - r_{2n}(w) l_n^\alpha(w)] \varphi(\mathrm{d}\alpha) \\ &= \int \left[(S_{n-1}^\alpha(w))^+ - b_n^\alpha(w) \right] \varphi(\mathrm{d}\alpha) \\ &\quad + \int [h_n^\alpha(w) - r_{2n}(w) l_n^\alpha(w)] \varphi(\mathrm{d}\alpha) \\ &= \tilde{M}_{n-1}(w) - p_n(w) \int Y_n^\alpha(w) \varphi(\mathrm{d}\alpha) \\ &\quad - \int [r_{2n}(w) l_n^\alpha(w) - h_n^\alpha(w)] \varphi(\mathrm{d}\alpha) \\ &= \tilde{M}_{n-1}(w) - \tilde{M}_n(w). \end{aligned}$$

Reasoning in the opposite direction, we see that Eq. (3.12) is also a necessary condition for the conservation principle $M_n(w) + \tilde{M}_n(w) = M_{n-1}(w) + \tilde{M}_{n-1}(w)$ to hold. ■

3.4. Equilibrium with exogenous interest rates

Interest rates are announced by the Central Bank, and can be viewed as exogenous in our model. (An interesting question for future research is how and to what extent the bank can control prices by its choice of interest rates. Lemma 7.5 below can be viewed as a first step in this direction.) In equilibrium, agents must be optimizing, given a rational forecast of interest rates and prices.

Let $\{r_{1n}, r_{2n}, p_n\}_{n=1}^\infty$ be a given system of interest rates and prices. The total expected utility to an agent α from a strategy π^α when $S_0^\alpha = s$, is given by:

$$I^\alpha(\pi^\alpha)(s) \triangleq \mathbb{E} \sum_{n=1}^{\infty} \beta^{n-1} \xi_n^\alpha(w),$$

in the notation of (3.4), and his optimal reward is:

$$V^\alpha(s) \triangleq \sup_{\pi^\alpha} I(\pi^\alpha)(s).$$

Definition 3.1. An *equilibrium* is a system of interest rates and prices $\{r_{1n}, r_{2n}, p_n\}_{n=1}^\infty$ and an admissible collection of strategies $\{\pi^\alpha, \alpha \in I\}$ such that:

(i) the prices $\{p_n\}_{n=1}^\infty$ satisfy Eq. (3.12), and
(ii) For every $\alpha \in I$, we have $I^\alpha(\pi^\alpha)(S_0^\alpha) = V^\alpha(S_0^\alpha)$ when every other agent $\beta \in I \setminus \{\alpha\}$ plays according to the strategy π^β.

Observe that we place no restrictions on interest rates in this definition. We are assuming implicitly that the bank sets interest rates arbitrarily, and has enough cash to cover all demands for loans and to meet all depositor requirements, in each period.

In this paper, we will not study the existence and structure of an equilibrium as general as that of Definition 3.1. We shall concentrate instead on the special case of a stationary equilibrium, which will be defined momentarily.

3.5. The distribution of wealth

An admissible collection of strategies $\{\pi^{\alpha},\ \alpha\in I\}$ together with an initial distribution for $\{S_0^{\alpha},\ \alpha\in I\}$ determines the random measures:

$$\nu_n(A,w)\triangleq\int 1_A(S_n^{\alpha}(w))\varphi(d\alpha),\ A\in\mathscr{B}(\mathbb{R}) \tag{3.13}$$

that describe the distribution of wealth across agents for $n=0,1,\ldots$

3.6. Stationary equilibrium

In order to obtain a stationary equilibrium, we must have a stationary economy. Thus, we shall assume from now on that the total production $Q_n(w)$ is constant, namely:

$$Q=\int Y_n(\alpha,w)\varphi(d\alpha)>0,\quad \text{for every}\quad w\in\Omega,\quad n\in\mathbb{N}. \tag{3.14}$$

A simple technique of Feldman and Gilles (1985) allows us to construct $\mathscr{B}(I)\otimes\mathscr{F}$-measurable functions:

$$(a,w)\mapsto Y_n^{\alpha}(w)=Y_n(\alpha,w):I\times\Omega\to[0,\infty) \tag{3.15}$$

for $n\in\mathbb{N}$, which have the desired properties.

Remark 3.1. *If, in particular, all the distributions $\lambda^{\alpha}\equiv\lambda$ $(\forall\alpha\in I)$ are the same, Feldman and Gilles (1985) show that the sequence of measurable functions Eq. (3.15) can be constructed in such a way that:*

(a) for every given $\alpha\in I$, the random variables $Y_1^{\alpha}(\cdot)$, $Y_2^{\alpha}(\cdot)$, ... are independent with common distribution λ;
(b) for every given $w\in\Omega$, the measurable functions $Y_1^{\bullet}(w)$, $Y_2^{\bullet}(w)$, ... are independent with common distribution λ; and
(c) Eq. (3.14) holds.

Thus, in this case, $Q=\int y\lambda(dy)>0$. ■

Definition 3.2. A *stationary Markov equilibrium* is an equilibrium $\{r_{1n}, r_{2n}, p_n\}_{n=1}^{\infty}$, $\{\pi^{\alpha}, \alpha \in I\}$ such that, in addition to conditions (i) and (ii) of Definition 3.1, the following are satisfied:

(iii) the interest rates $r_{1n}(w)$, $r_{2n}(w)$ and prices $p_n(w)$ have constant values r_1, r_2, and p, respectively;
(iv) the wealth distributions $\nu_n(\cdot, w)$ are equal to a constant measure μ;
(v) the quantities $M_n(w)$ and $\tilde{M}_n(w)$, corresponding to money held by the bank and by the agents, have constant values M and $\tilde{M}$, respectively; and
(vi) each agent $\alpha \in I$ follows a stationary Markov strategy π^{α}, which means that the bids b_n^{α} specified by π^{α} can be written in the form:

$$b_n^{\alpha}(w) = c^{\alpha}\left(\left(S_{n-1}^{\alpha}(w)\right)^{+}\right), \quad \text{for every } w \in \Omega, n \in \mathbb{N}$$

.

Here c^{α}: $[0, \infty) \to [0, \infty)$ is a measurable function such that $0 \le c^{\alpha}(s) \le s + k^{\alpha}$ for every $s \ge 0$.
The conditions (v) in Definition 3.2 are redundant, as is made clear by the following lemma.

Lemma 3.2. *In any equilibrium, conditions (i) and (iv) imply (v), and conditions (iv) and (v) imply (i).*

Proof. If (iv) holds, then $\tilde{M}_n(w) = \int (S_n^{\alpha}(w))^{+} \varphi(d\alpha) = \int s^{+} \nu_n(ds, w) = \int s^{+} \mu(ds)$ is the same for all n and w. Thus, both assertions follow from Lemma 3.1. ∎

If our model is in stationary Markov equilibrium, then *an individual agent faces a sequential optimization problem with fixed price and fixed interest rates*. After a detailed study of this one-person game in the next section, we shall return to the many-person model in Section 5.

4. The one-person game

Suppose that the model of the previous section is in stationary Markov equilibrium, and let us focus on the optimization problem facing a single agent. (We omit the superscript α in this section.) As we will now explain, this problem is a discounted dynamic programming problem in the sense of Blackwell (1965).

The interest rates $r_1 = 1 + \rho_1$, $r_2 = 1 + \rho_2$, and the discount factor β are assumed to satisfy Eq. (3.1) as before:

$$1 \le r_2 \le r_1 \quad \text{and} \quad r_2 < 1/\beta. \tag{4.1}$$

The *state space* $\mathscr{S}$ represents the possible wealth positions for the agent. Because the nonnegative number k is an *upper bound on loans*, the agent never owes more than $r_1 k$. Thus, we can take $\mathscr{S}=[-r_1 k, \infty)$. The price $p \in (0, \infty)$ remains fixed throughout. The agent's *utility function* u: $\mathbb{R} \to \mathbb{R}$ is, as before, concave and increasing with $u(0)=0$.

In each period, the agent begins at some state $s \in \mathscr{S}$. If $s<0$, the agent is punished by the amount $u(s/p)$ and is then allowed to play from state $s^+=0$. If $s \geq 0$, the agent chooses any *action* or *bid* $b \in [0, s+k]$, purchases b/p units of the commodity, and receives $u(b/p)$ in utility. In the terminology of dynamic programming, the *action set* is $B(s^+)$, where $B(s)=[0, s+k]$ for $s \geq 0$, and the daily *reward* of an agent at state s who takes action $b \in B(s)$ is:

$$r(s,b) = \begin{Bmatrix} u(b/p), & s \geq 0 \\ u(s/p)+u(b/p), & s<0 \end{Bmatrix}.$$

The remaining ingredient is the *law of motion* that specifies the conditional distribution $q(\cdot|s, b)$ of the next state s_1 by the rule:

$$s_1 = \begin{Bmatrix} -r_1(b-s^+)+pY, & s^+ \leq b \\ r_2(s^+-b)+pY, & s^+ \geq b \end{Bmatrix}.$$

Here Y is a nonnegative, integrable random variable with distribution λ. For ease of notation, we introduce the concave function:

$$g(x) \triangleq \begin{Bmatrix} r_1 x, & x \leq 0 \\ r_2 x, & x \geq 0 \end{Bmatrix} \equiv g(x;r_1,r_2). \tag{4.2}$$

Then the law of motion becomes $s_1 = g(s^+-b)+pY$.

A player begins the first day at some state s_0 and selects a *plan* $\pi=(\pi_1, \pi_2, \dots)$, where π_n makes a measurable choice of the action $b_n \in B(s_{n-1})$ as a function of $(s_0, b_1, s_1, \dots, b_{n-1}, s_{n-1})$. A plan π, together with the law of motion, determine the distribution of the stochastic process $s_0, b_1, s_1, b_2, \dots$ of states and actions. The *return* from π is the function:

$$I(\pi)(s) \triangleq \mathbb{E}^{\pi}_{s_0=s} \sum_{n=0}^{\infty} \beta^n r(s_n,b_{n+1}), \qquad s \in \mathscr{S}. \tag{4.3}$$

The *optimal return* or *value function* is:

$$V(s) \triangleq \sup_{\pi} I(\pi)(s), \qquad s \in \mathscr{S}. \tag{4.4}$$

A plan π is called *optimal* for the one-person game, if $I(\pi)=V$.

If the utility function $u(\cdot)$ is bounded, then so is $r(\cdot, \cdot)$, and our player's optimization problem is a discounted dynamic programming problem as in Blackwell (1965). In the general case, because $u(\cdot)$ is concave and increasing, we have:

$$u(-r_1 k) \leq u(s) \leq u(s^+) \leq u'_+(0)s^+ \tag{4.5}$$

for all $s \in \mathscr{S}$. This domination by a linear function is sufficient, as it was in [KSS1] (1994) and [KSS2] (1997), for many of Blackwell's results to hold in our setting as well. Thus, $V(\cdot)$ satisfies the *Bellman equation*:

$$V(s) = \sup_{b \in B(s)} \left[r(s,b) + \beta \mathbb{E} V(g(s^+ - b) + pY) \right]$$

$$V(s) = \begin{cases} \sup\limits_{0 \le b \le s+k} \left\{ u(b/p) + \beta \mathbb{E} V(g(s-b) + pY) \right\}; & s \ge 0 \\ u(s/p) + V(0); & s < 0 \end{cases}. \tag{4.6}$$

Equivalently, $V = TV$ where T is the operator:

$$(T\psi)(s) \triangleq \sup_{b \in B(s^+)} \left[r(s,b) + \beta \mathbb{E} \psi(g(s^+ - b) + pY) \right], \tag{4.7}$$

defined for measurable functions ψ: $\mathscr{S} \to \mathbb{R}$ that are bounded from below.

A plan π is called *stationary* if it has the form $b_n = c(s_{n-1}^+)$ for all $n \ge 1$, where c: $[0, \infty) \to [0, \infty)$ is a measurable function such that $c(s) \in B(s)$ for all $s \ge 0$. We call $c(\cdot)$ the *consumption function* for the stationary plan π.

The following characterization of optimal stationary plans, given by Blackwell (1965), extends easily to our situation, so we omit the proof.

Theorem 4.1. *For a stationary plan π with consumption function $c(\cdot)$, the following conditions are equivalent:*

(a) $I(\pi) = V$;
(b) $V(s) = r(s, c(s^+)) + \beta \mathbb{E} V(g(s^+ - c(s^+)) + pY)$, $s \in \mathscr{S}$;
(c) $T(I(\pi)) = I(\pi)$.

Under our assumptions, a stationary optimal plan exists but need not be unique. However, if the utility function $u(\cdot)$ is smooth and strictly concave, there is a unique optimal plan and the next theorem has some information about its structure.

For the rest of this section, we make the following assumption:

Assumption 4.1. *The utility function $u(\cdot)$ is concave and strictly increasing on $\mathscr{S}$, strictly concave on $[0, \infty)$, differentiable at all $s \ne 0$, and we have $u(0) = 0$, $u'_+(0) > 0$.*

Theorem 4.2. *Under Assumption 4.1, the following hold:*

(a) The value function $V(\cdot)$ is concave, increasing.
(b) There is a unique optimal stationary plan π corresponding to a continuous consumption function $c: [0, \infty) \to (0, \infty)$ such that $c(s) \in B(s)$ for all $s \in [0,\infty)$. Furthermore, the functions $s \mapsto c(s)$ and $s \mapsto s - c(s)$ are nondecreasing.
(c) For $s \in \mathscr{S} \setminus \{0\}$, the derivative $V'(s)$ exists and:

$$V'(s) = \begin{Bmatrix} \frac{1}{p}u'(c(s)/p); & s > 0 \\ \frac{1}{p}u'(s/p); & s < 0 \end{Bmatrix}.$$

(d) For $s > 0$, we have $c(s) > 0$. If $\beta r_l < 1$, then $c(0) > 0$.
(e) $\lim_{s \to \infty} c(s) = \infty$.
(f) If the limit $L \triangleq \lim_{s \to \infty}(s - c(s))$ is positive, it satisfies the equation:

$$\frac{u'(\infty)}{\beta r_2} = \mathbb{E}\left[u'(c(r_2 L + pY)/p)\right]$$

where $u'(\infty) \triangleq \lim_{x \to \infty} u'(x) = \inf_{x \in \mathbb{R}} u'(x)$; in particular, $L < \infty$ if $u' > 0$. If, furthermore, $c(\cdot)$ is strictly increasing and $u'(\infty) > 0$, then L is uniquely determined in $(0, \infty)$ by this equation.
(g) If $L > 0$ as in (f), then $s^* \triangleq \inf\{s > 0: \ s > c(s)\}$ is characterized by the equation:

$$u'(s^*/p) = \beta p r_2 \mathbb{E}[V'(pY)]$$

and we have $s^* > p\varepsilon_0$, where $\varepsilon_0 \triangleq \sup\{\varepsilon > 0: \ \mathbb{P}[Y \geq \varepsilon] = 1\}$.

Part (b) of the theorem asserts, inter alia, that under optimal play, an agent both consumes more and deposits more money, as his wealth increases. Part (c) is a version of the "envelope equation". Part (d) says that an agent with a positive amount of cash always spends a positive amount. However, because we have imposed no upper bound on the interest rate r_l, it could happen that $c(s) \leq s$ for all s, or, equivalently, that no borrowing occurs. Part (d) further asserts that if $\beta r_l < 1$, then there will be an active market for borrowing money.

The proof of Theorem 4.2 is presented in the following subsection. It is a bit lengthy; impatient readers may prefer to skip or skim it.

4.1. The proof of Theorem 4.2

The *n-day value function* $V_n(\cdot)$ represents the best a player can do in n days of playing the one-person game of Eqs. (4.3) and (4.4). It can be calculated by the induction algorithm:

$$V_1(s) = (T0)(s) = \begin{Bmatrix} u((s+k)/p), & s \geq 0 \\ u(s/p) + u(k/p), & s < 0 \end{Bmatrix},$$

$$V_{n+1}(s) = (TV_n)(s), \qquad s \in \mathscr{S}, n \geq 1. \tag{4.8}$$

Furthermore, it is not difficult to show, with the aid of Eq. (4.5), that:

$$\lim_{n \to \infty} V_n(s) = V(s), \qquad s \in \mathscr{S}. \tag{4.9}$$

The idea of the proof of Theorem 4.2 is to derive properties of the $V_n(\cdot)$ by a recursive argument based on Eq. (4.8), and then to deduce the desired properties of $V(\cdot)$. For the recursive argument, we consider functions w: $\mathscr{S} \to \mathbb{R}$ satisfying the following condition.

Condition 4.1. *The function $w(\cdot)$ is concave, increasing on $\mathscr{S}$, differentiable on $\mathscr{S} \setminus \{0\}$, with $w'_+(0) \leq (1/p)u'_+(0)$, and $w(s) = u(s/p) + w(0)$ for $s \leq 0$.*

Proposition 4.1. *If $w(\cdot)$ satisfies Condition 4.1, then so does $Tw(\cdot)$.*

Corollary 4.1. *For $n \geq 1$, $V_n(\cdot)$ satisfies Condition 4.1.*

Proof. Observe from Eq. (4.8) that $V_1(\cdot)$ satisfies the condition, and then apply the proposition and Eq. (4.8) again. ■

For the proof of Proposition 4.1 (established through a series of Lemmata 4.2–4.5 below), fix a concave, increasing function $w(\cdot)$ and set

$$\psi_s(b) = \psi(s,b) \triangleq u(b/p) + \beta \mathbb{E} w(g(s-b) + pY);$$

$$s \geq 0, \quad 0 \leq b \leq s+k. \tag{4.10}$$

Lemma 4.1. *Suppose that w: $\mathscr{S} \to \mathbb{R}$ is concave, increasing.*

(a) For each $s \geq 0$, $\psi_s(\cdot)$ is strictly concave on $[0, s+k]$ and attains its maximum at a unique point $c(s) \equiv c_w(s)$.
(b) $(s, b) \rightarrowtail \psi(s, b)$ is a concave function on the convex, two-dimensional set $\{(s, b): s \geq 0, 0 \leq b \leq s+k\}$.

Proof. Elementary, using the facts that $g(\cdot)$ of Eq. (4.2) and $w(\cdot)$ are concave, and $u(\cdot)$ is strictly concave on $[0, \infty)$; recall Assumption 4.1. ■

Now define $v(s) \equiv v_w(s) \triangleq (Tw)(s)$ for $s \in \mathscr{S}$. By Lemma 4.1(a), we can write:

$$v_w(s) = \begin{cases} u(c_w(s)/p) + \beta \mathbb{E} w(g(s - c_w(s)) + pY), & s \geq 0 \\ u(s/p) + v_w(0), & s < 0 \end{cases}. \quad (4.11)$$

It may be helpful to think of $v_w(\cdot)$ as the optimal return when an agent plays the game for one day and receives a terminal reward of $w(\cdot)$.

Lemma 4.2. *Suppose that* w: $\mathscr{S} \to \mathbb{R}$ *satisfies Condition 4.1. Then the function* $c_w(\cdot)$ *has the following properties:*

(a) Both $s \rightarrowtail c_w(s)$ *and* $s \rightarrowtail s - c_w(s)$ *are nondecreasing.*
(b) $c_w(\cdot)$ *is continuous.*
(c) For $s > 0$, *we have* $c_w(s) > 0$.
(d) If $\beta r_l < 1$, *then* $c_w(0) > 0$.
(e) $\lim_{s \to \infty} c_w(s) = \infty$.

Proof. For (a), note that the function $\tilde{w}(\cdot) \triangleq \beta \mathbb{E} w(g(\cdot) + pY)$ is concave, and thus the problem of maximizing $\psi_s(b) = u(b/p) + \tilde{w}(s - b)$ over $b \in [0, s + k]$ is a standard allocation problem for which (a) is well-known (see, for example, Theorem I.6.2 of Ross (1983)). Property (b) follows from (a). For (c), let $s > 0$; use Condition 4.1, the definition of $\psi_s(\cdot)$ in Eq. (4.10), and our standing assumption $\beta r_2 < 1$, to see that $p \cdot (\psi_s)'_+(0) = u'_+(0) - \beta r_2 p \mathbb{E} w'_-(r_2 s + pY) \geq u'_+(0) - \beta r_2 p \cdot w'_+(0) > 0$. To prove (d) notice that, for $s = 0$, the same calculation works with r_2 replaced by r_1. This is because of the definition of $g(\cdot)$ in Eq. (4.2). For the final assertion, see the proof of Theorem 4.3 in [KSS1] (1994), which relies crucially on the strict increase of $u(\cdot)$. ■

The proof that $v_w(\cdot)$ is concave will take three steps. The first is to show that $v_w(\cdot)$ is concave except possibly at the origin.

Lemma 4.3. *The function* $v_w = Tw$ *is concave on* $[-r_l k, 0]$ *and also on* $[0, \infty)$.

Proof. The concavity of $v_w(\cdot)$ on $[-r_1 k, 0]$ is clear from Eq. (4.11) and the concavity of $u(\cdot)$. For $s \geq 0$, $v_w(s) = \sup\{\psi(s, b)$: $0 \leq b \leq s + k\}$ is the supremum of a concave function (cf. Lemma 4.1(b)) over a convex set. It is well known that such an operation yields a concave function. ■

The second step is a version of the "envelope equation".

Lemma 4.4. *For $s \neq 0$, the function $v_w(\cdot)$ is differentiable at s and:*

$$v'_w(s) = \begin{Bmatrix} \frac{1}{p} u'(c_w(s)/p), & s>0 \\ \frac{1}{p} u'(s/p), & s<0 \end{Bmatrix}.$$

Proof. For $s<0$, the assertion is obvious from Eq. (4.11). Let us then fix $s>0$ and, for simplicity, write $v(\cdot)$ for $v_w(\cdot)$ and $c(\cdot)$ for $c_w(\cdot)$. Note first that, for $\varepsilon>0$, we have $v(s+\varepsilon) - v(s) \geq u((c(s)+\varepsilon)/p) - u(c(s)/p)$, since an agent with wealth $s+\varepsilon$ can spend $c(s)+\varepsilon$ and then be in the same position as an agent with s who spends the optimal amount $c(s)$. Hence, $\nu'_+(s) \geq (1/p)u'(c(s)/p)$. On the other hand, recall (c) and observe, for $0<\varepsilon<s \wedge c(s)$, that we have $v(s) - v(s-\varepsilon) \leq u(c(s)/p) - u((c(s)-\varepsilon)/p)$, since an agent with wealth $s-\varepsilon$ can spend $c(s)-\varepsilon$ and then be in the same position as an optimizing agent starting at s. Hence, $v'_-(s) \leq (1/p)u'(c(s)/p)$. Finally, $\nu'_-(s) \geq v'_+(s)$ because, by Lemma 4.3, $v(\cdot)$ is concave on $[0, \infty)$. ■

Lemma 4.5. *The function $v_w(\cdot)$ is concave on $\mathscr{S}$, and $(v_w)'_+(0) \leq (1/p)u'_+(0)$.*

Proof. By Lemma 4.3, it suffices for concavity to show that $(v_w)'_+(0) \leq (\nu_w)'_-(0)$. But by Lemma 4.4 and Eq. (4.11), we have $(v_w)'_+(0) = \lim_{s \downarrow 0}(1/p)u'_+(c_w(s)/p) = (1/p)u'_+(c_w(0)/p) \leq (1/p)u'_+(0) \leq (1/p)u'_-(0) = (v_w)'_-(0)$. ■

Proposition 4.1 follows from Eq. (4.11) and Lemmata 4.4 and 4.5. We are finally prepared to complete the proof of Theorem 4.2.

Proof of Theorem 4.2. From Corollary 4.1, the n-day value functions $V_n(\cdot)$ are concave, increasing; by Eq. (4.9), they converge pointwise to $V(\cdot)$. Hence, $V(\cdot)$ is also concave and increasing. By Lemma 4.1(a) with $w(\cdot) = V(\cdot)$ and Eq. (4.6), there is for each $s \geq 0$ a unique $c(s) \in [0, s+k]$ such that $V(s) = u(c(s)/p) + \beta \mathbb{E} V(g(s-c(s)) + pY)$. Set formally $c(s) \equiv c(0)$ for $-r_1 k \leq s < 0$, and it follows from Theorem 4.1 that $c(\cdot)$ corresponds to the unique optimal stationary plan.

Next, let $c_n(\cdot) \equiv c_{V_n}(\cdot)$, $n \geq 1$, where $c_{V_n}(\cdot)$ is the notation introduced in Lemma 4.1(a). It can be shown using the techniques of Schäl (1975) (see also Langen, (1981)), or by a direct argument, that $c_n(s) \to c(s)$ as $n \to \infty$ for each $s \in S$. Thus, the functions $c(\cdot)$ and $s \rightarrowtail s - c(s)$ are nondecreasing, because the same is true of $c_n(\cdot)$ and $s \mapsto s - c_n(s)$ for each n. In particular, $c(\cdot)$ in continuous. Now by Lemma 4.4, we can write:

$$p\left[V_{n+1}(s) - V_{n+1}(0)\right] = \int_0^s u'\left(c_n(x)/p\right)\mathrm{d}x, \qquad s \geq 0, n \in \mathbb{N},$$

and let $n \to \infty$, to obtain $p \cdot [V(s) - V(0)] = \int_0^s u'(c(x)/p)\mathrm{d}x$, $s \geq 0$. Differentiate to get part (c) of the theorem for $s > 0$. For $s < 0$, use Eq. (4.6).

Let $s \downarrow 0$ in (c) to get $pV'_+(0) = u'_+(c(0)/p) \leq u'_+(0)$. Thus, the value function V satisfies Condition 4.1. By the Bellman Eq. (4.6), $V = TV = v_V$ in the notation of Eq. (4.11) with $c = c_V$. Thus, parts (d) and (e) of the theorem follow from Lemma 4.2.

For parts (f) and (g), note that for all $s > 0$ such that $s > c(s)$, we have:

$$u'\left(c(s)/p\right) = \beta p r_2 \mathbb{E}\left[V'\left(r_2\left(s - c(s)\right) + pY\right)\right].$$

This leads to the characterization for s^* in part (g), as well as to:

$$\frac{1}{\beta r_2} u'\left(c(s)/p\right) \leq p\mathbb{E}\left[V'(pY)\right] = \mathbb{E}\left[u'\left(c(pY)/p\right)\right],$$

$$\text{for } s > 0, s > c(s),$$

thanks to (c). Therefore, letting $s \to \infty$, we obtain, in conjunction with (b), (c) and (e): $u'(\infty) \leq \beta r_2 \mathbb{E}[u'(c(pY)/p)]$ and:

$$\frac{u'(\infty)}{\beta r_2} = p\mathbb{E}\left[V'(Lr_2 + pY)\right] = \mathbb{E}\left[u'\left(c(r_2 L + pY)/p\right)\right].$$

Suppose now that $u'(\infty) > 0$, and that $c(\cdot)$ is strictly increasing; then the function $f(\cdot) \triangleq \mathbb{E}[u'(c(\cdot r_2 + pY)/p)]$ is strictly decreasing with $f(0+) = \mathbb{E}[u'(c(pY)/p)] \geq u'(\infty)\beta r_2$ and $f(\infty) = u'(\infty)$; thus, there is a *unique* root $l \in (0, \infty)$ of the equation $f(l) = u'(\infty)/\beta r_2$. Finally, for the inequality of (g), notice that we have:

$$V'(s^*) = \frac{1}{p}u'\left(c(s^*)/p\right) = \frac{1}{p}u'(s^*/p)$$
$$= \beta r_2 \mathbb{E}\left[V'(pY)\right] < \mathbb{E}\left[V'(pY)\right] \leq V'(p\epsilon_0),$$

and the inequality $s^* > p\varepsilon_0$ follows from the decrease of $V'(\cdot)$ on $(0, \infty)$. ■

4.2. The wealth-process of an agent

Suppose now that an agent begins the one-person game of this section with wealth $\mathscr{S}_0 = s_0$ and follows the stationary plan π of Theorem 4.2. The process $\{\mathscr{S}_n\}_{n \in \mathbb{N}}$ of the agent's successive wealth-levels then satisfies the rule:

$$S_n = g\left(S_{n-1}^+ - c\left(S_{n-1}^+\right)\right) + pY_n, \qquad n \geq 1, \tag{4.12}$$

where the endowment variables Y_1, Y_2, ... are independent with common distribution λ, and $g(\cdot)$ is the function of Eq. (4.2). Hence, $\{S_n\}_{n\in\mathbb{N}}$ is a Markov chain with state-space $\mathscr{S}=[-kr_1, \infty)$. An understanding of this Markov chain is essential to an understanding of the many-person game of Section 3. In particular, it is important to know when the chain has an invariant distribution with finite mean.

Theorem 4.3. *Under Assumption 4.1, the Markov chain* $\{\mathscr{S}_n\}_{n\in\mathbb{N}}$ *of Eq. (4.12) has an invariant distribution with finite mean, if either one of the following conditions holds:*

(a) $u'(\infty)>0,$

(b) $r_2=1$ *and* $\int y^2\lambda(dy)<\infty.$

Under the conditions (b), this invariant distribution is unique.

Sketch of Proof. (a) If $u'(\infty)>0$, one can show, as in Theorem 4.2(f) or as in Corollary 3.6 of [KSS2] (1997), that the function $g(s^+-c(s^+))$, $s\in\mathscr{S}$ is bounded, and then complete the proof as in Proposition 3.7 of [KSS2] (1997). For (b), one applies results of Tweedie (1988) as in the proof of Proposition 3.8 of [KSS2] (1997). For the uniqueness result under condition (b), one applies the arguments in the proof of Theorem 5.1 in [KSS2] (1997), pp. 992–994, with only very minor modifications, and taking advantage of the fact that the interval $R=[-r_1k, s^*]$ is a *regeneration set* for the Markov chain of Eq. (4.12). As in that proof, one shows that this set can be reached in finite expected time, using Theorem 4.2 ((e), (g)). ■

Remark 4.1. *For a given vector* $\boldsymbol{\theta}=(r_1, r_2, p)$ *of interest rates as in Eq. (4.1) and price* $p\in(0,\infty)$, *we denote by* $c_\theta(\cdot)\equiv c(\cdot;\boldsymbol{\theta})$, $\mu_\theta(\cdot)\equiv\mu(\cdot;\boldsymbol{\theta})$ *the optimal consumption function of Theorem 4.2 and the invariant measure of Theorem 4.3, respectively. If the bound on loans k is a function of* $\boldsymbol{\theta}$ *such that* $k(\boldsymbol{\theta})=k(r_1, r_2, p)=pk(r_1, r_2, 1)$, *then, as in Eq. (4.4) and Eq. (4.6) of [KSS1] (1994), we have the scaling properties:*

$$c_\theta(s)\equiv c(s;r_1,r_2,p)=pc\left(\frac{s}{p};r_1,r_2,1\right) \tag{4.13}$$

$$\mu_\theta(ds)\equiv\mu(ds;r_1,r_2,p)=\mu\left(\frac{ds}{p};r_1,r_2,1\right). \tag{4.14}$$

5. Conditions for stationary Markov equilibrium

We shall return in this section to the *Strategic Market Game* of Section 3 and show how to construct a stationary Markov equilibrium (Definition 3.2) for this game, using the basic building blocks of Section 4. This construction will rest on two basic assumptions (cf. Assumptions 5.1 and 5.2 below):

(i) Each agent uses a stationary plan, which is optimal for his own (one-person) game, and for which the associated Markov chain of wealth-levels Eq. (4.12) has an invariant distribution with finite mean.
(ii) The bank "balances its books", that is, selects r_1 and r_2 in such a way that it pays back (in the form of interest to depositors, and of loans to borrowers) what it receives (in the form of repayments with interest, from borrowers).

The construction is significantly simpler, at least analytically if not conceptually, when all the agents are "homogeneous", i.e., when they all have the same utility function $u^\alpha \equiv u$, income distribution $\lambda^\alpha \equiv \lambda$, and upper bound on loans $k^\alpha \equiv k$, $\forall \alpha \in I$. We shall deal with this case throughout, but refer the reader to [KSS1] (1994) and [KSS2] (1997) for aggregation techniques that can handle countably many types of homogeneous agents (and can be used in our present context as well).

Let us fix a price $p \in (0, \infty)$ for the commodity, and two interest rates $r_1 = 1 + \rho_2$ (from borrowers) and $r_2 = 1 + \rho_2$ (to depositors) as in Eq. (4.1).

Assumption 5.1. *The one-person game of Section 4 has a unique optimal plan π corresponding to a continuous consumption function c: $[0, \infty) \to [0, \infty)$, and the associated Markov chain of wealth-levels in Eq. (4.12) has an invariant distribution μ on $\mathscr{B}(\mathscr{S})$ with:*

$$\int s\mu(\mathrm{d}s) < \infty. \tag{5.1}$$

Assumption 5.2. *Under this invariant distribution μ of wealth-levels, the bank balances its books, in the sense that the total amount paid back by borrowers equals the sum of the total amount they borrowed, plus the total amount that the bank pays to lenders in the form of interest:*

$$\int\int\{py \wedge r_1 d(s^+)\}\mu(\mathrm{d}s)\lambda(\mathrm{d}y) = \int d(s^+)\mu(\mathrm{d}s) + \rho_2\int l(s^+)\mu(\mathrm{d}s). \tag{5.2}$$

Here, we have denoted by:

$$d(s) \triangleq (c(s) - s)^+, \quad l(s) \triangleq (s - c(s))^+ \tag{5.3}$$

the amounts of money borrowed and deposited, respectively, under optimal play in the one-person game, by an agent with wealth-level $s \geq 0$.

Theorems 4.2 and 4.3 provide sufficient conditions for Assumption 5.1 to hold. We shall derive in Section 7 similar, though somewhat less satisfactory, sufficient conditions for Assumption 5.2. In Section 6, we shall present several examples that can be solved explicitly. If the initial wealth distribution ν_0 is equal to μ, and if every agent uses the plan π, then Eq. (5.2) just says that the quantities M_0, M_1, ... of money held by the bank in successive periods are equal to a constant as in Definition 3.2(v). Thus, Assumption 5.2 is a necessary condition for the existence of a stationary Markov equilibrium.

Lemma 5.1. *Under the Assumptions 5.1 and 5.2, we have:*

$$p = \frac{1}{Q} \int c(s^+) \mu(\mathrm{d}s). \tag{5.4}$$

Proof. From Eqs. (4.12) and (5.2), we obtain $S_1 = g(S_0^+ - c(S_0^+)) + pY_1 = pY_1 + r_2 l(S_0^+) - r_1 d(S_0^+)$, so that:

$$\begin{aligned} E(S_1^+) &= \mathbb{E}\left[\left(pY_1 + r_2 l(S_0^+)\right) 1_{\{d(S_0^+)=0\}}\right] \\ &\quad + \mathbb{E}\left[\left(pY_1 - r_1 d(S_0^+)\right) 1_{\{0 < r_1 d(S_0^+) \leq pY_1\}}\right] \\ &= p\mathbb{E}(Y_1) - \mathbb{E}\left[pY_1 1_{\{r_1 d(S_0^+) > pY_1\}}\right] \\ &\quad + r_2 \mathbb{E}\left[l(S_0^+)\right] - \mathbb{E}\left[r_1 d(S_0^+) 1_{\{r_1 d(S_0^+) \leq pY_1\}}\right] \\ &= pQ + r_2 \mathbb{E} l(S_0^+) - \mathbb{E}\left[pY_1 \wedge r_1 d(S_0^+)\right]. \end{aligned}$$

Here S_0 and Y_1 are independent random variables with distributions μ and λ, respectively. From Eq. (5.2), the last expectation above is $\mathbb{E}[pY_1 \wedge r_1 d(S_0^+)] = \mathbb{E}[d(S_0^+) + \rho_2 l(S_0^+)]$, so that

$$\mathbb{E}(S_1^+) = pQ + \mathbb{E}\left[l(S_0^+) - d(S_0^+)\right] = pQ - \mathbb{E}c(S_0^+) + \mathbb{E}(S_0^+).$$

But, from Assumption 5.1, S_1 has the same distribution as S_0 (namely μ), so that in particular $\mathbb{E}(S_0^+) = \mathbb{E}(S_1^+)$, and thus $p = \mathbb{E}[c(S_0^+)]/Q$. ■

Theorem 5.1. *Suppose that for fixed interest rates r_1, r_2 as in Eq. (4.1), we can find a price $p \in (0, \infty)$ such that the consumption function $c(\cdot) \equiv c_\theta(\cdot)$ and the probability measure $\mu \equiv \mu_\theta$ (notation of Remark 4.1 with $\boldsymbol{\theta} = (r_1, r_2, p)$) satisfy the Assumptions 5.1 and 5.2. Let π be the corresponding optimal stationary plan; then the family $\Pi = \{\pi^\alpha\}_{\alpha \in I}$, $\pi_\alpha = \pi$ ($\forall \alpha \in I$) results in a stationary Markov equilibrium (p, μ_θ) with $\boldsymbol{\theta} = (r_1, r_2, p)$.*

Remark 5.1. *From the scaling properties Eqs. (4.13) and (4.14) and from Eq. (5.2), it is clear that if the procedure of Theorem 5.1 leads to Stationary Markov* *equilibrium for some $p \in (0, \infty)$, then it does so for every $p \in (0, \infty)$. For a given, constant level W_0 of total wealth in the economy, we can then determine the "right" price $p_\# \in (0, \infty)$ via:*

$$W_0 - M_0 = \int \left(S_0^\alpha(w)\right)^+ \varphi(\mathrm{d}\alpha) = \int s^+ \nu_0(\mathrm{d}s, w)$$

$$= \int s^+ \mu(\mathrm{d}s; r_1, r_2, p_\#) = \int s^+ \mu\left(\frac{\mathrm{d}s}{p_\#}; r_1, r_2, 1\right),$$

namely as:

$$p_\# = (W_0 - M_0) / \int s^+ \mu(\mathrm{d}s; r_1, r_2, 1). \tag{5.5}$$

Recall Eq. (3.12) and the discussion following it, as well as Eq. (4.14).

Proof of Theorem 5.1. From Remark 3.1, the Markov Chain:

$$S_n^\alpha(w) = g\left(\left(S_{n-1}^\alpha(w)\right)^+ - c_\theta\left(\left(S_{n-1}^\alpha(w)\right)^+\right) + pY_n^\alpha(w)\right), \qquad n \in \mathbb{N}$$

of Eq. (4.12) has the same dynamics for each fixed $\alpha \in I$, as for each fixed $w \in \Omega$. In particular, $\mu = \mu_\theta$ is a stationary distribution for the chain $\{S_n^\alpha(\cdot)\}_{n \in \mathbb{N}}$ for each given $\alpha \in I$, as well as for the chain $\{S_n^\bullet(w)\}_{n \in \mathbb{N}}$ for each given $w \in \Omega$. Assume that the initial price is $p_0 = p \in (0, \infty)$, and that the initial wealth-distribution ν_0 of Eq. (3.13) with $n = 0$, is $\nu_0 = \mu \equiv \mu_\theta$. Then from Eqs. (3.12) and (5.4):

$$p_1(w) = \frac{1}{Q} \int_I c_\theta\left(\left(S_n^\alpha(w)\right)^+\right) \varphi(\mathrm{d}\alpha) = \frac{1}{Q} \int_S c(s^+) \mu(\mathrm{d}s) = p.$$

On the other hand, since μ is invariant for the chain, we have $\nu_1 = \mu$ as well. By induction, $p_n = p$ and $\nu_n = \mu$ ($\forall n \in \mathbb{N}$).
Condition (i) of Definition 3.1 is true by assumption, and we have verified (iii) and (iv) of Definition 3.2, whereas (vi) holds by our choice of $\pi^\alpha = \pi$. Condition (ii) of Definition 3.1 follows from the optimality of π in the one-person game and from the fact that a change of strategy by a single player cannot alter the price. Condition (v) of Definition 3.2 follows from Lemma 3.2. ∎

6. Examples

We consider in this section three examples, for which the one-person game of Section 4 and the stationary Markov equilibrium of Theorem 5.1 can be computed explicitly.

Example 6.1. Suppose that all agents have the same utility function:

$$u(x) = \begin{cases} x; & x \le 1 \\ 1; & x > 1 \end{cases}, \tag{6.1}$$

the same upper-bound on loans $k = \delta$, and the same income distribution:

$$\mathbb{P}[Y=0] = 1-\delta, \quad \mathbb{P}[Y=2] = \delta \quad \text{for some} \quad 0 < \delta < 1/2, \tag{6.2}$$

so that $Q = \mathbb{E}(Y) = 2\delta < 1$. Suppose also that the bank sets interest rates:

$$r_1 = \frac{1}{\delta}, r_2 = 1. \tag{6.3}$$

We claim then that, for sufficiently small values of the discount parameter, namely $\beta \in (0, \delta)$, and with price:

$$p = 1, \tag{6.4}$$

the optimal policy in the one-person game of Section 4 is given as:

$$c(s) = \begin{cases} s+\delta, & 0 \le s \le 1-\delta \\ 1, & s \ge 1-\delta \end{cases}; \tag{6.5}$$

that the invariant measure μ of the corresponding (optimally controlled) Markov chain Eq. (4.12) has $\mu_k \equiv \mu(\{k\})$ given by:

$$\mu_{-1} = (1-\delta)(1-\eta), \quad \mu_0 = (1-\delta)(1-\eta)\eta,$$
$$\mu_k = (1-\eta)\eta^k \qquad (k \in \mathbb{N}) \tag{6.6}$$

with $\eta \triangleq \delta/(1-\delta)$; and that the pair (p, μ) of Eqs. (6.4) and (6.6) then corresponds to a stationary Markov equilibrium as in Theorem 5.1.

With $c(\cdot)$ given by Eq. (6.5), the amounts borrowed and deposited are given by:

$$d(s) = \begin{cases} \delta; & 0 \le s \le 1-\delta \\ 1-s; & 1-\delta \le s \le 1 \\ 0; & s \ge 1 \end{cases} \quad \text{and} \quad l(s) = \begin{cases} 0; & 0 \le s \le 1 \\ s-1; & s \ge 1 \end{cases}, \tag{6.7}$$

respectively, in the notation of Eq. (5.3), whereas the Markov chain of Eq. (4.12) takes the form:

$$S_{n+1} = \begin{cases} -1 + Y_{n+1}; & 0 \le S_n^+ \le 1-\delta \\ -(1-S_n^+)/\delta + Y_{n+1}; & 1-\delta \le S_n^+ \le 1 \\ S_n^+ - 1 + Y_{n+1}; & S_n^+ \ge 1 \end{cases}.$$

After a finite number of steps, this chain only takes values in $\{-1, 0, 1, 2, \dots\}$ with transition probabilities:

$$p_{-1,-1} = 1-\delta, \quad p_{-1,1} = \delta; \quad p_{n,n+1} = \delta, \quad p_{n,n-1} = 1-\delta \qquad (n \in \mathbb{N}_0).$$

The probability measure μ of Eq. (6.6) is the unique invariant measure of a Markov Chain with these transition probabilities.

Consider now the return function $Q(s) = I(\pi)(s)$, $S_0 = s$ corresponding to the stationary strategy π of Eq. (6.5) in the one-person game. This function satisfies $Q(s) = u(c(s)) + \beta \mathbb{E} Q(g(s - c(s)) + Y)$, $s \geq 0$ and $Q(s) = u(s) + Q(0)$, $s \leq 0$, or equivalently:

$$Q(s) = \begin{Bmatrix} s + Q(0); & s \leq 0 \\ (s+\delta) + \beta \mathbb{E} Q(-1+Y); & 0 \leq s \leq 1-\delta \\ 1 + \beta \mathbb{E} Q\left(\dfrac{s-1}{\delta} + Y\right); & 1-\delta \leq s \leq 1 \\ 1 + \beta \mathbb{E} Q(s-1+Y); & s \geq 1 \end{Bmatrix}. \tag{6.8}$$

In order to check the optimality of this strategy for the one-person game, it suffices to show (by Theorem 4.1) that Q satisfies the Bellman equation $Q = TQ$ (in the notation of Eq. (4.7)). This verification is carried out in Appendix A of [GKSS].

Let us check now the balance Eq. (5.2); it takes the form:

$$\int \int \left\{ y \wedge \frac{d(s^+)}{\delta} \right\} \mu(\mathrm{d}s) \lambda(\mathrm{d}y) = \int d(s^+) \mu(\mathrm{d}s), \tag{6.9}$$

which is satisfied trivially, since both sides are equal to $\delta(\mu(\{-1\}) + \mu(\{0\}))$, from Eqs. (6.2–6.7. Thus, the Assumptions 5.1 and 5.2 are both satisfied, and the pair (p, μ) of Eqs. (6.4) and (6.6) corresponds to a stationary Markov equilibrium.

Example 6.2. Suppose that all agents have the same utility function:

$$u(x) = \begin{Bmatrix} x; & x \leq 1 \\ 1 + \eta(x-1); & x > 1 \end{Bmatrix} \tag{6.10}$$

for some $0 < \eta < 1$, the same upper-bound on loans $k = 1$, and the same income distribution:

$$\mathbb{P}[Y=0] = 1-\delta, \quad \mathbb{P}[Y=5] = \delta \qquad \text{for some} \quad \frac{1}{3} < \delta < \frac{1}{2}. \tag{6.11}$$

Suppose also that the bank fixes the interest rates $r_1 = 1/\delta$, $r_2 = 1$ as in Eq. (6.3). We claim that for sufficiently small values of the discount-parameter, namely $\beta \in (0, 1/3)$, for suitable values of the slope-parameter $\eta \in (0, 1)$, and with price $p = 1$ as in Eq. (6.4), the optimal policy in the one-person game of Section 4 is given as

$$c(s) = \begin{Bmatrix} 1; & 0 \leq s \leq 2 \\ s-1; & s \geq 2 \end{Bmatrix}; \tag{6.12}$$

the invariant measure of the corresponding Markov chain in Eq. (4.12) has $\mu_k = \mu(\{k\})$ given by:

$$\mu_{-1/\delta} = (1-\delta)^3, \quad \mu_0 = \delta(1-\delta)^2, \quad \mu_1 = \delta(1-\delta),$$
$$\mu_{5-1/\delta} = \delta(1-\delta)^2, \quad \mu_5 = \delta^2(1-\delta), \quad \mu_6 = \delta^2; \tag{6.13}$$

and the pair (p, μ) of Eqs. (6.13) and (6.4) corresponds to a stationary Markov equilibrium for the strategic market game.

For the consumption strategy of Eq. (6.12), the amounts borrowed and deposited by an agent with wealth $s \geq 0$ are given as $d(s) = (1-s)^+$ and $l(s) = (s-1)1_{[1,2]}(s) + 1_{(2,\infty)}(s)$, respectively, and the Markov Chain of Eq. (4.12) becomes:

$$S_{n+1} = \begin{Bmatrix} (S_n^+ - 1)/\delta + Y_{n+1}; & 0 \leq S_n^+ \leq 1 \\ S_n^+ - 1 + Y_{n+1}; & 1 \leq S_n^+ \leq 2 \\ 1 + Y_{n+1}; & S_n^+ \geq 2 \end{Bmatrix}. \tag{6.14}$$

After a finite number of steps, the chain $\{S_n\}$ takes values in the set $\{-1/\delta, 0, 1, 5-1/\delta, 5, 6\}$ with transition probabilities given by the matrix:

$$\begin{pmatrix} 1-\delta & 0 & 0 & \delta & 0 & 0 \\ 1-\delta & 0 & 0 & \delta & 0 & 0 \\ 0 & 1-\delta & 0 & 0 & \delta & 0 \\ 0 & 0 & 1-\delta & 0 & 0 & \delta \\ 0 & 0 & 1-\delta & 0 & 0 & \delta \\ 0 & 0 & 1-\delta & 0 & 0 & \delta \end{pmatrix}. \tag{6.15}$$

It is not hard to check that the measure μ of Eq. (6.13) is the unique invariant measure for a Markov chain with the transition probability matrix of Eq. (6.15). The optimality of the strategy Eq. (6.12) for the one-person game, is verified in Appendix B of [GKSS]. On the other hand, the balance Eq. (5.2) takes again the form Eq. (6.9) and is again satisfied trivially, since both sides are now equal to $\mu_{-1/\delta} + \mu_0 = (1-\delta)^2$. Therefore, Assumptions 5.1 and 5.2 are both satisfied and the pair (p, μ) of Eq. (6.4), Eq. (6.13) is a stationary Markov equilibrium, from Theorem 5.1.

Example 6.3. Suppose that all agents have the same utility function:

$$u(x) = \begin{Bmatrix} 2x; & x \leq 0 \\ x; & x \geq 0 \end{Bmatrix}, \tag{6.16}$$

the same upper bound on loans $k = 1$, and the same income distribution:

$$\mathbb{P}[Y=0] = \mathbb{P}[Y=2] = \frac{1}{2}. \tag{6.17}$$

In particular, $Q = \mathbb{E}(Y) = 1$. Suppose also that the bank sets interest rates:

$$r_1 = r_2 = 2. \tag{6.18}$$

We claim then that, with $0 < \beta < 1/3$ and $p = 1$, the optimal consumption policy is:

$$c(s) = s + 1, \qquad s \geq 0 \tag{6.19}$$

(borrow up to the limit, and consume everything). The corresponding Markov Chain of Eq. (4.12) becomes trivial, namely $S_{n+1} = 2(S_n^+ - c(S_n^+)) + Y_{n+1} = -2 + Y_{n+1}$, $n \geq 0$ and has invariant measure $\mu_0 = \mu_{-2} = 1/2$. In equilibrium, everybody borrows $k = 1$, half the agents pay back 2, the other half pay back nothing, and so the bank balances its books (Eq. (5.2) is satisfied).

On the other hand, with $1/3 < \beta < 1/2$ and $p = 1$, we claim that the optimal consumption policy is:

$$c(s) = s, \qquad s \geq 0, \tag{6.20}$$

i.e., neither to borrow nor to lend, and to consume everything at hand. The Markov Chain of Eq. (4.12) is again trivial, $S_{n+1} = Y_{n+1}$, $n \geq 0$ and has invariant measure $\mu_0 = \mu_2 = 1/2$; again the books balance (Eq. (5.2) is satisfied), because there are neither borrowers nor lenders. These claims are verified in Appendix C of [GKSS].

7. Two existence theorems

From Theorem 5.1, we know that a stationary Markov equilibrium (Definition 3.2) for our strategic market game exists, if (i) each agent's optimally controlled Markov chain has a stationary distribution with finite mean, and (ii) the bank balances its books. Condition (i) follows from natural assumptions about the model, as in Theorems 4.2 and 4.3. However, condition (ii) is more delicate, and so it is of interest to have existence results that do not rely on this assumption. We provide two such results in Theorems 7.1 and 7.2 below.

Theorem 7.1. *Suppose that the following hold:*

(i) Assumption 4.1.
(ii) Agents have common utility function $u(\cdot)$, upper bound k on loans, and income distribution λ.
(iii) $\lambda(\{0\}) = 1 - \delta$, $\lambda([a, \infty)) = \delta$ for some $0 < \delta < 1$, $0 < a < \infty$; $\int y^2 \lambda(dy) < \infty$.

Then, with interest rates $r_1 = 1/\delta$ and $r_2 = 1$, the pair (p, μ_θ) corresponds to a stationary Markov equilibrium, for any $p \in [k/a\delta, \infty)$ and with $\boldsymbol{\theta} = (r_1, r_2, p)$.

Proof. Theorem 4.3 guarantees that, under conditions (i) and (iii), the optimally controlled Markov Chain of Eq. (4.12) has an invariant distribution $\mu = \mu_\theta$ with finite mean, where $\boldsymbol{\theta} = (1/\delta,\ 1, p)$, $\forall p \geq k/a\delta$. Thus, Assumption 5.1 is satisfied, and in order to prove the result it suffices (by Theorem 5.1) to check the balance Eq. (5.2), now in the form:

$$\int_{[a,\infty]}\int_{\mathscr{S}}\left[py \wedge \frac{1}{\delta}d(s^+)\right]\mu(\mathrm{d}s)\,\lambda(\mathrm{d}y) = \int_{\mathscr{S}} d(s^+)\,\mu(\mathrm{d}s).$$

(As $\rho_2 = 0$, the bank pays no interest to depositors, and balancing its books means that the bank gets back from the borrowers exactly what they had received in loans.) Now, for any $s \in \mathscr{S}$, $y \geq a$ and $p \geq k/a\delta$, we have $py \geq (k/a\delta)\cdot a = (k/\delta) \geq d(s^+)/\delta$ and, thus, the left-hand side of Eq. (5.2) equals, by assumption:

$$\frac{1}{\delta}\int\int d(s^+)\,\mu(\mathrm{d}s)\,\lambda(\mathrm{d}y) = \frac{\lambda([a,\infty))}{\delta}\int_S d(s^+)\,\mu(\mathrm{d}s) = \int_S d(s^+)\,\mu(\mathrm{d}s).$$

■

The conclusion of Theorem 7.1 holds for Examples 6.1 and 6.2; these satisfy its conditions (ii) and (iii), though not its condition (i). Observe also that, under the conditions of Theorem 7.1, we have:

$$Q = \int y\lambda(\mathrm{d}y) \geq a\lambda([a,\infty)) = a\delta \geq \frac{k}{p}. \tag{7.1}$$

Theorem 7.2. *Suppose the following hold:*

(i), (ii) as in Theorem 7.1.
(iii) $\lambda([0,\ y^*]) = 1$, for some $y^* \in (0,\ \infty)$.
(iv) $u'(\infty) > 0$.

Then there exist interest rates $r_1 \in [1,\ y^*/Q]$, $r_2 = 1$, a price $p \in (k/Q,\ \infty)$ and a probability measure μ_θ on $\mathscr{B}(\mathscr{S})$, such that the pair $(p,\ \mu_\theta)$ with $\boldsymbol{\theta} = (r_1,\ r_2,\ p)$ corresponds to a stationary Markov equilibrium.

Note that Example 6.2 satisfies conditions (ii), (iii) and (iv), as well as the conclusion, of this result.

It seems likely that all of the assumptions of Theorem 7.2 can be weakened. In particular, it should be possible to replace (ii) by the assumption there are finitely many types of utility functions and income distributions. A more challenging generalization would be to eliminate (iv), and perhaps replace (iii) by the assumption that λ has finite second moment.

The rest of this section is devoted to the proof of Theorem 7.2, which will rely on Kakutani's fixed point theorem. Before applying Kakutani's theorem, we shall deal with three technical problems: (1) bounding the Markov Chain corresponding to an optimal plan and thereby bounding the stationary distribution, (2) bounding the price, and (3) finding interest rates to balance the books.

7.1. Bounding the Markov chain

Let $\boldsymbol{\theta}=(r_1, r_2, p)$ be a vector of parameters for the one-person game of Section 4. The discount factor will be held constant, but the upper bound on loans will be a function of p, namely:

$$k(p)=pk_1, \qquad \text{for some} \quad 0<k_1<Q, \tag{7.2}$$

where k_1 is the bound when the price p is equal to 1. The inequality $k(p)/p \equiv k_1 < Q$ says that the bank imposes a loan limit strictly less than an agent's expected income. In order to guarantee that the books balance, it is intuitively clear that *the upper-bound on loans cannot exceed the monetary value of expected income*, as was observed already in Eq. (7.1).

To show their dependence on $\boldsymbol{\theta}$, we now write $c_\theta(\cdot)$ for the optimal consumption function of Theorem 4.2 as in Remark 4.1 and use $V_\theta(\cdot)$ to denote the value function. Likewise, the function $g(\cdot)$ of Eq. (4.2) is written $g_\theta(\cdot)$ to indicate its dependence on the interest rates r_1 and r_2. We also write $\mathscr{S}_\theta$ for the state-space $[-r_1 k(p), \infty)$.

Let $\{S_n\}_{n\in\mathbb{N}}$ be the Markov chain of successive wealth-levels for an agent who uses $c_\theta(\cdot)$ in the one-person game with parameter $\boldsymbol{\theta}$. We can rewrite Eq. (4.12) to show the dependence on $\boldsymbol{\theta}$ as:

$$S_n=g_\theta\left(S_{n-1}^+-c_\theta\left(S_{n-1}^+\right)\right)+pY_n, \qquad n\geq 1, \tag{7.3}$$

where $Y_1, Y_2, \ldots$ are independent with common distribution λ. These random variables are uniformly bounded by condition (iii) of Theorem 7.2, so that bounding the chain is tantamount to bounding the function $s \mapsto s^+-c_\theta(s^+)$. (The bounding of the price p is treated in the next subsection.) It will also be important to obtain a uniform bound over an appropriate collection of $\boldsymbol{\theta}$-values. Let us fix $p^*\in(0,\infty)$, $r_2^*\in[1, 1/\beta)$, $r_1^*\in[r_2^*,\infty)$ and introduce the parameter-space:

$$\Theta \triangleq \left\{(r_1,r_2,p):\ 1\leq r_2\leq r_1\leq r_1^*, \qquad r_2\leq r_2^*, \qquad 0<p\leq p^*\right\}. \tag{7.4}$$

Lemma 7.1. $\eta^* \triangleq \sup\{|s^+-c_\theta(s^+)|:\ \boldsymbol{\theta}\in\Theta,\ s\in\mathscr{S}_\theta\}<\infty.$

Proof. Since $0<c_\theta(s)\leq s+k(p)=s+pk_1\leq s+p^*k_1$, we need only consider those values of s and $\boldsymbol{\theta}$ for which $s^+-c_\theta(s^+)>0$ and, in particular, $s>0$. Furthermore, we have $s-c_\theta(s)=p[s/p-c(s/p;\ r_1, r_2, 1)]$ from Eq. (4.13), so that the supremum of $s-c_\theta(s)$ over Θ is the same as that over the compact set $K\triangleq\{\boldsymbol{\theta}\in\Theta:\ p=p^*\}$.

Now let $s > 0$ and $s > c_\theta(s)$. By Theorem 4.2(d–f), we have that $c_\theta(s) > 0$, and that the number $L_\theta \triangleq \lim_{s\to\infty}(s - c_\theta(s)) \in (0, \infty)$ satisfies

$$\frac{u'(\infty)}{\beta r_2^*} = \mathbb{E}\left[u'\left(\frac{c_\theta(r_2 L_\theta + pY)}{p}\right)\right] \leq u'\left(\frac{c_\theta(r_2 L_\theta)}{p}\right).$$

With I: $(u'(\infty), u'_+(0)) \to (0, \infty)$ denoting the inverse of the function $u'(\cdot)$ on $(0, \infty)$, we get then

$$c_\theta(r_2 L_\theta) \leq pI\left(\frac{u'(\infty)}{r_2 \beta}\right) \leq p^* I\left(\frac{u'(\infty)}{r_2^* \beta}\right),$$

since $u'(\infty) < u'(\infty)/\beta r_2^* \leq u'(\infty)/\beta r_2 < \infty$. Define:

$$\eta(\theta) \triangleq \sup\left\{s \geq 0 \ : \ c_\theta(s) \leq p^* I\left(\frac{u'(\infty)}{r_2^* \beta}\right)\right\}. \tag{7.5}$$

Then $\eta(\boldsymbol{\theta}) < \infty$ for each $\boldsymbol{\theta} \in K$ because $c_\theta(s) \to \infty$ as $s \to \infty$ (Theorem 4.2(e)). Also, as in [KSS2, Proposition 3.4], $\boldsymbol{\theta} \rightarrowtail c_\theta(s)$ is continuous for fixed s. This fact, together with the continuity and monotonicity of $c_\theta(\cdot)$, can be used to check that $\eta(\cdot)$ is upper-semicontinuous. Therefore, for $\boldsymbol{\theta} \in K$ and $s > c_\theta(s)$, $s > 0$, we have $s - c_\theta(s) \leq r_2(s - c_\theta(s)) \leq r_2 L_\theta \leq \sup_K \eta(\boldsymbol{\theta}) < \infty$. ■

Lemma 7.2. *Let* $\{S_n\}_{n\in\mathbb{N}}$ *be the Markov chain Eq. (7.3) corresponding to optimal play in the one-person game with parameter* $\boldsymbol{\theta} \in \Theta$. *Then, whatever the distribution of* S_0, *the distributions of* S_n^+, $n \geq 1$, *are supported on the interval* $[0, r_1^* \eta^* + p^* y^*]$.

Proof. Immediate from Eqs. (7.2) and (7.3), and Lemma 7.1.

7.2. Bounding the price

Assume that the total amount of fiat money in our many-person model is the positive quantity W. (Recall from Lemma 3.1 that W is preserved from period to period.) Let γ be the distribution of fiat money among agents. Notice that γ differs from the distribution of wealth positions μ, in that those agents with negative wealth positions hold no fiat money. Thus, $\mu(A) = \gamma(A \cap [0, \infty))$ for $A \in \mathscr{B}(\mathbb{R})$.

Suppose that in a certain period the parameters of the model are given by the vector $\boldsymbol{\theta} = (r_1, r_2, p)$ and that all agents bid according to $c_{\boldsymbol{\theta}}(\cdot)$. The newly formed price is:

$$\tilde{p} = \tilde{p}(\theta, \gamma) \triangleq \frac{1}{Q}\int_{[0,\infty]} c_\theta(s)\gamma(\mathrm{d}s) = \frac{1}{Q}\int_{\mathbb{R}} c_\theta(s^+)\mu(\mathrm{d}s). \tag{7.6}$$

Let Θ be as in Eq. (7.4) with:

$$p^* \triangleq \frac{W^*}{Q-k_1} \quad \text{and} \quad r_1^* \triangleq \frac{y^*}{Q}, \tag{7.7}$$

where W^* is a given constant, $0 < W^* \leq W$. (Recall that $0 < k_1 < Q$ and that y^* is an upper bound on the income variable Y.)

Define $\mathscr{M}$ to be the collection of all probability measures γ on $\mathscr{B}([0, r_1^*\eta^* + p^* y^*])$ with $\int_{(0,\infty)} s\gamma(\mathrm{d}s) = W^*$. We need a technical lemma.

Lemma 7.3. *(a) Suppose $\theta_n \to \boldsymbol{\theta}$ as $n \to \infty$, where $\boldsymbol{\theta}$, θ_1, θ_2, ... lie in Θ. Then $c_{\theta_n}(s) \to c_\theta(s)$ uniformly on compact sets.*
(b) The function $\tilde{p}(\boldsymbol{\theta}, \gamma)$ of Eq. (7.6) is continuous and everywhere positive on $\Theta \times \mathscr{M}$. Furthermore, $\tilde{p}$ has a continuous, everywhere positive extension to the compact set $\overline{\Theta} \times \mathscr{M}$, where:

$$\overline{\Theta} = \{(r_1, r_2, p) \quad : \quad 1 \leq r_2 \leq r_1 \leq r_1^*, \qquad r_2 \leq r_2^*, \qquad 0 \leq p \leq p^*\}.$$

Proof. (a) Similar to Proposition 3.4 of [KSS2] (1997).
(b) The continuity of $\tilde{p}$ on $\Theta \times \mathscr{M}$ follows from (a), since every $\gamma \in \mathscr{M}$ is supported by the compact set $K \triangleq [0, r_1^*\eta^* + p^* y^*]$. Also, $\tilde{p}$ is strictly positive on $\Theta \times \mathscr{M}$ because, by Theorem 4.2(d), $c_\theta(s) > 0$ for all $s > 0$. To extend to $\overline{\Theta} \times \mathscr{M}$, let $\boldsymbol{\theta} = (r_1, r_2, 0) \in \overline{\Theta}$, and first set $c_\theta(s) \equiv c(s; r_1, r_2, 0) \triangleq s$. Then, for $\gamma \in \mathscr{M}$, let:

$$\tilde{p}(\theta, \gamma) \triangleq \frac{1}{Q} \int_{(0,\infty)} s\gamma(\mathrm{d}s) = \frac{W^*}{Q}.$$

Obviously our extension is positive. To check its continuity, fix $\boldsymbol{\theta} = (r_1, r_2, 0) \in \overline{\Theta}$, $\gamma \in \mathscr{M}$, and suppose that $\lim_{n\to\infty}(\theta_n, \gamma_n) = (\boldsymbol{\theta}, \gamma)$, where $(\theta_n, \gamma_n) \in \Theta \times \mathscr{M}$ for all n. It suffices to show that:

$$\tilde{p}(\theta_n, \gamma_n) = \frac{1}{Q} \int c_{\theta_n}(s)\gamma_n(\mathrm{d}s) \to \frac{1}{Q} \int s\gamma(\mathrm{d}s) = \frac{W^*}{Q}, \qquad \text{as } n \to \infty.$$

Suppose $\theta_n = (r_1^{(n)}, r_2^{(n)}, p_n)$. By Eq. (7.4) and Lemma 7.1, we have $|c_{\theta_n}(s) - s| = p_n|c(s/p_n; r_1^{(n)}, r_2^{(n)}, 1) - s/p_n| \leq p_n\eta^* \to 0$, as $n \to \infty$, and the result follows. ■

Define:

$$p_* \triangleq \inf\{\tilde{p}(\theta, \gamma) : (\theta, \gamma) \in \Theta \times \mathscr{M}\}. \tag{7.8}$$

Lemma 7.4. *For every $(\boldsymbol{\theta}, \gamma) \in \Theta \times \mathscr{M}$, we have*

$$0 < p_* \leq \tilde{p}(\boldsymbol{\theta}, \gamma) \leq p^* = (W^*/Q - k_1).$$

Proof. The first inequality is by Lemma 7.3(b), and the second by Eq. (7.8). For the third, use Eqs. (7.2), (7.6) and (7.7 to get $Q\tilde{p}(\boldsymbol{\theta}, \gamma) = \int c_\theta(s)\gamma(\mathrm{d}s) \leq \int(s + pk_1)\gamma(\mathrm{d}s) \leq W^* + p^* k_1 = p^* Q$. ■

7.3. Finding interest rates that balance the books

Let the sets Θ and $\mathscr{M}$ be as in the previous section so that, in particular, p^* and r_1^* satisfy Eq. (7.7). Suppose that $\gamma \in \mathscr{M}$ is the distribution of money across agents at some stage of play. Assume also that all agents believe a certain $\boldsymbol{\theta} = (r_1, r_2, p) \in \Theta$ to be the vector of parameter-values. If they further believe the game to be in equilibrium, then they will play according to $c_\theta(\cdot)$. Our objective in this section is to see that in such a situation the bank can find new interest rates $(\tilde{r}_1, \tilde{r}_2)$ that will balance the books. To do so, we need expressions for the total amounts of fiat money borrowed and paid back.

An agent with money $s \geq 0$ will borrow the amount $d_\theta(s) = (c_\theta(s) - s)^+$, so the total amount borrowed is:

$$D \equiv D(\theta,\gamma) \triangleq \int d_\theta(s)\gamma(\mathrm{d}s). \tag{7.9}$$

If the bank sets the interest rate $\tilde{r}_1$ for borrowers, then the amount paid back by an agent, who begins the period with fiat money s and receives income $\tilde{p}y$, is $\tilde{p}y \wedge \tilde{r}_1 d_\theta(s)$, where $\tilde{p}$ is the newly formed price as in Eq. (7.6). The total amount paid back is $R(\tilde{r}_1)$, where:

$$R(r) \equiv R(r,\theta,\gamma) \triangleq \int\int\{\tilde{p}y \wedge rd_\theta(s)\}\gamma(\mathrm{d}s)\lambda(\mathrm{d}y) \tag{7.10}$$

and λ is the distribution of the generic income variable Y. Let:

$$J \equiv J(\theta,\gamma) \triangleq \{r \in [1,r_1^*]: R(r,\theta,\gamma) = D(\theta,\gamma)\}. \tag{7.11}$$

Lemma 7.5. *For all $\boldsymbol{\theta} \in \Theta$ and $\gamma \in \mathscr{M}$, the set $J(\theta, \gamma)$ is a closed, nonempty subinterval of $[1, r_1^*]$. In particular, there exists $r \in [1, r_1^*]$ such that*

$$R(r, \boldsymbol{\theta}, \gamma) = D(\boldsymbol{\theta}, \gamma).$$

Proof. The function $r \mapsto R(r) = R(r, \boldsymbol{\theta}, \gamma)$ of Eq. (7.10) is obviously nondecreasing, and is continuous by the dominated convergence theorem; thus J is clearly a closed subinterval of $[1, r_1^*]$. It remains to prove that J is nonempty, and for this it suffices to show that:

$$R(1) \leq D \leq R(r_1^*).$$

The first inequality is trivial, because $\tilde{p}y \wedge 1 d_\theta(s) \leq d_\theta(s)$. To prove the second inequality, let $c^* = c_\theta(0)$. By Theorem 4.2(b), we have $c_\theta(s) \geq c_\theta(0) = c^*$ and

$d_\theta(s) \le d_\theta(0) = c^*$, $s \ge 0$. Thus, Eq. (7.6) implies $Q\tilde{p} \ge c^*$. Fix $s \ge 0$, $y \in [0, y^*]$ and observe:

$$\tilde{p}y \wedge r_1^* d_\theta(s) \ge \frac{c^*}{Q} y \wedge \frac{y^*}{Q} d_\theta(s) \ge \frac{y}{Q}[c^* \wedge d_\theta(s)] = \frac{y}{Q} d_\theta(s).$$

Hence:

$$R(r_1^*) \ge \frac{1}{Q} \int\int \{d_\theta(s) y\} \gamma(\mathrm{d}s) \lambda(\mathrm{d}y) = \frac{D}{Q} \int y\lambda(\mathrm{d}y) = D.$$

Remark 7.1. *By Lemma 7.5, we see that the bank can choose $\tilde{r}_1$ in a given period so that borrowers, as a group, pay back precisely the amount that they borrowed. The bank can then set $\tilde{r}_2 = 1$, which means that depositors, as a group, get back exactly what they received: the bank is able to balance its books. It would be interesting to have conditions that make it possible for the bank to pay a positive interest rate $\rho_2 > 0$ ($\tilde{r}_2 \equiv 1 + \rho_2 > 1$) to depositors, and still balance its books. It seems unlikely that this will always be possible without price inflation, or growth in the economy.*

7.4. The proof of Theorem 7.2

Consider the set:

$$\tilde{\Theta} = \{(r_1, r_2, p) : 1 \le r_1 \le r_1^*, r_2 = 1, p_* \le p \le p^*\},$$

where p_*, r_1^* and p^* are given by Eqs. (7.7) and (7.8), respectively. Let $\mathscr{M}$ be the set of probability measures γ on $\mathscr{B}([0, r_1^*\eta^* + p^* y^*])$ with $\int s\gamma(\mathrm{d}s) = W^*$, as in Section 7.2 and Section 7.3. Define a set-valued mapping ψ on the compact set $\tilde{\Theta} \times \mathscr{M}$ as follows: for $(\boldsymbol{\theta}, \gamma) = ((r_1, r_2, p), \tilde{\gamma}) \in \tilde{\Theta} \times \mathscr{M}$, $\psi((r_1, r_2, p), \gamma)$ is the set of all $(\tilde{\theta}, \tilde{\gamma}) = ((\tilde{r}_1, \tilde{r}_2, \tilde{p}), \tilde{\gamma})$ such that $\tilde{r}_1 \in J(\boldsymbol{\theta}, \gamma)$, $\tilde{r}_2 = 1$, $\tilde{p} = \tilde{p}(\boldsymbol{\theta}, \gamma)$, and $\tilde{\gamma}$ is the distribution of $[g_\theta(S_0^+ - c_\theta(S_0^+)) + \tilde{p})Y]^+$. Here S_0^+ and Y are independent. By Lemmata 7.2, 7.4, and 7.5, $\psi(\boldsymbol{\theta}, \gamma)$ is a compact, convex subset of $\tilde{\Theta} \times \mathscr{M}$. Furthermore, it is straightforward to verify that:

$$\left\{\left((\theta,\gamma),(\tilde{\theta},\tilde{\gamma})\right) : \left(\tilde{\theta},\tilde{\gamma}\right) \in \psi(\theta,\gamma)\right\}$$

$$\text{is a closed subset of} \quad (\tilde{\Theta} \times \mathscr{M}) \times (\tilde{\Theta} \times \mathscr{M}).$$

Hence, ψ is upper semicontinuous (cf. Theorem 10.2.4 of Istrătescu (1981)), and thus, by Kakutani's fixed point theorem (Corollary 10.3.10 in Istrătescu (1981)), there exists $(\boldsymbol{\theta}, \gamma) = ((r_1, r_2, p), \gamma) \in \tilde{\Theta} \times \mathscr{M}$ such that $(\boldsymbol{\theta}, \gamma) \in \psi(\boldsymbol{\theta}, \gamma)$.

This fixed point $((r_1, r_2, p), \gamma) = (\boldsymbol{\theta}, \gamma)$ determines a stationary Markov equilibrium in which the bank sets the interest rates to be r_1 and $r_2 = 1$, the price is p, and the distribution of wealth-levels μ is related to γ, the distribution of fiat money, by the rule that μ is the distribution of $g_\theta(S_0^+ - c_\theta(S_0^+)) + pY$; here, S_0^+

and Y are independent. Clearly, we have $1 = r_2 \leq r_1 \leq r_1^* = y^*/Q$, and $k = k(p) = pk_1 < pQ$, from Eq. (7.2).

8. The game with a money-market

We shall discuss in this section the strategic market game when there is no outside bank, but instead agents can borrow or deposit money through a *money-market* at interest rates r_1 and r_2, respectively, with $r_1 > r_2$. In contrast to the situation of an outside bank, which fixes and announces interest rates for borrowing and depositing money, here r_1 and r_2 are going to be determined *endogenously*.

In order to see how this can be done, imagine that agent $\alpha \in I$ enters the day $t = n$ with wealth position $S_{n-1}^{\alpha}(w)$ from the previous day — in particular, with fiat money $(S_{n-1}^{\alpha}(w))^{+}$. His information $\mathscr{F}_{n-1}^{\alpha}$ (at the beginning of day $t = n - 1$) measures, in addition to the quantities mentioned in Section 3, past interest rates $r_{1,k}$ and $r_{2,k}$, $k = 0, \ldots, n-1$ for borrowing and depositing, respectively. The agent can decide either to deposit money:

$$l_n^{\alpha}(w) \in \left[0, \left(S_{n-1}^{\alpha}(w)\right)^{+}\right] \tag{8.1}$$

into the money-market, or to offer a bid of:

$$j_n^{\alpha}(w) \in [0, k^{\alpha}] \tag{8.2}$$

in I.O.U. notes for money, or to do neither, but not both:

$$j_n^{\alpha}(w)\, l_n^{\alpha}(w) = 0. \tag{8.3}$$

The total amount deposited is:

$$L_n(w) \triangleq \int_I l_n^{\alpha}(w)\, \varphi(\mathrm{d}\alpha); \tag{8.4}$$

the total amount offered in I.O.U. notes is:

$$J_n(w) \triangleq \int_I j_n^{\alpha}(w)\, \varphi(\mathrm{d}\alpha); \tag{8.5}$$

and the money-market is declared *active* on day $t = n$, if:

$$J_n(w)\, L_n(w) > 0 \tag{8.6}$$

(*inactive*, if $J_n(w) L_n(w) = 0$).

After agents have thus made their bids in the money-market, a new interest rate for borrowing money is formed, namely:

$$r_{1,n}(w) \triangleq \begin{Bmatrix} \dfrac{J_n(w)}{L_n(w)}; & \text{if } J_n(w)\, L_n(w) > 0 \\ 1; & \text{otherwise} \end{Bmatrix}. \tag{8.7}$$

Agent $\alpha \in I$ receives his I.O.U. notes' worth $j_n^\alpha(w)/r_{1,n}(w)$ in fiat money, and bids the amount:

$$b_n^\alpha(w) \triangleq \left(S_{n-1}^\alpha(w)\right)^+ + \begin{cases} \dfrac{j_n^\alpha(w)}{r_{1,n}(w)} - l_n^\alpha(w); & \text{if } J_n(w)L_n(w) > 0 \\ 0; & \text{otherwise} \end{cases} \tag{8.8}$$

in the commodity market. Thus, the total amount of money bid for commodity is:

$$\begin{aligned} B_n(w) &\triangleq \int_I b_n^\alpha(w)\varphi(\mathrm{d}\alpha) \\ &= W_{n-1}(w) + \begin{cases} \dfrac{J_n(w)}{r_{1,n}(w)} - L_n^\alpha(w); & \text{if } J_n(w)L_n(w) > 0 \\ 0; & \text{otherwise} \end{cases} \\ &= W_{n-1}(w) \end{aligned} \tag{8.9}$$

from Eq. (8.7), where:

$$W_k(w) \triangleq \int_I \left(S_k^\alpha(w)\right)^+ \varphi(\mathrm{d}\alpha) \tag{8.10}$$

is the total amount of money across agents on day $t = k \in \mathbb{N}_0$.

Next, the various agents' commodity endowments $Y_n^\alpha(w)$, $\alpha \in I$ for that day $t = n$ are revealed (same assumptions and notation as in the beginning of Section 3). A new *commodity price*:

$$p_n(w) \triangleq \frac{B_n(w)}{Q} = \frac{W_{n-1}(w)}{Q} \tag{8.11}$$

is formed, and agent $\alpha \in I$ receives his bid's worth $x_n^\alpha(w) \triangleq b_n^\alpha(w)/p_n(w)$ in units of commodity. He consumes this amount, and derives utility $\xi_n^\alpha(w)$ as in Eq. (3.4). The borrowers pay back their debts — with interest $r_{1,n}(w)$ — to the extent that they can; the rest is forgiven, but "punishment in the form of negative-utility" is incurred if they enter the next day with $S_n^\alpha(w) < 0$, as in Eq. (3.4). A new interest rate for deposits is formed:

$$r_{2,n}(w) \triangleq \begin{cases} \dfrac{1}{L_n(w)} \displaystyle\int_I \{ j^\alpha(w) \wedge p_n(w) Y_n^\alpha(w) \} \varphi(\mathrm{d}\alpha); & \text{if } J_n(w)L_n(w) > 0 \\ 1; & \text{otherwise} \end{cases} \tag{8.12}$$

and agent $\alpha \in I$ moves to the new wealth position:

$$\begin{aligned} S_n^\alpha(w) &\triangleq \left[r_{2,n}(w) l_n^\alpha(w) - j_n^\alpha(w) \right] 1_{\{J_n(w)L_n(w) > 0\}} + p_n(w) Y_n^\alpha(w), \\ S_n^\alpha(w) &= g\left(\left(S_{n-1}^\alpha(w)\right)^+ - b_n^\alpha(w);\ \ r_{1,n}(w), r_{2,n}(w) \right) + p_n(w) Y_n^\alpha(w) \end{aligned} \tag{8.13}$$

in the notation of Eq. (4.2).

Remark 8.1. *Indeed, suppose that the money-market is active on day $t=n$. If agent $\alpha \in I$ is a depositor ($l_n^\alpha(w)>0$, $j_n^\alpha(w)=0$), he bids the amount $b_n^\alpha(w)=(S_{n-1}^\alpha(w))^+ - l_n^\alpha(w) < (S_{n-1}^\alpha(w))^+$ in the commodity market, and ends up with fiat money*

$$S_n^\alpha(w) = r_{2,n}(w)\left[S_{n-1}^\alpha(w))^+ - b_n^\alpha(w)\right] + p_n(w)Y_n^\alpha(w),$$

after he has received his endowment's worth and his deposit back with interest. If agent α is a borrower ($j_n^\alpha(w)>0$, $l_n^\alpha(w)=0$), he bids in the commodity market the amount $b_n^\alpha(w)=(S_{n-1}^\alpha(w))^+[1/r_{1,n}(w)]j_n^\alpha(w) > (S_{n-1}^\alpha(w))^+$, and his new wealth position is:

$$S_n^\alpha(w) = -j_n^\alpha(w) + p_n(w)Y_n^\alpha(w) = r_{1,n}(w)\left[(S_{n-1}^\alpha(w))^+ - b_n^\alpha(w)\right] + p_n(w)Y_n^\alpha(w).$$

If the agent is neither borrower nor depositor (or if the money-market is inactive) on day $t=n$, he bids $b_n^\alpha(w)=(S_{n-1}^\alpha(w))^+$ for commodity and ends up at the new wealth position $S_n^\alpha(w)=p_n(w)Y_n^\alpha(w)$.

Remark 8.2. *These rules preserve* the total amount of fiat money in the economy, *and guarantee that* the price of the commodity remains constant from period to period. *Indeed, if the money-market is inactive on day $t=n$, we have $W_n(w)=p_n(w)\int_I Y_n^\alpha(w)\varphi(d\alpha)=Qp_n(w)=W_{n-1}(w)$ in the notation of Eq. (8.11), from Eqs. (3.14) and (8.10). On the other hand, if the money-market is active on day $t=n$, we obtain from Eqs. (8.13) and (8.12):*

$$\begin{aligned}
W_n(w) &= \int_I (S_n^\alpha(w))^+ \varphi(\mathrm{d}\alpha) \\
&= \int_I \left[r_{2,n}(w) l_n^\alpha(w) + p_n(w) Y_n^\alpha(w) 1_{\{j_n^\alpha(w)=0\}}\right] \varphi(\mathrm{d}\alpha) \\
&\quad + \int_I \left[p_n(w) Y_n^\alpha(w) - j_n^\alpha(w)\right] 1_{\{0<j_n^\alpha(w)\le p_n(w)Y_n^\alpha(w)\}} \varphi(\mathrm{d}\alpha) \\
&= r_{2,n}(w) L_n(w) + \int_I p_n(w) Y_n^\alpha(w) 1_{\{j_n^\alpha(w)\le p_n(w)Y_n^\alpha(w)\}} \varphi(\mathrm{d}\alpha) \\
&\quad - \int_I j_n^\alpha(w) 1_{\{j_n^\alpha(w)\le p_n(w)Y_n^\alpha(w)\}} \varphi(\mathrm{d}\alpha) \\
&= \int_I \Big(\left[j_n^\alpha(w) \wedge p_n(w) Y_n^\alpha(w)\right] \\
&\quad + \left[p_n(w) Y_n^\alpha(w) - j_n^\alpha(w)\right] 1_{\{j_n^\alpha(w)\le p_n(w)Y_n^\alpha(w)\}} \Big) \varphi(\mathrm{d}\alpha) \\
&= p_n(w) \int_I Y_n^\alpha(w) \varphi(\mathrm{d}\alpha) = Qp_n(w) = W_{n-1}(w),
\end{aligned}$$

again. In either case:

$$W_n = W_0 = :W, \quad p_n = p_0 \triangleq \frac{W}{Q}, \qquad \forall n \in \mathbb{N}. \tag{8.14}$$

Definition 8.1. A *strategy* π^α for agent $\alpha \in I$ specifies $w \rightarrowtail l_n^\alpha(w)$, $w \rightarrowtail j_n^\alpha(w)$ as $\mathscr{F}_{n-1}^\alpha$-measurable (thus also $\mathscr{F}_{n-1}$-random variables that satisfy Eqs. (8.1) and (8.2) for every $n \in \mathbb{N}$.

A strategy π^α is called *stationary*, if it is of the form:

$$j_n^\alpha(w) = j^\alpha\left(\left(S_{n-1}^\alpha(w)\right)^+; p_{n-1}(w), r_{1,n-1}(w), r_{2,n-1}(w)\right),$$
$$l_n^\alpha(w) = l^\alpha\left(\left(S_{n-1}^\alpha(w)\right)^+; p_{n-1}(w), r_{1,n-1}(w), r_{2,n-1}(w)\right) \tag{8.15}$$

$\forall\ n \in \mathbb{N}$; here j^α are measurable mappings of $[0, \infty) \times \check{\Theta}$ into $\mathbb{R}$ with:

$$0 \le j^\alpha(s;\theta) \le k^\alpha, \quad 0 \le l^\alpha(s;\theta) \le s, \quad j^\alpha(s;\theta) l^\alpha(s;\theta) = 0;$$
$$\text{for every} \quad \forall (s,\theta) \in [0,\infty) \times \check{\Theta}, \quad \text{where}$$
$$\check{\Theta} \triangleq \{r_1, r_2, p): 1 \le r_2 \le r_1 < \infty, r_2 < 1/\beta, p > 0\}. \tag{8.16}$$

(Such a strategy requires, of course, the specification of an initial vector of interest rates and price $\boldsymbol{\theta}_0 = (r_{1,0}, r_{2,0}, p_0) \in \check{\Theta}$, in order for j_1^α, l_1^α to be well-defined.)

A collection of strategies $\Pi = \{\pi^\alpha: \alpha \in I\}$ is called *admissible* for the money-market game if, for every $n \in \mathbb{N}$, the functions $(\alpha, w) \rightarrowtail l_n^\alpha(w)$, $(\alpha, w) \rightarrowtail j_n^\alpha(w)$ are $\mathscr{G}_{n-1} \equiv \mathscr{B}(I) \otimes \mathscr{F}_{n-1}$-measurable, in the notation of Section 3.1.

Definition 8.2. We say that an admissible collection of stationary strategies $\overline{\Pi} = \{\tilde{\pi}^\alpha: \alpha \in I\}$ results in *stationary Markov equilibrium* (r_1, r_2, p, μ) for the money-market game, with $\boldsymbol{\theta} = (r_1, r_2, p) \in \check{\Theta}$ and μ a probability measure on $\mathscr{B}(\mathscr{S})$ if, starting with initial vector $(r_{1,0}, r_{2,0}, p_0) = \boldsymbol{\theta}$, and with $\nu_0 = \mu$ in the notation of Eq. (3.13), we have:

(i) $(r_{1,n}, r_{2,n}, p_n) = \boldsymbol{\theta}$, $\nu_n = \mu$ $(\forall n \in \mathbb{N})$ when agents play according to the strategies $\tilde{\pi}^\alpha$, $\alpha \in I$; and
(ii) as in Definition 3.1. ■

In an effort to seek sufficient conditions for such a stationary Markov equilibrium, let us assume from now on that all agents have the same utility function $u^\alpha(\cdot) \equiv u(\cdot)$, the same upper bound on loans $k^\alpha \equiv k$, and the same income distribution $\lambda^\alpha \equiv \lambda$. By analogy with Assumptions 5.1 and 5.2, consider now the following two assumptions.

Assumption 8.1. *Support that there exists a triple* $\theta = (r_1, r_2, p) \in \check{\Theta}$ *for which the one-person game of Section 4*

(i) *has a unique optimal stationary plan* π, *corresponding to a continuous consumption function* $c \equiv c_\theta: [0, \infty) \to [0, \infty)$, *and* $j(s;\, \boldsymbol{\theta}) = r_1(c_\theta(s) - s)^+ \equiv r_1 d(s)$, $l(s;\, \boldsymbol{\theta}) \equiv l(s) = (s - c_\theta(s))^+$ *as in Eqs. (8.14) and (5.3); and*
(ii) *the associated Markov Chain of wealth-levels in Eq. (4.10) has an invariant measure* $\mu \equiv \mu_\theta$ *on* $\mathscr{B}(\mathscr{S})$ *with* $\int s\mu(ds) < \infty$.

Assumption 8.2. *The quantities of Assumption 8.1 satisfy the balance equations:*

$$\int d(s^+)\,\mu(\mathrm{d}s) = \int l(s^+)\,\mu(\mathrm{d}s) > 0 \tag{8.17}$$

("total amount borrowed is positive, and equals total amount deposited, in equilibrium") and

$$r_2 \int l(s^+)\,\mu(\mathrm{d}s) = \int\!\!\int \left[r_1 d(s^+) \wedge py \right] \mu(\mathrm{d}s)\,\lambda(\mathrm{d}y) \tag{8.18}$$

("total amount paid back to depositors equals total amount paid back by borrowers, in equilibrium").

The reader should not fail to notice that we have now two balance Eqs. (8.16) and (8.17), instead of the single balance Eq. (5.2) for the outside bank. This reflects the fact that the bank needs to balance its books only once, whereas a money-market has to clear twice:

(i) before the agents' endowments are announced — by the formation of the "ex ante" interest rate Eq. (8.7), which guarantees that the deposits $L_n(w)$ exactly match the payments to borrowers $J_n(w)/r_{1,n}(w)$;
(ii) and after — by the formation of the "ex post" interest rate Eq. (8.12), which matches exactly the amount $\int_I [j_n^\alpha(w) \wedge p_n(w) Y_n^\alpha(w)] \varphi(\mathrm{d}\alpha)$ paid back to borrowers, with the amount $r_{2,n}(w) L_n(w)$ that has to be paid to depositors.

In light of these remarks, it is no wonder that stationary Markov equilibrium with a money-market is much more delicate, and difficult to construct, than with an outside bank. This extra difficulty will also be reflected in the Examples that follow.

Here are now the analogues of Lemma 5.1 and Theorem 5.1; their proofs are left as an exercise for the diligent reader.

Lemma 8.1. *Under Assumptions 8.1 and 8.2:*

$$p = \frac{1}{Q} \int c(s^+)\,\mu(\mathrm{d}s) = \frac{1}{Q} \int s^+\,\mu(\mathrm{d}s). \tag{8.19}$$

Theorem 8.1. *Under Assumptions 8.1 and 8.2, the family $\Pi = \{\pi^{\alpha} : \alpha \in I\}$ with $\pi^{\alpha} = \pi$ $(\forall \alpha \in I)$ results in Stationary Markov Equilibrium (r_1, r_2, p_n, μ) for a money-market.*

For fixed $\boldsymbol{\theta} = (r_1, r_2, p) \in \check{\Theta}$, Theorems 4.2 and 4.3 provide fairly general sufficient conditions that guarantee the validity of Assumption 8.1. However, we have not been able to obtain results comparable to Theorems 7.1 and 7.2, providing reasonably general sufficient conditions for Assumption 8.2 to hold. We shall leave this subject to further research, but revisit in our new context the Examples of Section 6.

Example 6.1 (continued). Recall the setup of Eqs. (6.1)–(6.4), the consumption strategy $c(\cdot)$ of Eq. (6.5), and the invariant measure μ of Eq. (6.6). The balance Eq. (5.2), for an outside-bank stationary Markov equilibrium, was satisfied for all values of the Bernoulli parameter $0 < \delta < 1/2$; however, the balance Eqs. (8.16) and (8.17) are satisfied if and only if $\delta = 1/4$.
Thus, for this value $\delta = 1/4$, the vector $\boldsymbol{\theta} = (r_1, r_2, p) = (4, 1, 1)$ and the measure $\mu(\{-1\}) = 1/2$, $\mu(\{0\}) = 1/6$, $\mu(\{k\}) = 2/3^{k+1}$, $k \in \mathbb{N}$ of Eq. (6.6), form a stationary Markov equilibrium for the money-market.

Example 6.2 (continued). Recall the setup of Eqs. (6.11)–(6.14), the consumption function of Eq. (6.13), and the invariant probability measure of Eq. (6.14). The balance Eq. (5.2) for an outside bank holds for all values of the Bernoulli parameter $\delta \in (1/3, 1/2) = (0.33, 0.5)$ and all values of the discount and slope parameters β, η as in (B.10). The balance Eqs. (8.16) and (8.17) for a money-market will be satisfied, if and only if the total amount borrowed in equilibrium, namely $\int d(s^+)\mu(ds) = \mu_{-1/\delta} + \mu_0 = (1-\delta)^2$, equals the total amount $\int l(s^+)\mu(ds) = \mu_{5-1/\delta} + \mu_5 + \mu_6 = \delta$ deposited in equilibrium; in other words, if and only if $\delta = (3-\sqrt{5})/2 = 0.382$. With this value of δ in Eqs. (6.12) and (6.15), the vector $\boldsymbol{\theta} = (r_1, r_2, p) = (2.62, 1, 1)$ and the measure μ of Eq. (6.14) form a stationary Markov equilibrium for the money-market.

Example 6.3 (continued). In the setting of Eqs. (6.16)–(6.18) and with $\boldsymbol{\theta} = (r_1, r_2, p) = (2, 2, 1)$, $\mu(\{0\}) = 1/2$, the pair $(\boldsymbol{\theta}, \mu)$ leads to stationary Markov equilibrium if $1/3 < \beta < 1/2$; no such equilibrium exists for $\boldsymbol{\theta} = (2, 2, 1)$ and $0 < \beta < 1/3$.

Acknowledgements

Research supported by the Santa Fe Institute, and by the National Science Foundation under grants DMS-97-32810 (Karatzas) and DMS-97-03285 (Sudderth). We are deeply indebted to the Referee and the Co-Editor for the care with which they read the original version of the paper, and for their many helpful remarks.

References

Blackwell, D., 1965. Discounted dynamic programming. Ann. Math. Stat. 36, 226–235.

Feldman, M., Gilles, C., 1985. An expository note on individual risk without aggregate uncertainty. J. Econ. Theory 35, 26–32.

Geanakoplos, J., Karatzas, I., Shubik, M., Sudderth, W., 1999. [GKSS]. A Strategic Market Game with Active Bankruptcy. Santa Fe Institute Publications 99-04-025, Santa Fe.

Istrătescu, V.I., 1981. Fixed Point Theory. Boston.

Karatzas, I., Shubik, M., Sudderth, W., 1994. Construction of stationary Markov equilibria in a strategic market game. Math. Oper. Res. 19, 975–1006, [KSS1].

Karatzas, I., Shubik, M., Sudderth, W., 1997. A strategic market game with secured lending. J. Math. Econ. 28, 207–247, [KSS2].

Langen, H.-J., 1981. Convergence of dynamic programming models. Math. Oper. Res. 6, 493–512.

Miller, J., Shubik, M., 1994. Some dynamics of a strategic market game with a large number of agents. Economica 60, 1–28.

Ross, S.M., 1983. Introduction to Stochastic Dynamic Programming. Academic Press, San Diego.

Schäl, M., 1975. Conditions for optimality in dynamic programming, and for the limit of n-stage optimal policies to be optimal. Z. Wahrsch. 32, 179–196.

Tweedie, R.L., 1988. Invariant measures for Markov chains with no irreducibility assumptions. J. Appl. Probab. 25A, 275–286.