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BY

PRADEEP DUBEY, JOHN GEANAKOPLOS
AND
MARTIN SHUBIK

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We extend the standard model of general equilibrium with incomplete markets to allow for default and punishment by thinking of assets as pools. The equilibrating variables include expected delivery rates, along with the usual prices of assets and commodities. By reinterpreting the variables, our model encompasses a broad range of adverse selection and signalling phenomena in a perfectly competitive, general equilibrium framework.

Perfect competition eliminates the need for lenders to compute how the size of their loan or the price they quote might affect default rates. It also makes for a simple equilibrium refinement, which we propose in order to rule out irrational pessimism about deliveries of untraded assets.

We show that refined equilibrium always exists in our model, and that default, in conjunction with refinement, opens the door to a theory of endogenous assets. The market chooses the promises, default penalties, and quantity constraints of actively traded assets.

KEYWORDS: Default, incomplete markets, adverse selection, moral hazard, equilibrium refinement, endogenous assets.

1. INTRODUCTION

GENERAL EQUILIBRIUM THEORY has for the most part not made room for default. In the Arrow–Debreu model of general equilibrium with complete contingent markets (GE), and likewise in the general equilibrium model with incomplete markets (GEI), agents keep all their promises by assumption. More specifically, in the GE model, agents never promise to deliver more goods than they personally own. In the GEI model, the definition of equilibrium allows agents to promise more of some goods than they themselves have, provided they are sure to get the difference elsewhere. Agents there too must honor their commitments, though no longer exclusively out of their own endowments. Each agent can keep his promises because other agents keep their promises to him.

We build a model that explicitly allows for default, but is broad enough to incorporate conventional general equilibrium theory as a special case. We call the model $GE(R, \lambda, Q)$ because each asset j is defined by its promise R_j, the penalty rate λ_j, which determines the utility punishment for default on the promise, and the quantity restriction Q_j attendant on those who sell it. When

[1]This work is supported in part by NSF Grant DMS-87-05294 and SES-881205. The first version of this paper (Dubey, Geanakoplos, and Shubik (1990)) appeared in 1990. That version, containing essentially the basic model and Theorems 1 and 2 from this paper, was never published, though our model has been frequently used and cited: see, for example, Araujo, Monteiro, and Pascoa (1998), Bisin and Gottardi (1999), Santos and Scheinkman (2001), and Zame (1993). An expanded version, of which this paper is a part, was circulated in 1994, and again as a Cowles Foundation Discussion Paper No. 1247 in 2000.

λ and Q are set to infinity (or made sufficiently high), the model reduces to GEI.[2]

Fixing exogenously the set $\mathcal{A}$ of tradeable assets,

$$\mathcal{A} = \{(R_j, \lambda_j, Q_j) : (R_j, \lambda_j, Q_j) \text{ is tradeable}\},$$

we solve for equilibrium $E(\mathcal{A})$. The equilibrating variables include anticipated delivery rates K_j on assets, along with the usual prices (π_j, p_ℓ) of assets and commodities. In keeping with the spirit of perfect competition, we suppose that every agent regards (p, π, K) as fixed.

One of the central features of our model is that assets are thought of as pools.[3] Different sellers of the same asset will typically default in different events, and in different proportions. The buyers of the asset receive a pro rata share of all the different sellers' deliveries, just as an investor today does in the securitized mortgage market, or in the securitized credit card market. When the pools are large, an (infinitesimal) buyer can reasonably assume that both the price π_j and pool delivery rate K_j are unaffected by the number of shares he buys.

We have in mind the huge, anonymous markets now becoming so common on Wall Street. Mortgages today are promises sold by homeowners to banks, who then sell them into mortgage pools (totalling around $3 trillion). The bank plays a minor, administrative role, collecting payments and verifying the eligibility of the homeowners (according to criteria specified by the pool, not the bank). The bank receives a "servicing fee" for its efforts, and passes the default and prepayment risk on to the shareholders in the pool. The analysis therefore properly shifts from the one-on-one interaction of banker and homeowner to the pool level of anonymous shareholders (lenders) and borrowers, which is more akin to perfect competition.[4]

Even though our pools are perfectly competitive, heterogeneous default still creates adverse selection. Sellers with a proclivity for default have incentive to sell disproportionately many promises into the pool, thereby worsening the pool's delivery rate. We show in Section 8 that the adverse selection and signalling phenomena described by Akerlof (1970), Spence (1973), and

[2]Many authors (including Townsend (1979), Diamond (1984), Gale and Hellwig (1985), Hart and Moore (1998), and Allen and Gale (1998)) have studied models of equilibrium default. But none of these models yields GEI as a special case while explicitly incorporating nonpecuniary punishment for default.

[3]To the best of our knowledge, Green (1974) was the first to introduce pooling in the context of default. His contribution is all the more remarkable, coming as it did, before pooling became so prevalent in practice.

[4]If the banks "cherry pick" their loans, selling only the worst ones into the pools, or if the agencies which organize the pools likewise retain the best loans, then game theoretic considerations come to the foreground, and the analysis becomes vastly more complicated. The system currently in place in the United States is designed to eliminate or at least discourage such cherry picking. To the extent it is successful, perfect competition becomes a plausible idealization of reality.

Rothschild and Stiglitz (1976) can all be captured in our perfectly competitive framework.[5]

An important consequence of default is that the subset $\mathcal{A}^* \equiv \mathcal{A}^*(E(\mathcal{A})) \subset \mathcal{A}$ of actively traded assets

$$\mathcal{A}^* = \{(R_j, \lambda_j, Q_j) \in \mathcal{A} : (R_j, \lambda_j, Q_j) \text{ is positively traded in } E(\mathcal{A})\}$$

is usually much smaller than $\mathcal{A}$. The reason is that, with default, the sale of an asset is not the negative of its purchase. The buyer receives only what is delivered, but the seller gives up in addition penalties for what is not delivered. The wedge between the marginal utility of buying and the marginal disutility of selling is like a transactions cost. Assets which yield gains to trade greater than this cost will still be traded, while the rest will not. In GEI the selection of assets is usually regarded as outside the model, because typically every (nonredundant) asset is actively traded, so $\mathcal{A} = \mathcal{A}^*$. However, with default, there will be many assets in $\mathcal{A} \setminus \mathcal{A}^*$ that are priced by the market, but neither bought nor sold.[6] The promises, penalties, and sales limitations corresponding to assets in $\mathcal{A}^*$ can thus be regarded as endogenously emerging out of $\mathcal{A}$.

At equilibrium the market is "open" for every untraded asset j in $\mathcal{A} \setminus \mathcal{A}^*$ via its price π_j and its expected delivery rate K_j, though agents choose voluntarily not to go there. For active markets $j \in \mathcal{A}^*$, the rate K_j is determined via rational expectations of the actual deliveries made in pool j. But if $j \in \mathcal{A} \setminus \mathcal{A}^*$ is inactive, how are we to assign K_j? Allowing K_j to be arbitrarily low would by itself render j inactive. How are we to eliminate such irrational pessimism?[7]

We introduce an equilibrium refinement in which the government intervenes to sell infinitesimal quantities ε_j of each asset and fully delivers on its promises. We take $K_j = \lim K_j(\varepsilon)$ as $\varepsilon \to 0$. This touch of intervention generates enough optimism to rule out all spurious inactivity on asset markets.

Our refinement is very simple. Agents do not have to speculate on reactions to reactions to... to untried actions, as in the contract theory literature, where a small number of parties are in face-to-face strategic interaction (see, e.g., Cho and Kreps (1987)). They need only think about the observable macro variable $K_j(\varepsilon)$.[8]

[5]Moral hazard also enters the picture: first because sellers have a choice not to deliver, and second, because a seller of many assets will be less able to fully deliver on any of them than if he had refrained from overextending himself. For other general equilibrium models of adverse selection, see Gale (1992), Hellwig (1987), and Prescott and Townsend (1984).

[6]In some applications we might choose to limit $\mathcal{A}$ exogenously; the point is that even if $\mathcal{A}$ is inclusive, $\mathcal{A}^*$ will still be limited.

[7]In Green (1974) expectations of future delivery rates were taken to be completely exogenous. Therefore the issue of rational expectations on active markets, much less that of pessimism on inactive markets, could not arise. This eliminated the need for equilibrium refinement, and the possibility of endogenous asset selection and signalling.

[8]In Section 3.2 we distinguish our refinement from the trembles used in game theory.

If agents have the mental powers to anticipate future rates of default (contingent on future events), just as they are presumed by conventional equilibrium theory to have the mental powers to anticipate future prices (contingent on future events), then default is consistent with the orderly function of markets. In Section 4 we prove the existence of refined equilibrium with default under exactly the same conditions necessary to prove the existence of equilibrium in the GEI model (where default is ruled out by assumption). More precisely, we show that our refined equilibrium $E(\mathcal{A})$ exists for every collection $\mathcal{A}$ of assets (R, λ, Q) for which $Q < \infty$, or for which $Q = \infty$ but the promises R are all paid in the same numeraire.

Recall that each asset (R_j, λ_j, Q_j) is characterized by three dimensions. If the set $\mathcal{A}$ of available assets is comprehensive (i.e., all conceivable levels and combinations of the three asset dimensions are present in $\mathcal{A}$), then we prove in Section 5 that $\mathcal{A}^*$ will in effect select the Arrowian levels: completely spanning promises, with infinite penalties, and nonbinding quantity constraints. On the other hand, if one of the dimensions in $\mathcal{A}$ is *exogenously* fixed far from its Arrowian level, then the forces of supply and demand will *endogenously* select the levels in the remaining dimensions in $\mathcal{A}^*$ to be far from Arrowian. In Sections 6–8, we illustrate the endogeneity of $\mathcal{A}^*$ by fixing each pair of dimensions and investigating how the market endogenously picks the third.

In Section 6 we consider an example with all the Arrow promises, plus the riskless promise. When penalties are exogenously restricted to be low, and quantity limits fixed at their infinite Arrowian levels, we show that none of the Arrow securities is actively traded, so that the market endogenously chooses the riskless promise. When the penalties are raised sufficiently, the Arrow promises become active.

In Section 7, we suppose the span of promises is exogenously restricted, while the quantity limits remain Arrowian. We first ask how harsh default penalties *should* be. We find in our example that they should be so low that agents sometimes default even when they have resources to deliver, but not so low that everybody always defaults. We next ask how harsh the penalties *will* be that endogenously emerge in $\mathcal{A}^*$. We find that the forces of supply and demand do select a unique penalty, which in the example turns out to be the optimal penalty.

We show in Section 8 that if promises and penalties are fixed exogenously, then $\mathcal{A}^*$ endogenously selects quantity limits Q_j. Fixing the penalties in a particular way, we can incorporate insurance as a special case of default, which displays adverse selection and signalling in a pure form. We then obtain primary and secondary insurance policies as part of the refined equilibrium. Unlike the previous sections, we get multiple, Pareto comparable equilibria. If we further impose the Rothschild–Stiglitz exclusivity constraint, prohibiting agents from taking out more than one insurance policy, we get a unique equilibrium, equivalent to their separating equilibrium, moreover without jeopardizing the existence of equilibrium.

Finally, in Section 9 we ask why the right asset is necessarily used in Section 7, but the wrong assets are used in most of the equilibria of Section 8. The answer has to do with adverse selection. In refined equilibrium, the delivery rates K_j of untraded assets j depend only on the reliability of the agents most eager to sell (the "on-the-verge sellers"). Depending on the allocation achieved via the traded assets, the most eager to sell might be the most unreliable or the most reliable.

2. DEFAULT IN EQUILIBRIUM: THE $GE(R, \lambda, Q)$ MODEL

2.1. *The Economy*

As in the canonical model of general equilibrium with incomplete markets (GEI), we consider a two-period economy, where agents know the present but face an uncertain future. In period 0 (the present) there is just one state of nature (called state 0), in which H agents trade in L commodities and J assets. Then chance moves and selects one of S states that occur in period 1 (the future). Commodity trades take place again, and assets pay off. The difference from GEI is that in our $GE(R, \lambda, Q)$ model, assets pay off in accordance with what agents opt to deliver. Our notation for the exogenous variables is:

$\ell \in L = \{1, \ldots, L\}$ = set of commodities;
$s \in S = \{1, \ldots, S\}$ = set of states in period 1;
$S^* = \{0\} \cup S$ = set of all states;
$h \in H = \{1, \ldots, H\}$ = set of agents;
$e^h \in \mathbb{R}_+^{S^* \times L}$ = initial endowment of agent h;
$j \in J = \{1, \ldots, J\}$ = set of assets;
$R_j \in \mathbb{R}_+^{S \times L}$ = promise per unit of asset j of each commodity $\ell \in L$ in each state $s \in S$;
$u^h : \mathbb{R}_+^{S^* \times L} \to \mathbb{R}$ = utility function of agent h;
$\lambda_{sj}^h \in \overline{\mathbb{R}}_+ \equiv \mathbb{R}_+ \cup \{\infty\}$ = real default penalty on agent h for asset j in state s;
$Q_j^h \in \mathbb{R}_+$ = bound on sale of asset j by agent h.

We assume that no agent has the null endowment, and that all named commodities are present in the aggregate, i.e., $e_s^h = (e_{s1}^h, \ldots, e_{sL}^h) \neq 0$ for all $h \in H$ and $s \in S^*$, and $e_{s\ell} = \sum_{h \in H} e_{s\ell}^h > 0$ for all $s\ell \in S^* \times L$. Also each u^h is continuous, concave, and strictly increasing in each of its $S^* \times L$ variables. Having assumed strict monotonicity and concavity, there is no further loss of generality in assuming that $u^h(x) \to \infty$ whenever $\|x\|_\infty \to \infty$.[9]

[9]Let u^h be concave, continuous, and strictly monotonic. Let $\square = \{x \in \mathbb{R}_+^{S^* L} : \|x\|_\infty \leq 2\|\sum_h e^h\|_\infty\}$. Let $\mathcal{L}$ be the set of affine functions $L : \mathbb{R}_+^{S^* L} \to \mathbb{R}$ such that $L(x) \geq u^h(x)$ for all $x \in \mathbb{R}_+^{S^* L}$, and $L(x) = f(x)$ for some $x \in \square$. Define $\tilde{u}^h(x) \equiv \inf_{L \in \mathcal{L}} L(x)$. Then equilibria with u^h and $\tilde{u}^h$ coincide, and $\tilde{u}^h$ has the desired property.

Agents h have heterogeneous, state-dependent endowments $e_s^h \in \mathbb{R}_+^L$ and disutilities of default λ_{sj}^h.

A promise $R_j \equiv (R_{s\ell j} : s \in S,\ \ell \in L) \equiv (R_{sj})_{s\in S} \in (\mathbb{R}_+^L)^S$ for $j \in J$ specifies bundles of goods (or services) to be delivered in each state.

Each kind of asset prescribes a limit, Q_j^h, on the sales each agent h can make of it. Such limits are natural in any realistic model of credit.[10] If $Q_j^h = 0$, then agent h is essentially forbidden from selling asset j. If the limits $Q_j^h = \infty$ or are very large, they may be entirely irrelevant, as they are in the examples of Sections 6 and 7.[11] But if they are small, then they may be used as a signal that the sellers are not making many promises, and hence that the promises are reliable. We explore signalling in Section 8.

An economy is defined as a vector

$$\mathcal{E} = \big((u^h, e^h)_{h\in H}, \big(R_j, ((\lambda_{sj}^h)_{s\in S}, Q_j^h)_{h\in H}\big)_{j\in J}\big).$$

Note again that assets consist of promises, penalties for default, and limits on sales.

2.2. *Equilibrium*

2.2.1. *Macro variables and individual choice variables*

Our endogenous variables consist of three macro variables and four individual choice variables:

$p \in \mathbb{R}_{++}^{S^*\times L}$ = commodity prices;
$\pi \in \mathbb{R}_+^J$ = asset prices;
$K \in [0,1]^{S\times J}$ = expected delivery *rates* on assets;
$x^h \in \mathbb{R}_+^{S^*\times L}$ = consumption of h;
$\theta^h \in \mathbb{R}_+^J$ = asset purchases of h;
$\varphi^h \in \mathbb{R}_+^J$ = asset sales of h;
$D^h \in \mathbb{R}_+^{(S\times L)\times J}$ = deliveries by agent h on assets $j \in J$.

The possibility of default forces us to add delivery rates K as macro variables. In keeping with anonymity, the promises φ_j^h and $\varphi_j^{\tilde h}$ of different sellers h and $\tilde h$ are not allowed to be distinguished, even though they may deliver differently, $D_j^h \neq D_j^{\tilde h}$. Just as the sales of promises φ_j^h are pooled at the market for asset j, so we suppose the deliveries on j are also pooled. The buyers (shareholders) of

[10] Evidence abounds that finite bounds are always imposed in the extension of credit. Even the best "name" among borrowers has a limited credit line.

[11] In Section 4 we are able to prove the existence of equilibrium even when $Q_j^h = \infty$, provided $\lambda \gg 0$ and the R_j all deliver in the same good.

pool j receive a pro rata share of all its different sellers' deliveries. Each share of pool j delivers the fraction

$$K_{sj} = \frac{\sum_{h \in H} p_s \cdot D_{sj}^h}{\sum_{h \in H} p_s \cdot R_{sj} \varphi_j^h}$$

of its promise $p_s \cdot R_{sj}$ in state s. For large pools an agent will take K_{sj} as fixed (just as he takes π_j as fixed). The shareholder of pool j does not know, or need to know, the identities of the sellers or the quantities of their sales. All that matters to him is the price π_j of the share and the anticipated delivery rates $K_j \equiv (K_{sj})_{s \in S}$.

The terms $(R_j, ((\lambda_{sj}^h)_{s \in S}, Q_j^h)_{h \in H})$ of pool j are set exogenously, just as the location, date, and quality of a commodity are in traditional general equilibrium theory. The prices π_j, the delivery rates K_j, and the trades $(\theta_j^h, \varphi_j^h)_{h \in H}$ at each pool j are all determined endogenously at equilibrium by the market forces of supply and demand.

An agent's ability to keep a promise depends on how many promises he sells, both of the same kind j, and of other kinds $j' \neq j$. Moral hazard enters the picture, since a buyer of an asset (i.e., lender) does not know which other promises the seller (i.e., borrower) has made, and because borrowers have the option to default.

Adverse selection enters the picture because agents have different endowments out of which to keep their promises, and also different disutilities of default. Some agents, whom we may think of as unreliable, have more incentive to default and to make large sales of promises into the pool.

Signalling enters the picture via the quantity limits Q_j. If no agent can sell more than Q_j promises into the pool, and if reliable agents can be counted on to sell the full quota Q_j, then buyers need not worry that the unreliable sellers are disproportionately represented.

2.2.2. *Household budget and payoff*

The *budget set* $B^h(p, \pi, K)$ of agent h is given by

$$\begin{aligned} B^h(p, \pi, K) &= \Big\{ (x, \theta, \varphi, D) \in \mathbb{R}_+^{S^* \times L} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{J \times S \times L} : \\ &\quad p_0 \cdot (x_0 - e_0^h) + \pi \cdot (\theta - \varphi) \leq 0;\ \varphi_j \leq Q_j^h \text{ for } j \in J; \text{ and,} \\ &\quad \forall s \in S,\ p_s \cdot (x_s - e_s^h) + \sum_{j \in J} p_s \cdot D_{sj} \leq \sum_{j \in J} \theta_j K_{sj} p_s \cdot R_{sj} \Big\}. \end{aligned}$$

The budget set allows agent h to deliver whatever he pleases. On the other hand, the agent expects to receive a fraction K_{sj} of the promises bought by him via asset j in state s. The first constraint says that agent h cannot spend more on purchases of commodities x_0 and assets θ than the revenue he receives from the sale of commodities e_0^h and assets φ. Moreover, he can never sell more than Q_j^h of any asset j. The second constraint applies separately in each state $s \in S$. It says that agent h cannot spend more on the purchase of commodities x_s and asset deliveries $\sum_j D_{sj}$ in state s than the revenue he gets in state s from commodity sales e_s^h and asset receipts $\sum_j \theta_j K_{sj} p_s R_{sj}$.

The only reason that agents deliver anything on their promises is that they feel a disutility λ_{sj}^h from defaulting. The payoff of (x, θ, φ, D) given prices p, to agent h is

$$w^h(x, \theta, \varphi, D, p) = u^h(x) - \sum_{j \in J} \sum_{s \in S} \frac{\lambda_{sj}^h [\varphi_j p_s \cdot R_{sj} - p_s \cdot D_{sj}]^+}{p_s \cdot v_s},$$

where $v_s \in \mathbb{R}_+^L$ is exogenously specified with $v_s \neq 0$. Note that $[\varphi_j p_s \cdot R_{sj} - p_s \cdot D_{sj}]^+ \equiv \max\{0, \varphi_j p_s \cdot R_{sj} - p_s \cdot D_{sj}\}$ is exactly the money value of the default of h on his promise to deliver on asset j in state s. Dividing it by $p_s \cdot v_s$ measures it in real terms.

Notice that the budget set is convex,[12] and the payoff function w^h is concave, in the household choice variables (x, θ, φ, D).[13]

2.2.3. *Default penalties*

Once we allow for default it is evident that society has much to gain from punishing those agents who fail to keep their promises. In a multiperiod world, market forces themselves might provide some incentive to keep promises, since agents who acquired a bad reputation for previous defaults might find it more difficult to obtain new loans. Collateral is also a very important device for guaranteeing at least partial payment (see Geanakoplos (1997)), but here we ignore it. For reasons of simplicity and tractability, we confine attention to a two period model with exogenously specified default penalties that are increasing in the size of the default. These penalties might be interpreted as the sum of third party punishment such as prison terms, pangs of conscience, (unmodeled) reputation losses, and (unmodeled) garnishing of future income.

Default in our model can either be strategic or due to ill fortune. Penalties are imposed on agents who fail to deliver, whatever the cause. Debtors choose

[12]It is also worth noting a *scaling property* of the budget set (which is immediate from its definition and the fact that $e_s^h \neq 0$ and $p_s \gg 0$ for all $s \in S^*$): $(x, \theta, \varphi, D) \in B^h(p, \pi, K)$ and $0 < \alpha < 1 \Rightarrow (\alpha x, \alpha\theta, \alpha\varphi, \alpha D) \in B^h(p', \pi', K')$ for all (p', π', K') sufficiently close to (p, π, K). This property will often be useful to us.

[13]Had we expressed these choices with other (apparently natural) variables, such as $\delta_{sj}^h \equiv$ delivery per unit promised, the budget set would no longer be convex, nor would w^h be concave.

whether to repay or to bear the penalty for defaulting; creditors cannot observe why default occurs. Agents who have no resources to repay will be punished as severely as they would if they had the resources but chose not to repay.[14] The consequences of default penalties are therefore two-fold: they tend to induce agents to keep promises when they are able, and they tend to discourage agents from making promises that they know in advance they will not always be able to keep.

Although in practice the severity of the penalty (e.g., a felony vs. a misdemeanor) depends on the nominal amount, and that is only adjusted slowly in the face of inflation, we suppose the adjustment is instantaneous, so that the penalties depend on the "real" default. Accordingly, we divide nominal defaults by the market price in state s of a fixed basket of goods v_s.

For simplicity (and for the facility of doing comparative statics) we have taken the default penalty to be linear and separable in the amount of default, as in Shubik and Wilson (1977). But we can easily accommodate more general payoffs w^h that allow for the marginal rate of substitution between goods to depend on the level of default. All that is needed for Theorem 1 is the continuity of w^h in all its variables, and concavity of w^h in (x, θ, φ, D). For Theorem 2 we need to assume, in addition, that given any x, $w^h(x, \theta, \varphi, D, p) < u^h(e^h)$ if the default in any state, on any asset, is sufficiently large.

One could easily imagine a legal system that imposes penalties that are nonconcave and even discontinuous in the size of the default, for example, trigger penalties that jump to a minimum level at the first infinitesimal default. One could also imagine confiscation of commodities in case of default. Our model does not explicitly allow for these possibilities. But as we show in our working paper (Dubey, Geanakoplos, and Shubik (2000)), with a continuum of households, such modifications to the default penalties do not destroy the existence of equilibrium.

2.2.4. *Market clearing*

We are now in a position to define a $GE(R, \lambda, Q)$ equilibrium. It is a list $\langle p, \pi, K, (x^h, \theta^h, \varphi^h, D^h)_{h\in H}\rangle$ such that (1)–(4) below hold.

(1) For $h \in H$, $(x^h, \theta^h, \varphi^h, D^h) \in \arg\max w^h(x, \theta, \varphi, D, p)$ over $B^h(p, \pi, K)$,

(2) $$\sum_{h\in H}(x^h - e^h) = 0,$$

(3) $$\sum_{h\in H}(\theta^h - \varphi^h) = 0,$$

[14] In our model default penalties do not distinguish fraud from ill fortune. In reality they are hard to separate, but ever since Las siete Partitas of Don Alfonso X "the wise," bankruptcy law has sought to distinguish them.

$$(4)\qquad K_{sj} = \begin{cases} \sum_{h\in II} p_s \cdot D^h_{sj} \Big/ \sum_{h\in II} p_s \cdot R_{sj}\varphi^h_j & \text{if } \sum_{h\in II} p_s \cdot R_{sj}\varphi^h_j > 0, \\ \text{arbitrary} & \text{if } \sum_{h\in H} p_s \cdot R_{sj}\varphi^h_j = 0. \end{cases}$$

Condition (1) says that all agents optimize; (2) and (3) require commodity and asset markets to clear. Condition (4), together with the definition of the budget set, says that each potential lender (i.e., buyer) of an asset is correct in his expectation about the fraction of promises that do in fact get delivered. Moreover, his expectation of the rate of delivery does not depend on anything he does himself; in particular, it does not depend on the amount θ^h_j he loans (i.e., purchases) of the asset. Every lender gets the same rate of delivery.

Since heterogeneous borrowers may be selling the same asset, the realized rate of delivery K_{sj} is an average of the rates of delivery of each of the borrowers, weighted by the quantity of their sales. It might well happen that those borrowers with the highest rates of default are selling most of the asset, and this is the adverse selection and moral hazard that rational lenders must forecast.

We believe that our definition of $GE(R, \lambda, Q)$ equilibrium embodies the spirit of perfect, anonymous competition, and represents a significant fraction of the mass asset markets of a modern enterprise economy.

2.2.5. *Chain reactions of default*

Our general equilibrium formulation enables us to evaluate the system-wide consequences of default. In a world in which promises can exceed physical endowments, each default can begin a chain reaction. A creditor in one asset where payment does not occur is deprived of the means of delivery in another asset where he is the debtor, thereby causing a further default in some other asset, etc. The indirect effects of default might be as important as the direct effects, but they are missed in partial equilibrium models.

In modern financial economies, agents often are long and short in many different assets. They rely on revenues from their loans to keep their own promises. But these revenues are only as reliable as the loans other agents have made to yet different parties, thus opening the possibility of a chain reaction of defaults. If α defaults against β, forcing β to default against γ, forcing γ to default against δ, then in our definition of equilibrium, α, β, and γ will pay default penalties, and the total utility loss from defaults will be large. Curiously this phenomenon is at its most dangerous when the financial system is at an intermediate level of development, with smoothly functioning markets, that permit agents to go short, but with missing asset markets, which force agents to hold complicated portfolios of assets to achieve the risk spreading they desire.

Consider a world with four agents and three possible future events, each consisting of many different states of the world. Suppose β wants to consume in the first event, γ in the second event, and δ in the third event. Suppose

agents β, γ, and δ have no endowment in the future states. Suppose α wants to consume in the present, but has a considerable endowment of good in the future, except in one unlikely state ω in the third event.

If there were an advanced financial system of Arrow securities, agent α would in effect sell directly to each of the other three agents. For example, with just three Arrow event-contingent securities, each one paying off exclusively in a different one of the three events, agent α would sell the first security to β, the second to γ, and the third to δ. Agent α by himself would default in state ω, and he alone would pay a default penalty.

Suppose, however, that in a less advanced financial system there are again three securities available. R^{123} promises 1 dollar in every state, R^{23} promises 1 dollar in (every state in) events 2 and 3, and R^3 promises 1 dollar in (every state in) event 3. Then in equilibrium we could expect α to sell R^{123}, β to buy R^{123} and to sell R^{23}, γ to buy R^{23} and to sell R^3, and δ to buy R^3. In the bad state ω in event three, the chain of defaults indicated above will take place. The penalty that α pays for starting the chain reaction may be very small compared to the total penalty incurrent by the rest of the defaulters.

Notice that the asset span of $\{R^{123}, R^{23}, R^3\}$ is exactly the same as with the three Arrow event-contingent securities. What makes the chain of defaults possible is the interlocking asset trade, with some investors holding assets that other investors short, in a long chain. With Arrow securities this chain would never include more than one link and one default.

One way around these chain reactions is to encourage market intermediation that nets payouts, as discussed in Dubey, Geanakoplos, and Shubik (2000).

3. EQUILIBRIUM REFINEMENT

3.1. *Untraded Assets*

It is a curious fact that many of the large asset markets that our model seeks to describe have been initiated not by entrepreneurs but by government intervention. The government, for example, began the GNMA mortgage program by guaranteeing delivery on the promises of all borrowers eligible for the program (but not the timing[15] of delivery). It is likely, however, that these mortgage markets would function smoothly even without government guarantees. Private companies indeed do sell insurance on non-GNMA mortgages. A reasonable question to ask is why the securitized mortgage market did not begin on its own?

One possible explanation is provided by our model. When assets are actively traded, expected deliveries K_{sj} must be equal to actual deliveries. Expectations cannot therefore be unduly pessimistic. But for assets that are not actively traded, our model makes no assumption about expectations of delivery

[15]A default induces the government to prepay the loan immediately, even if the lender would prefer the scheduled payments.

(see (4)). In the real world, investors with no experience in observing default rates might tend to overestimate their probability. This can create serious problems, in practice as in our model. In the model, so far, there is nothing to stop the expectations from being absurdly pessimistic, which in turn will support trivial equilibria with no trade in the asset. The point is easily seen by a simple example. Consider an equilibrium of an economy. Introduce new assets j, but choose their prices π_j close to zero. Then no agent will be willing to sell them, for he gets very little in exchange, but undertakes a relatively large obligation either to deliver commodities or to pay default penalties. Also choose the K_{sj} to be positive but even smaller. Then in spite of their low price, no agent will be willing to buy the assets since he expects them to deliver virtually nothing. Thus we have obtained trivial equilibria in which there is no trade of the new assets on account of arbitrarily pessimistic expectations regarding their deliveries.

We believe that unreasonable pessimism prevents many real world markets from opening, and provides an important role for government intervention. But it is interesting to study equilibrium in which expectations are always reasonably optimistic. It is of central importance for us to understand which markets are open and which are not, and we do not want our answer to depend on the agents' whimsical pessimism.

3.2. *Refined Equilibrium*

Expectations for deliveries by assets that are not traded are analogous to beliefs in game theory "off the equilibrium path." Selten (1975) dealt with the game theory problem by forcing every agent to tremble and play all his strategies with probability at least $\varepsilon > 0$, and then letting $\varepsilon \to 0$. We shall also invoke a tremble, but in quite a different spirit. Our tremble will be "on the market" and not on households' (players') strategies. Indeed, it might well be that no household could tremble the way we want.

Consider an external ε-agent who sells and buys $\varepsilon = (\varepsilon_j)_{j \in J} \gg 0$ of every asset, and fully delivers on his promises. (One might interpret this agent as a government which guarantees delivery on the first infinitesimal promises.) This touch of honesty banishes whimsical pessimism.[16]

[16]In the strategic market games literature it has been observed that markets can be arbitrarily shut because each agent expects that no other agent will go there, and hence does not go himself. Those markets are truly opened simply by announcing any price: trade will necessarily be induced, unless there were no gains to trade to begin with. Unfortunately, in our model, the problem is not so simple. As we saw in our thought experiment, it is *always* possible to announce some prices (π, K) that eliminate all buying and selling, arbitrarily shutting the asset markets. To truly open them, we need to pick *appropriate* (π, K). This is achieved by our carefully chosen boost.

In strategic market games an external agent was introduced simply to trade, because any trade necessarily led to the formation of a price, which is all that was required. He could easily be replaced by a tremble on household strategies that forced them to trade instead. This expedient does not work for asset markets in the presence of default. Our external agent must trade and

An equilibrium $E(\varepsilon)$ obtained with the ε-agent is called an *ε-boosted equilibrium*. Thus any such $E(\varepsilon) = \langle p(\varepsilon), \pi(\varepsilon), K(\varepsilon), (x^h(\varepsilon), \theta^h(\varepsilon), \varphi^h(\varepsilon), D^h(\varepsilon))_{h\in H}\rangle$ must satisfy:

$$(1^*) \qquad \big(x^h(\varepsilon), \theta^h(\varepsilon), \varphi^h(\varepsilon), D^h(\varepsilon)\big) \in \arg\max w^h(x, \theta, \varphi, D, p(\varepsilon))$$
$$\text{over} \quad B^h(p(\varepsilon), \pi(\varepsilon), K(\varepsilon)),$$

$$(2^*) \qquad \sum_{h\in H}(x_s^h(\varepsilon) - e_s^h) = \begin{cases} 0 & \text{if } s = 0, \\ \sum_{j\in J} \varepsilon_j (1 - K_{sj}(\varepsilon)) R_{sj} & \text{if } s \in S, \end{cases}$$

$$(3^*) \qquad \sum_{h\in H}(\theta^h(\varepsilon) - \varphi^h(\varepsilon)) = 0,$$

$$(4^*) \qquad K_{sj}(\varepsilon) = \begin{cases} \dfrac{p_s(\varepsilon)\cdot R_{sj}\varepsilon_j + \sum_{h\in H} p_s(\varepsilon)\cdot D_{sj}^h(\varepsilon)}{p_s(\varepsilon)\cdot R_{sj}\varepsilon_j + \sum_{h\in H} p_s(\varepsilon)\cdot R_{sj}\varphi_j^h(\varepsilon)} & \text{if } p_s(\varepsilon)\cdot R_{sj} > 0, \\ 1 & \text{if } p_s(\varepsilon)\cdot R_{sj} = 0. \end{cases}$$

Since the ε-agent buys and sells ε_j units of each asset j, asset market clearing (3*) is as before. But since he delivers fully $\varepsilon_j R_{sj}$ on his promises, and gets delivered only $\varepsilon_j K_{sj}(\varepsilon) R_{sj}$, on net he injects the vector of commodities $\sum_{j\in J} \varepsilon_j (1 - K_{sj}(\varepsilon)) R_{sj}$ into the economy in each state $s \in S$. This explains (2*). Finally, condition (4*) says that delivery rates are boosted by the external agent. (The delivery rate is irrelevant when promises $p_s(\varepsilon)\cdot R_{sj} = 0$, and we have arbitrarily set it equal to 1.) As $\varepsilon \to 0$, this boost disappears for assets that are positively traded in the limit. But if $\varepsilon_j / \sum_{h\in H} \varphi_j^h(\varepsilon)$ does not go to zero, the limiting rates K_{sj} will be boosted (unless there is no default by the real agents).

An equilibrium $E = \langle p, \pi, K, (x^h, \theta^h, \varphi^h, D^h)_{h\in H}\rangle$ is called a *refined equilibrium* if there exists a sequence of ε-boosted equilibria $E(\varepsilon)$ with $\varepsilon \to 0$ and $E(\varepsilon) \to E$.

In (an unrefined) equilibrium, the price π_j of an untraded asset j might be so low that no agent is close to wanting to sell it, or so high that no agent is close to wanting to buy it. In any refined equilibrium, if an untraded asset j is expected to default somewhere, i.e., $K_{sj} p_s \cdot R_{sj} < p_s \cdot R_{sj}$ for some $s \in S$, then there must be at least one agent who is selling it in $E(\varepsilon)$, otherwise $K_{sj} = 1$, and is therefore *on-the-verge of selling* it in E (in the sense that he would sell it if its price were any higher). If *every* agent h on the verge of selling j is "strictly conscientious," i.e., has a penalty λ_{sj}^h greater than the marginal utility of his consumption in state s, then clearly $K_{sj} = 1$, since deliveries will be full in $E(\varepsilon)$ for all small ε. If every agent is strictly conscientious in at least m states

deliver fully. Replacing him with households who tremble (and deliver less than fully on average), will lead to more equilibria, defeating the purpose of our refinement.

on asset j, then $K_{sj} \geq m/S$ for at least one state s. These properties, and a few more described in the Appendix, yield what we call *on-the-verge equilibria*. They form a superset of refined equilibria, but are much easier to compute. If the superset is a singleton, then Theorems 1 and 2 guarantee that we can compute refined equilibrium as the unique solution of these on-the-verge conditions, without having to bother with the infinite sequence $E(\varepsilon)$. We exploit this in our examples.

We could have imagined an external agent who delivers only 70% of his promises, instead of 100%. It is clear that any "100% refined equilibrium allocation" is a "70% refined equilibrium allocation," thus explaining why our choice of 100% deliveries gives the sharpest refinement.

In Sections 5–8, on the endogeneity of the asset structure, we show that the equilibrium refinement plays a crucial role in determining whether an asset j is positively traded ($j \in \mathcal{A}^*$) or not ($j \in \mathcal{A} \setminus \mathcal{A}^*$).

3.3. *Perfectly Competitive Pooling vs. Negotiation*

Pooling dramatically reduces the information needed to buy a diversified portfolio of risks. Instead of forecasting individual deliveries K_{sj}^h for many different individuals h, a buyer need only concern himself with a single average delivery K_{sj}. Figuring out K_{sj}^h for one individual is typically no less difficult than estimating K_{sj} for a pool with a large population. Buyers facing K_{sj} are aware that unreliable sellers have put more into the pool, but they need not worry that they will get a worse selection than other buyers.

Perfectly competitive pooling reduces information requirements even further. Buyers need not worry about either π_j or K_{sj} varying as they change their expenditures on j.

Nor do they need to forecast how the delivery at any pool would vary if the price were changed, since no one of them can change the price. In the contract theory tradition, buyers try to lure a better clientele by offering higher prices. But then they must foresee delivery rates $K(\pi, R, \lambda, Q)$ over 4 dimensions, while with pooling they need only foresee K over 3 dimensions (R, λ, Q).[17]

Thus pooling ameliorates the costly information processing problems inherent in multiple bilateral negotiations, which is one reason why it is becoming so prevalent in modern economies.[18]

[17]The reader may worry that the paucity of prices will compromise the potential of the model to incorporate the interesting features of adverse selection and signalling. While our simplification precludes the analysis of price-setting in one-on-one bargaining situations, adverse selection and signalling are clearly still present in the model. As we show in Section 8, our simple model does produce interesting equilibria, even for insurance markets.

[18]The informational difference between modern pools and traditional negotiation is beautifully illustrated by the mortgage banker played by Jimmy Stewart in the movie "It's a Wonderful Life." His clients have no qualifications for loans, but he chooses to lend to them on the standard terms, based on his extraordinary intuitive insight into their character. His brilliance is rewarded

Pooling also vastly increases liquidity. When an asset is very finely defined, so as to require delivery in exactly those states most appropriate for a small group of people, then it is not likely to be heavily traded. A seller may have to wait a long time to find a suitable buyer, and vice versa. And when such a seller is found, he will exercise some temporary monopoly power. When there is pooling, the volume of trade is high and nobody has monopoly power.

4. THE ORDERLY FUNCTION OF MARKETS WITH DEFAULT

Our first goal in this paper is to establish that default is completely consistent with the orderly function of markets. To that end we prove that under fairly general conditions, refined equilibrium always exists in our model.

The universal existence of equilibrium is somewhat surprising because of the historical tendency to associate default with disequilibrium (or more accurately, to make full delivery part of the definition of equilibrium). Furthermore, endogeneity of the asset payoff structure is known to complicate the existence of equilibrium with incomplete markets. But we show that no new existence problems arise from the endogeneity of the asset payoffs due to default.

The universal existence of equilibrium with default is also surprising because the pioneering papers placing adverse selection in a model of competition, by Akerlof (1970) on the market for lemons, and Rothschild and Stiglitz (1976) on insurance markets, purportedly showed that adverse selection is quite commonly inconsistent with equilibrium. (We discuss Rothschild–Stiglitz in Section 8.)

We are now ready to state our main theorem, which is that $GE(R, \lambda, Q)$ equilibrium always exists, even if we insist on the equilibrium refinement discussed in Section 3. Its proof is given in the Appendix.

THEOREM 1: *For any* $\lambda \in \overline{\mathbb{R}}_+^{HSJ}$ *and* $Q \in \mathbb{R}_+^{HJ}$, *a refined equilibrium exists, where* $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \infty$.

Our proof uses the fact that $\varphi_j^h \leq Q_j^h$ by assumption. Later the Q_j^h will play an important role as signals, but now the reader may wonder what would happen if they were eliminated, or taken to be enormously large. Recall that there is a pathology that occasionally occurs even when there is no default, for example in the GEI model. Sometimes two assets j and j' that promise different commodities nevertheless become nearly equivalent at some spot prices $(p_s)_{s \in S}$ because they then promise nearly the same money. At these prices the asset span suddenly drops, and demand blows up as agents try to go infinitely long

in the movie. Unfortunately, in modern day competitive pools, there is no role left for Jimmy Stewart's skills.

in asset j' and infinitely short in asset j (or vice versa). This destroys the existence of equilibrium. The bounds Q_j^h prevent this, as Radner (1972) long ago pointed out for the GEI model.

Without the bounds, GEI equilibrium can only be guaranteed if all the assets promise payoffs exclusively in the same good (say L) in each state $s \in S$. (See Geanakoplos and Polemarchakis (1986).)

Default accentuates the asset span problem: two assets that usually make different deliveries might, given certain macro variables (p, π, K), actually deliver the same money. For example, if $R_j = R_{j'}$, and agent h sells asset j and buys j', defaulting slightly in state s on j, while agent h' sells j' and buys j, defaulting slightly in state s', then h and h' have effectively created a new asset trading off consumption between s and s'. (We shall see that augmenting the span of asset deliveries is the raison d'être of lenient default penalties.) As the default rates get lower, one might think the agents might go unboundedly longer and shorter in the two assets, creating the same problem of asset span as in GEI, since in the limit there would be no default and the span would drop. One should therefore wonder if default introduces additional difficulties in proving the existence of equilibrium. We have just seen that in the presence of the bounds Q_j^h it does not. Even without the bounds Q_j^h, we can show that equilibrium exists, just as in GEI, provided that all assets deliver in the same commodity.

THEOREM 2: *Let all promises R_j be exclusively in good L for all $s \in S$ and let $R_j \neq 0$ for all $j \in J$. Define $GE(R, \lambda) = GE(R, \lambda, Q)$ with $Q_j^h = \infty \; \forall h \in H$, $j \in J$. Then $GE(R, \lambda)$ exists for any vector $\lambda \in \overline{\mathbb{R}}_+^{HSJ}$ with $\sum_{s \in S} \lambda_{sj}^h R_{sLj} > 0$ for all $h \in H$ and $j \in J$.*

In the course of the proof (see the Appendix), we show that if an agent h is buying a portfolio θ^h and selling an equally expensive portfolio φ^h that makes the same promises, and if there is the slightest default on an asset in θ^h, then agent h must default completely in some state on one of the assets in φ^h. The problematic scenario just described with a slight default in two assets j and j' making identical promises could not occur, because h and h' would be defaulting completely (not slightly) in some states s and s'.

5. ENDOGENOUS EMERGENCE OF ARROW SECURITIES

In some contexts it has become customary to think of endogenizing the asset structure by allowing atomic agents to invent new assets (often one at a time) to upset a prevailing equilibrium. These asset-creating agents are hypothesized to be motivated by payoffs that might depend on the perceived volume of trade that would take place in their new asset if no other prices changed (or in the new trading equilibrium, after all prices equilibrated), or in some other way on their perceived profits from introducing the new asset. When the status quo

assets are chosen so that none of these agents has an incentive to introduce a new asset, the asset structure is said to have been endogenously determined. This approach to endogenizing the asset structure inevitably involves a combination of price taking behavior and oligopolistic-Nash thinking on the part of the asset-creating agents.

By contrast we follow a relentlessly competitive approach to the problem of endogenous assets.[19] Every agent is a price taker. An asset is endogenously missing in our approach if it is not in $\mathcal{A}^*$, i.e., if there is a price at which no agent wants to sell or buy it.

Recall that an asset is specified not just by its vector R_j of promises across states, but also by the associated default penalties λ^h_{sj}, and quantity constraints Q^h_j. The Arrow (1953) security, for state s, is an asset $i(s)$ that promises one unit of good L in state s (and nothing else), with penalty $\lambda^h_{si(s)} = \infty$ and $Q^h_{i(s)} = \infty$ for all $h \in H$.

If the government could simultaneously and without limitations choose assets, it would pick all the Arrow securities. We show in Theorem 3 that the market would do the same. Given an arbitrary collection of assets that includes all the Arrow securities (but possibly many other assets with low penalties or low quantity constraints), equilibrium will necessarily be the same as if only the Arrow securities were available. No asset with $K_{sj} < 1$ will be actively traded. The set of actively traded assets $\mathcal{A}^*$ can always be taken to be just the Arrow securities, no matter how big $\mathcal{A}$ is.

THEOREM 3: *Let* $\mathcal{E} = ((u^h, e^h)_{h\in H}, (R_j, ((\lambda^h_{sj})_{s\in S}, Q^h_j)_{h\in H})_{j\in J})$ *be an economy that includes all the Arrow securities* $\{i(s) : s \in S\}$. *Then for any* $GE(R, \lambda, Q)$ *equilibrium* $\eta = ((p, \pi, K), (x^h, \theta^h, \varphi^h, D^h)_{h\in H})$, *we can find prices* $q \in \mathbb{R}^{(1+S)L}_{++}$ *such that* $(q, (x^h)_{h\in H})$ *is an Arrow–Debreu equilibrium. Moreover, if* $\lambda \gg 0$, *no agent defaults on any actively traded asset in* η, *even if there are assets* $j \in J$ *with low* λ^h_{sj}. *Finally, there is an equilibrium* η', *possibly* η *itself, with the same* $((p, \pi, K), (x^h)_{h\in H})$ *such that the only actively traded assets in* η' *are Arrow securities.*

PROOF: Let η be given. Let $q_0 = p_0$ and let $q_s = \pi_{i(s)}(p_s/p_{sL})\ \forall s \in S$. Let

$$v^h(q) \equiv \max\{u^h(x) : q \cdot x \le q \cdot e^h,\ x \in \mathbb{R}^{S^*\times L}_+\}.$$

Observe that $K_{sj} = 1$ for each asset j with $\lambda^h_{sj} = \infty\ \forall h, s$, if $R_s \neq 0$, since no agent will default in the refinement and the external agent will be fully delivering. It follows that by never defaulting, each agent h could, by selling and buying the Arrow securities, achieve at least $v^h(q)$, that is,

$$u^h(x^h) \ge u^h(x^h) - \text{default penalty} \ge v^h(q).$$

[19]For other competitive approaches for endogenous assets, see Allen and Gale (1988, 1991), Pesendorfer (1995), and Townsend (1979).

It follows that $q \cdot x^h \geq q \cdot e^h \ \forall h \in H$. Since η is a $GE(R, \lambda, Q)$ equilibrium, $\sum_{h\in H} x^h = \sum_{h\in H} e^h$. Hence, $q \cdot x^h = q \cdot e^h \ \forall h \in H$, and $(q, (x^h)_{h\in H})$ is an Arrow–Debreu equilibrium, and the default penalty actually borne by each agent $h \in H$ is zero.

Clearly each agent is indifferent to achieving x^h via the actively traded assets in η, or via Arrow securities. If every agent trades exclusively via Arrow securities, then supply will equal demand, and we achieve the desired equilibrium η'. *Q.E.D.*

If $\mathcal{A}$ does not include all the Arrow securities, equilibrium will still select actively traded assets. When default is expected on an asset and $K_j < 1$, buyers receive only what is delivered, but sellers give up in addition penalties for what is not delivered. This effective transactions cost limits the number of assets that can be actively traded, and leads to $\mathcal{A}^*$ much smaller than $\mathcal{A}$. This is so even though we confine attention to refined equilibria in which optimistic expectations tend to boost trade in every asset.

In the following three sections we give examples showing that if two of the three dimensions (R, λ, Q) of assets in $\mathcal{A}$ are fixed, the third dimension is endogenously determined in equilibrium via $\mathcal{A}^*$. In our examples one of the two exogenous dimensions is fixed far from Arrowian (to steer clear of Theorem 3), and we find that the endogenous dimension is then chosen also far from Arrowian.

6. ENDOGENOUS PROMISES

Here we give an example showing that if penalties are low, the market will choose to actively trade only a single riskless promise, instead of the Arrow promises. In every state $s \in S$, there is some agent who does not intend to deliver and is relatively unworried about his punishment in that state (because he thinks the state is relatively unlikely). He will have incentive to sell the corresponding Arrow promise j and debase its K_{sj}, and therefore its price π_j. This will effectively prevent agents intending to deliver in state s from selling j. By raising the general level of default penalties, this phenomenon is discouraged. As penalties are made harsher, $\mathcal{A}^*$ increases.

EXAMPLE 1: Suppose there are just two future states $S = \{1, 2\}$, one commodity $L = 1$, and three asset promises $R_0 = (1, 1)$, $R_1 = (1, 0)$, $R_2 = (0, 1)$ with $Q_j^h = \infty$ for all h, j. Let there be two agents $H = \{1, 2\}$ with period one endowments $e^1 = (1, 0)$, $e^2 = (0, 1)$, and payoffs that depend only[20] on con-

[20]To keep our examples as simple as possible, we suppose no utility in period 0. This violates the strict monotonicity assumption of our model, but our theorems can be extended to cover this, and the other examples. We refrain from doing so for ease of exposition.

sumption (x_1, x_2) in period 1 and on penalties:

$$W^1(x, \theta, \varphi, D) = \frac{2}{3}\left[\log x_1 - \sum_{j=0}^{2} \lambda(R_{1j}\varphi_j - D_{1j})^+\right] + \frac{1}{3}\left[\log x_2 - \sum_{j=0}^{2} \lambda(R_{2j}\varphi_j - D_{2j})^+\right],$$

$$W^2(x, \theta, \varphi, D) = \frac{1}{3}\left[\log x_1 - \sum_{j=0}^{2} \lambda(R_{1j}\varphi_j - D_{1j})^+\right] + \frac{2}{3}\left[\log x_2 - \sum_{j=0}^{2} \lambda(R_{2j}\varphi_j - D_{2j})^+\right].$$

(We take the penalty deflator $v_s = 1$, as we shall in all our remaining examples as well.) Note that each agent h effectively assigns probability 2/3 to his good state $s = h$, and probability 1/3 to his bad state $s \neq h$. The penalty rate is λ in each state, on all three assets. We shall show that for low values of λ only asset R_0 will be actively traded in equilibrium.

In any (symmetric) equilibrium, each agent will consume $1 - x$ in his good state, and x in his bad state. If $\lambda \geq 3$, it is easy to check that the Arrow–Debreu allocation $(x^1 = (2/3, 1/3), x^2 = (1/3, 2/3))$ is achieved via trade in the Arrow securities R_1, R_2. But for $\lambda \leq 7/3$, the Arrow securities are inactive in any equilibrium. To quickly check this is so for $\lambda < 2$, simply note that the marginal disutility of selling R_2 is $\frac{1}{3}\lambda$ for agent $h = 1$ (since $x_2^1 \equiv x \leq 1/3$, so $1/x \geq 3 > 2 > \lambda$). For agent $h = 2$ it is

$$\min\left\{\frac{2}{3}\lambda, \frac{2}{3}\frac{1}{1-x}\right\} \geq \min\left\{\frac{2}{3}\lambda, \frac{2}{3}\right\} > \frac{1}{3}\lambda \quad \text{if} \quad \lambda < 2.$$

Thus agent $h = 2$ is *not* selling R_2. But agent $h = 1$ would default completely if he sold R_2; hence R_2 is not sold actively in equilibrium. Similarly R_1 is not actively traded. We leave it to the reader to show that there is active trade in asset R_0 in refined equilibrium for any $1 < \lambda \leq 7/3$ (the computation is similar to Example 2 in Section 7, where the equilibrium is calculated in detail). For these λ the market endogenously chooses asset promises R_0.

For $7/3 < \lambda < 3$, both agents sell *both* Arrow securities R_1 and R_2, giving delivery rates less than 1. As λ rises to 3, delivery rates converge to 1 and trades rise to Arrow–Debreu levels.

7. ENDOGENOUS DEFAULT PENALTIES

We turn to the dual of the problem in the last section, and show that when asset promises are exogenously restricted in $\mathcal{A}$, the market will endogenously choose intermediate default penalties in $\mathcal{A}^*$, even though higher and lower levels are available in $\mathcal{A}$.

We begin by asking how high the penalties *should* be, when promises are restricted.

7.1. *The Economic Advantages of Intermediate Default Penalties with Incomplete Markets*

There are four fundamental drawbacks to reducing the default penalties λ so far that some agents choose to default in at least some states in equilibrium: (i) creditors, rationally anticipating that they might not be repaid (on account of direct and indirect reasons), are less likely to lend; (ii) borrowers may not repay even in contingencies that have been foreseen, and even though they are able; (iii) imposing penalties is a deadweight loss; (iv) the default of unreliable agents imposes an externality on reliable agents who, because they cannot distinguish themselves from the unreliable agents, are forced to borrow on less favorable terms.

Despite myriad reasons why default is socially costly, the benefits from permitting some default often outweigh all of these costs. These benefits are basically twofold, and both stem from the fact that markets are incomplete to begin with. First, an agent who defaults on a promise is in effect tailoring the given security and substituting a new security that is closer to his own needs, at a cost of the default penalty. With incomplete markets one set of assets may lead to a socially more desirable outcome than another set. Second, since *each* agent may be tailoring the same given security to his special needs, one asset is in effect replaced by as many assets as there are agents, and so the dimension of the asset span is greatly enlarged. A larger asset span is likely to improve social welfare (although this gain must be weighed against the deadweight loss of the default penalties that are thereby incurred). In short, permitting default allows for a plethora of additional assets that do not have to be specified in advance.

A third benefit from allowing default, which is closely related to the first two, is that agents can go long and short in the same security, thereby doubling their asset span. We make use of this in the following example, which shows that the optimal default penalty is intermediate, even though it causes all the disadvantages (i)–(iv).

EXAMPLE 2: Let $H = \{1, 2, 3\}$, $S = \{1, 2, 3\}$, and $L = \{1\}$. Suppose agents care only for consumption at $t = 1$, and have the same utility[21]

$$u(x_1, x_2, x_3) = \sum_{s=1}^{3} \log(x_s).$$

The endowments of the agents are $e^1 = (0, 1, 1)$, $e^2 = (1, 0, 1)$, $e^3 = (1, 1, 0)$. We assume there is one asset 0 promising $R_0 = (1, 1, 1)$. We take default penalties to be $\lambda^h_{s0} = \lambda > 0 \ \forall h, s$, with the penalty deflator $v_s = 1 \ \forall s \in S$. We take $Q^h_0 = \infty \ \forall h$.

We can calculate the equilibrium for any value of $\lambda \in (1, \infty)$. When $\lambda \leq 1$, buyers realize that sellers will not deliver anything, so demand will be zero and equilibrium will involve no trade. When $\lambda > 1$, there can be no inactive refined equilibrium.[22] When $\lambda \to \infty$ buyers will anticipate full delivery, but sellers will realize that with probability $1/3$ they will not be able to avoid a crushing penalty, and so again equilibrium trade goes to 0. By setting an intermediate level of default penalties we can make everybody better off. We graph the situation schematically in welfare space in Figure 1.

In equilibrium different sellers default differently. The buyers of the asset receive the average deliveries of all the sellers. For instance, when $\lambda = \lambda^* = 6/5$, sellers in their good states deliver fully, and sellers in their bad state default completely, even though they have goods on hand. Thus our example illustrates the pooling aspect of assets, namely that investors buy shares of a pool of individually distinct deliveries.

At $\lambda = 6/5$, $x^1 = (1/3, 5/6, 5/6)$, $x^2 = (5/6, 1/3, 5/6)$, and $x^3 = (5/6, 5/6, 1/3)$, $\theta^h = \varphi^h = \varphi = 1/2 \ \forall h$, and $K_{s0} = K = 2/3 \ \forall s$. By buying and selling $1/2$ unit of the asset R_0, agent h gains $1/3 = (2/3)(1/2) = K\theta^h$ when $s = h$ and gains $(-1/6) = (2/3)(1/2) - 1/2 = K\theta^h - \varphi^h$ in the two states $s \neq h$. Agent h delivers fully when $s \neq h$ because his marginal utility of consumption after delivery is $1/(5/6) = 6/5 = \lambda^*$. When $s = h$, agent h defaults completely since his marginal utility of consumption $1/(1/3) = 3 > \lambda^*$. Since for any $s \in S$ we have 2 agents with $h \neq s$, $K_{s0} = 2/3$. Thus the asset promise $R_0 = (1, 1, 1)$ actually delivers $(2/3, 2/3, 2/3)$ per unit promise. Agent $h = 1$ delivers $1/2 \cdot (0, 1, 1)$, agent $h = 2$ delivers $1/2 \cdot (1, 0, 1)$, and agent $h = 3$ delivers $1/2 \cdot (1, 1, 0)$. The reason each agent buys and sells only $1/2$ a unit of

[21] All our examples satisfy the assumptions of Theorem 2. Hence we can be sure a refined equilibrium exists. Actually $\log(x)$ is not continuous at 0, so by "$\log x$" we really mean

$$\log x = \begin{cases} \ln x & \text{if } x \geq \delta, \\ \dfrac{1}{\delta} x + \ln \delta - 1 & \text{if } 0 \leq x \leq \delta, \end{cases} \quad \text{for some very small } \delta > 0.$$

[22] Suppose trades are very small in the perturbation. Then each agent will be delivering fully in his good states; hence $K_{s0} \geq 2/3$ for at least one s. But then $h = s$ will want to trade a nonvanishing amount, a contradiction.

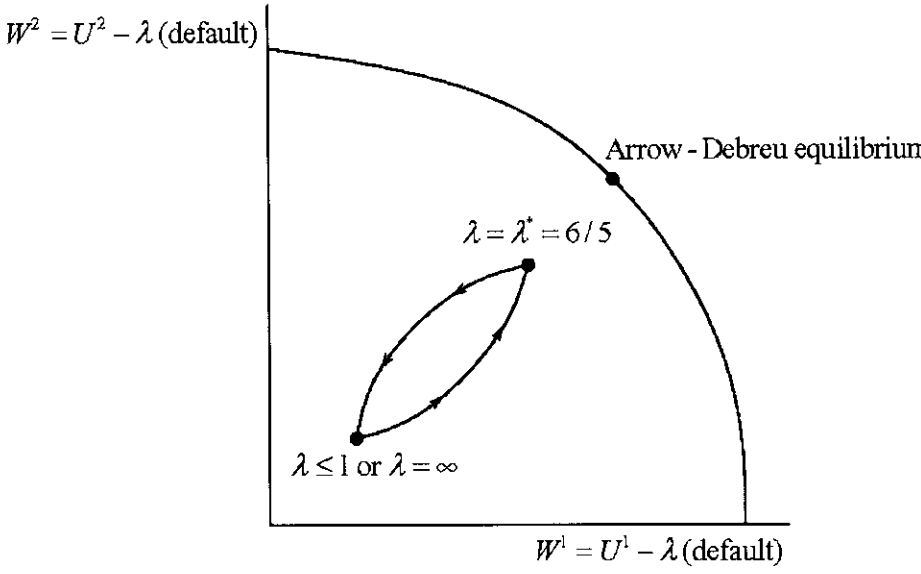

FIGURE 1.

asset R_0 instead of a full unit to get to the Arrow–Debreu allocation is that the sale of φ units of the asset is accompanied by the loss of $\varphi\lambda$ utiles for the inevitable default in state $s = h$. The marginal utility from buying the asset is $(2/3)(6/5) + (2/3)(6/5) + (2/3) \cdot (3) = 18/5$; the marginal disutility from selling is also $(6/5) + (6/5) + (6/5) = 18/5$. (It is therefore more convenient to take $\pi_0 = 18/5$.)

A consequence of pooling is that the volume of trade is high. In equilibrium (when $\lambda = 6/5$), each agent sells $1/2$ unit of the asset, giving a total volume of trade equal to $3 \cdot 1/2 = 3/2$, much greater than the volume of trade per asset in the Arrow–Debreu equilibrium.

When $1 < \lambda < 6/5$, agents default in every state, delivering nothing in their bad state and delivering $D(\lambda)$ only up to the point where the marginal utility of consumption equals λ in their good state. The reader can verify that $K(\lambda) = (6\lambda - 6)/(4\lambda - 3)$, $D(\lambda) = 3 - (3/\lambda)$, $\varphi(\lambda) = (4/3) - (1/\lambda)$, and $x^1(\lambda) = (2(1 - (1/\lambda)), 1/\lambda, 1/\lambda)$, etc. Clearly as[23] $\lambda \to 1$, $x^1(\lambda) \to (0, 1, 1)$, $D(\lambda) \to 0$, and $K(\lambda) \to 0$. (Asset trade $\varphi(\lambda)$ does not go to 0 as $\lambda \to 1$ because the log utility is $-\infty$ at zero consumption.) As $\lambda \uparrow 6/5$, $\varphi(\lambda)$, $D(\lambda)$, and $K(\lambda)$ are monotonically increasing, as is the utility of final consumption.

For $\lambda \geq 6/5$, the agents always deliver fully in their good states, while still defaulting completely in their bad states. Thus K_{s0} is maintained at $2/3$, but

[23]Recalling that $\log x = \ln x$ only for $x \geq \delta$, we really require $2(1 - (1/\lambda)) \geq \delta$, that is, $\lambda \geq 2/(2 - \delta)$. By taking δ small, $2/(2 - \delta)$ is just about 1.

asset trade again begins to drop because the inevitable punishment makes selling less attractive. The formulas are messy and we do not bother to present them here. An increase in the penalty rate beyond $\lambda = 6/5$ does not improve risk bearing (since φ begins to drop), and it also increases the deadweight loss from punishing agents who cannot deliver anyway. It thus strictly lowers welfare.

Furthermore, observe that as λ rises from 1 to $\lambda = 6/5$, the deadweight utility loss from default $\lambda\varphi + 2\lambda(\varphi - D) = (4/3)\lambda - 1 - (10/3)\lambda + 4 = 3 - 2\lambda$ actually falls, to 3/5. Since the allocation is improving, and the default penalty is falling, we deduce that $\lambda^* = 6/5$ leads to the Pareto best outcome among all economies with $\lambda_{sj}^h = \lambda$.

Example 2 illustrates that the optimal default penalty might be low enough to encourage some real default, despite the attendant deadweight loss, when markets are incomplete. In fact, the optimal penalty is so low that agents do not deliver anything in their bad state, even though the receipts they obtain from their asset holdings are on hand for delivery. In short, there is strategic default. Diamond (1984) presented a principal-agent model in which the optimal default penalty is intermediate. But the agent always delivered everything he had on hand. Default in his case was not strategic, but only due to bad fortune.[24] Example 2 also illustrates that the possibility of default makes the asset payoffs endogenous, since we do not know before an equilibrium is calculated what the default rates will turn out to be. If we change the utilities or endowments of the agents, or the default penalties, the equilibrium will change, the default rates will change, and the asset payoffs will be different.

7.2. *Market Choice of Default Penalties*

In Example 2 we asked how severe the default penalties *should* be to promote economic efficiency. Since our model allows for the possibility that different punishment regimes coexist at the same time, we can also ask how harsh the punishment scheme *will* be that endogenously emerges in equilibrium. For example, an agent could indicate his intention to perform a service, he could orally commit to performing the service, he could put in writing that he promised to perform a service, or he could draw up a contract with a lawyer announcing his promise to perform a service. If all four of these promises are treated equally by the courts, then there is no issue of selecting a punishment. But if the punishment in case of default is different for these different manners of making the same promise, then in effect the parties to the agreement are choosing the severity of default penalties attached to the promise.

[24]Hess (1983) gave an example in which it would be Pareto improving if an agent were allowed to default in a particular state, without penalty, provided he delivered fully in the others.

We shall now show that in our example, the forces of supply and demand select the optimal default penalty. The example is noteworthy for two reasons: it shows that equilibrium forces can select a single default penalty at which all assets will trade, and that this penalty is optimal. We are unable to prove a general theorem establishing either point, but in Section 7.3 we do give circumstances under which the market will indeed choose the optimal assets.

EXAMPLE 3: Consider Example 2 with only one asset promise $R_0 = (1, 1, 1)$ and $\lambda_{s0}^h = \lambda^* = 6/5\ \forall h \in H$ and $\forall s \in S$. It is natural to regard the penalty λ^* as imposed by a beneficent and knowledgeable government. But we may also regard λ^* as emerging from the equilibrium forces of supply and demand.

Now let there be a finite number of additional assets R_j, all making the same promises $R_j = (1, 1, 1)$, but with default penalties $\lambda_j = \lambda_{sj}^h$ for all $h \in H$, $s \in S$, ranging at intervals of $\lambda^*/100$ from 0 to $100\lambda^*$. The symmetry of the utilities, endowments, and penalties guarantees (by symmetrizing the proof of Theorem 2) that a symmetric, refined equilibrium must exist. (Symmetry implies that $\varphi_j^h = \theta_j^h = \varphi_j$ for all $h \in H$ and $j \in J$, and that deliveries are the same up to relabeling states, and hence that K_{sj} is invariant across s.) We shall now show that despite the myriad of available assets, in every (symmetric) refined equilibrium, all trade will be conducted in the asset j^* for which $\lambda_{sj^*}^h = \lambda^*$. We begin by describing an equilibrium of this type, and then we show it is essentially the only (symmetric) equilibrium satisfying the "on the verge" condition (described in the Appendix as a shortcut to computing equilibrium).

The equilibrium will involve exactly the same prices, delivery rates, trades, and consumption as described in Example 2 for the case $\lambda = \lambda^* = 6/5$. There we found that $x^1 = (1/3, 5/6, 5/6)$, $x^2 = (5/6, 1/3, 5/6)$, $x^3 = (5/6, 5/6, 1/3)$, and $\varphi_{j^*}^h = 1/2$ for all h, and $K_{sj^*} = 2/3$ for all s, $\pi_{j^*} = 18/5 =$ the marginal utility of buying or selling asset j^*. We must now extend that equilibrium to define prices π_j and delivery rates K_{sj} for all the new assets. The "on the verge" condition requires some agents to be on the verge of buying and others to be on the verge of selling, each asset. This uniquely specifies all these (π_j, K_j) for $j \neq j^*$. Set $\pi_j = \min\{\lambda_j, 6/5\} + \min\{\lambda_j, 6/5\} + \min\{\lambda_j, 3\}$ for $j \neq j^*$, which is the marginal disutility of selling asset j. At these prices agents are just indifferent between selling j and j^*, so it is optimal to supply zero of j.

The marginal utility of buying asset j must be equal to $(18/5)/\pi_{j^*} = 1$, i.e.,

$$\frac{\frac{6}{5}K_j + \frac{6}{5}K_j + 3K_j}{\pi_j} = 1.$$

Hence, by on-the-verge trading, $K_j = (5/27)\pi_j$ or else $K_j = 1$. For $\lambda_j > 3$, $\pi_j > 27/5$, so $K_j \geq 1$, i.e., $K_j = 1$. For $3 \geq \lambda_j > \lambda^* = 6/5$, $\pi_j > 18/5$, hence $K_j > 2/3$, consistent with two out of three types being strictly conscientious. Thus on-the-verge boosting holds.

By concavity, since the first-order conditions are satisfied, each agent is indeed maximizing by trading exclusively via asset j^*. We have thus displayed an on-the-verge equilibrium in which (almost) any default penalty is available, yet only a single one (namely the Pareto efficient penalty) is used in equilibrium.

We now argue that there can be no other symmetric on-the-verge equilibrium. In any (symmetric) equilibrium we have consumption $x^1 = (2x, 1-x, 1-x)$, and similarly $x^2 = (1-x, 2x, 1-x)$, and $x^3 = (1-x, 1-x, 2x)$. If $x = 1/6$, then all (π_j, K_j) are defined, as in the last paragraph, by the "on-the-verge" condition and in this case only asset j^* will be actively traded[25] (aside from trivial wash sales in assets j with $\lambda_j > 3$ and $K_j = 1$). If $x > 1/6$, then agent 1 has delivered up to a point in states 2 and 3 where his marginal utility of consumption $1/(1-x) > 6/5$. He would not have done that unless he was selling an asset with default penalty $\lambda_j \geq 1/(1-x) > 6/5$. If asset j delivers fully in every state, then it is irrelevant, since by symmetry each agent is buying and selling an equal amount of it. But from the argument in the proof of Theorem 2, if the asset did not fully deliver everywhere, then any agent buying and selling it would default completely in at least one state. Since by symmetry every agent buys and sells it, $K_j \leq 2/3$. The marginal utility to purchasing asset j is at most

$$\frac{2}{3}\left(\frac{1}{2x} + \frac{1}{1-x} + \frac{1}{1-x}\right) = \frac{2}{3}\frac{3x+1}{(1-x)2x}$$

$$= \frac{1}{1-x} + \frac{1}{(1-x)3x} < \frac{3}{1-x}$$

(if $x > 1/6$) in period 1. The marginal disutility of selling asset j is at least

$$\frac{1}{1-x} + \frac{1}{1-x} + \frac{1}{1-x} = \frac{3}{1-x},$$

a contradiction.

If $x < 1/6$, we shall show there can be no equilibrium price π^* for asset $j = j^*$. The marginal disutility of selling asset j^* is

$$\frac{1}{1-x} + \frac{1}{1-x} + \frac{6}{5},$$

since $1/(1-x) < 6/5 = \lambda^*$. Hence, the marginal disutility of selling is less than 18/5. It is also the case that every agent would deliver in each of his

[25]For $6/5 < \lambda_j < 3$, no agent will deliver anything on asset j in his bad state, since he consumes 1/3 and $3 > \lambda_j$. Hence if j is actively traded, $K_j \leq 2/3$, contradicting our formula $K_j = (5/27)\pi_j = (5/27)(6/5 + 6/5 + \lambda_j) > 2/3$. If $\lambda_j < 6/5$, and yet consumption in the good state is 5/6, then no agent who actively sells j will deliver anything on j in any state. Hence if j were active, K_j would be zero, contradicting our formula for K_j.

two good states if he were selling asset j^*. Hence, from on-the-verge boosting, $K_{s0}\varepsilon \geq 2/3$ for some s, and so by symmetry, $\forall s \in S$. The marginal utility of buying asset j^* is then at least

$$\frac{2}{3}\frac{1}{1-x} + \frac{2}{3}\frac{1}{1-x} + \frac{2}{3}\frac{1}{2x}.$$

For $x < 1/6$, the marginal utility of buying is always larger than 18/5, hence larger than the marginal disutility of selling, a contradiction. This proves there is a unique symmetric on-the-verge equilibrium. Hence it must be the unique symmetrically refined equilibrium (which we know exists by symmetrizing the proof of Theorems 1 and 2).

8. ENDOGENOUS QUANTITY CONSTRAINTS

We saw in Section 7 that the forces of supply and demand could endogenously select unique default penalties that are active in equilibrium, out of an arbitrarily large array of possibilities. Here we give an analogous example for quantity constraints.

EXAMPLE 4: Consider our standard example, but now with 6 households whose endowments are $e^1 = (0, 1, 1)$, $e^2 = (1, 0, 1)$, $e^3 = (1, 1, 0)$, $e^4 = (1, 0, 0)$, $e^5 = (0, 1, 0)$, and $e^6 = (0, 0, 1)$. The utilities of all households are identical: $u(x) = \sum_{s=1}^{3} \log x_s$. Their default penalties are given by

$$\lambda_{sj}^h = \begin{cases} \infty & \text{if } e_s^h = 1, \\ 0 & \text{if } e_s^h = 0, \end{cases} \quad \text{for all } h \in H,\ s \in S,\ j \in J.$$

All assets $j \in J = \{1, 2, \ldots, 100\}$ entail the same promises $R_j = (1, 1, 1)$, but different quantity constraints $Q_j = j/30$.

These penalties lead to full delivery in each agent's good state(s), and to full default, without any penalty, in each agent's bad state(s). If a household buys and sells equal quantities of an asset j, with delivery rates $K_j = (\kappa_j, \kappa_j, \kappa_j)$, he in effect obtains insurance. By giving up (on net) a dollar in his good state, he obtains $\kappa_j/(1 - \kappa_j)$ dollars in his bad state. Default, with the proper penalties, can thus encompass insurance.

Example 4 satisfies the conditions of Theorem 1, so refined equilibria exist. One equilibrium involves $\theta_{12}^h = \varphi_{12}^h = Q_{12} = 12/30 = 2/5$ for all $h \in H$, $\theta_{30}^h = \varphi_{30}^h = 18/30 < Q_{30}$ for all $h \in \{4, 5, 6\}$. Prices and delivery rates are given by $\kappa_{12} = \pi_{12} = 1/2$, and $\kappa_j = \pi_j = 1/3$ for all other j. In effect all households take out primary insurance $j = 12$ up to its quantity limit $Q_{12} = 2/5$, at the rate 1/2, reflecting equal proportions of reliable and unreliable in the pool. The unreliable households, desperate for more insurance, take an additional

secondary policy $j = 30$, but at a much worse rate of $1/3$, since they alone constitute the pool $j = 30$. Our refinement leads to $\kappa_j = 1/3$ for all inactive assets $j \notin \{12, 30\}$, as can easily be verified using our on-the-verge condition. At this equilibrium both reliable and unreliable households feel constrained by the primary limit $Q_{12} = 2/5$, and reliable households are just indifferent to taking out the first dollar of secondary insurance. We therefore call it the pivotal equilibrium.

Notice that out of a whole menu of potential quantity signals, the market chooses just two at which there is active trade.

Primary and secondary insurance are well-known features of insurance markets. We pursue the details in a sequel paper (Dubey and Geanakoplos (2003)). Let us mention, however, that there are multiple equilibria, in contrast to the previous examples. Any quantity limit $0 \leq Q_{j^*} \leq 2/3$ can serve as the primary market maximum. In each of these equilibria, all agents take out primary insurance up to its maximum limit, giving $\pi_{j^*} = \kappa_{j^*} = 1/2$, while $\pi_j = K_j = 1/3$ for all other $j \neq j^*$. (Only in the maximal equilibrium with $Q_{j^*} = 2/3$ are reliable agents taking out all the primary insurance they want at the going rate of $1/2$.) If the primary quantity limit satisfies $2/3 \geq Q_{j^*} > 2/5$ then, as in the pivotal equilibrium, only unreliable agents will take out further secondary insurance. If $0 \leq Q_{j^*} < 2/5$, then all households will take out further secondary insurance, at a rate $1/3 < \kappa < 1/2$ (since unreliable agents take out more secondary insurance than the reliable). The equilibrium with primary limit $Q_{j^*} = 2/3$ Pareto dominates the equilibria with primary limits $2/5 \leq Q_{j^*} < 2/3$. In the aforementioned sequel paper we introduce a further refinement, capturing the hierarchical nature of insurance contracts, and find that only the pivotal equilibrium survives, along with the pure pooling equilibrium in which all agents join in the same secondary pool (the primary limit is $Q_{j^*} = 0$).

In their famous paper on insurance, Rothschild and Stiglitz (1976) imposed an exclusivity assumption, that agents can take out only one policy (sell one asset), and they found that equilibrium might not exist. Exclusivity destroys the convexity of the budget set, so our existence theorem does not directly apply. But in another companion paper, Dubey and Geanakoplos (2002), we showed that equilibrium in fact does always exist, and is unique. Indeed it is precisely the separating equilibrium of Rothschild and Stiglitz! (In our numerical example, reliable agents sell and buy $\varphi_9^h = Q_9 = 9/30$ units of asset $i = 9$, at price $\pi_i = 2/3 = \kappa_i$. Unreliable agents sell and buy $\varphi_{30}^h = Q_{30} = 30/30$ units of asset $j = 30$, at price $\pi_j = \kappa_j = 1/3$. The pricing of the inactive assets implied by our refinement is strictly monotonic in Q_j, over a large interval, and is described in detail in Dubey and Geanakoplos (2002).) The universal existence and uniqueness of the exclusivity insurance equilibrium is made possible by our perfectly competitive framework.[26]

[26]Cho and Kreps (1987) also showed in their game-theoretic model that the separating equilibrium must always exist. As we have said, their model has much greater complexity.

9. $\mathcal{A}$-EFFICIENCY AND THE MARKET CHOICES $\mathcal{A}^*$

In Theorem 3 and also in Example 3 we found a match between what assets the market *ought* to choose, and what assets it *does* choose. This is not always (or even usually) the case. Example 4 displays multiple refined equilibria that can be Pareto compared; in the bad equilibria, the wrong assets are used. Why is a socially useful asset j sometimes not used in equilibrium?

The major problem is that in refined equilibrium, the delivery rates K_{sj} for an untraded asset j depend on the delivery rates of the agents most eager to sell it (i.e., the agents on-the-verge of selling it). If the more reliable agents have higher disutilities of selling j, their higher delivery rates will not be reflected in K_j. The asset may make useful promises, but still not be used by the market, because its deliveries are debased by adverse selection.

Very often there will be only one seller type that is most eager to sell. If there are several, then again the market cannot screen out unreliable sellers. We call an asset *unambiguously beneficial* if it enables a Pareto improvement no matter what the selection of sales from among the agents most eager to sell it. We shall prove that assets that are not unambiguously beneficial can always be left inactive at some on-the-verge equilibrium. Call an asset *super beneficial* if it remains unambiguously beneficial even when its penalty rates are slightly reduced. We shall prove that super beneficial assets can never be left inactive in any refined equilibrium.

More precisely, let $\widetilde{\mathcal{A}} \subset \mathcal{A}$, and let $\tilde{E} = (\tilde{p}, \tilde{\pi}, \tilde{K}, (\tilde{x}^h, \tilde{\theta}^h, \tilde{\varphi}^h, \tilde{D}^h)_{h\in H})$ be a refined equilibrium of the economy with assets $\widetilde{\mathcal{A}}$. Let $j \in \mathcal{A} \setminus \widetilde{\mathcal{A}}$ and let σ_j denote the set of agents on the verge of selling asset j, i.e., the agents with the lowest marginal disutilities of selling j, given prices in the $\widetilde{\mathcal{A}}$-equilibrium. (See the section On-the-Verge Equilibria in the Appendix.) We say that asset j is *unambiguously beneficial* at $\tilde{E}$ if, $\forall$ sufficiently small $(\varphi_j^h)_{h\in H}$ with $\varphi_j^h = 0$ for all $h \notin \sigma_j$, $\exists$ asset purchases and incentive-compatible deliveries $(\theta_j^h, D_j^h)_{h\in H}$ and an allocation $(y^h)_{h\in H}$ satisfying:

(i) $u^h(y^h) \geq u^h(\tilde{x}^h)$ for all $h \in H$, with at least one strict inequality;

(ii) $\sum_{h\in H} y_0^h = \sum_{h\in H} e_0^h$;

(iii) $\sum_{h\in H} \theta_j^h = \sum_{h\in H} \varphi_j^h$;

(iv) $y_s^h = \tilde{x}_s^h + \dfrac{\theta_j^h}{\sum_{i\in H} \theta_j^i} \sum_{i\in H} D_{sj}^i - D_{sj}^h$ for all $s \in S$;

(v) $(D_{sj}^h)_{s\in S} \in \underset{(D_{sj})_{s\in S}}{\arg\max} \left\{ u^h\left(y_0^h, \left(\tilde{x}_s^h + \dfrac{\theta_j^h}{\sum_{i\in H} \theta_j^i} \sum_{i\in H} D_{sj}^i - D_{sj}\right)_{s\in S}\right) - \sum_{s\in S} \lambda_{sj}^h \dfrac{\tilde{p}_s \cdot (D_{sj} - \varphi_k^h R_{sj})}{p_s \cdot v_s} \right\}$ for all $h \in H$.

Condition (v) says that h's deliveries are incentive-compatible for him; and (iv) says that consumption is altered in period 1 only via deliveries on asset j.

If the asset j would still be unambiguously beneficial at $\tilde{E}$ even after replacing all its penalties λ_{sj}^h by $(1-\delta)\lambda_{sj}^h$, for some $\delta > 0$, then we call it *super beneficial*.

We call $\tilde{E}$ *$\mathcal{A}$-efficient* if no asset $k \in \mathcal{A} \setminus \tilde{\mathcal{A}}$ is unambiguously beneficial at $\tilde{E}$.[27] We call $\tilde{E}$ *strongly $\mathcal{A}$-inefficient* if some asset $k \in \mathcal{A} \setminus \tilde{\mathcal{A}}$ is super beneficial at $\tilde{E}$.

THEOREM 4: *Let $\tilde{\mathcal{A}} \subset \mathcal{A}$, and let $\tilde{E} = (\tilde{p}, \tilde{\pi}, \tilde{K}, (\tilde{x}^h, \tilde{\theta}^h, \tilde{\varphi}^h, \tilde{D}^h)_{h \in H})$ be a refined equilibrium of the economy with assets $\tilde{\mathcal{A}}$ that is $\mathcal{A}$-efficient. If utilities are smooth,[28] then $\tilde{E}$ can be extended to an equilibrium satisfying the on-the-verge conditions for the economy with assets $\mathcal{A}$, with the same active assets as before, $\mathcal{A}^* = \tilde{\mathcal{A}}^*$. If the $\mathcal{A}$ economy has a unique on-the-verge equilibrium, then the extended equilibrium is also a refined equilibrium (and therefore is the unique refined equilibrium).*

Conversely, if $\tilde{E}$ is strongly $\mathcal{A}$-inefficient, and if utilities u^h are smooth and additively separable between time periods 0 and 1, then there is no refined equilibrium of $\mathcal{A}$ that extends $\tilde{E}$, leaving all assets in $\mathcal{A} \setminus \tilde{\mathcal{A}}$ untraded.

The on-the-verge conditions and the proof of Theorem 4 are in the Appendix.

Theorem 4 helps explain Examples 3 and 4. In Example 3, by symmetry, every agent is always on-the-verge of selling every untraded asset, and since they all deliver 2/3 of the time, there is no adverse selection, and refined equilibrium chooses the right asset. Every equilibrium that uses an asset $\lambda \neq \lambda^*$ must be strongly $\mathcal{A}$-inefficient and thus will be upset by the λ^* asset. On the other hand, once the λ^* asset is used, no other asset is unambiguously beneficial.

In Example 4, by contrast, the unreliable agents are always strictly more eager to sell than the reliable agents. Thus K_{sj} is debased by the adverse selection to 1/3 for every untraded asset, no matter what the equilibrium. Untraded assets have little power to upset an equilibrium, and so we get multiple, low-welfare equilibria.

If Example 4 is modified by an exclusivity restriction, limiting every agent to selling at most one asset, then the identity of the most eager seller depends on the equilibrium $\tilde{E}$. If reliable and unreliable agents are pooled together at $\tilde{E}$, then the reliable agents are the most eager to trade a new asset with the

[27] Our notion of $\mathcal{A}$-efficient is different from the constrained efficiency defined in Geanakoplos and Polemarchakis (1986).

[28] Smoothness (in the sense of Debreu, which includes differentiability and the hypothesis that each agent will be worse off than at his initial endowment if he consumes 0 of any good) guarantees interior consumption, so that all derivatives are well defined.

right quantity limit, upsetting any pooling equilibrium. If reliable and unreliable agents are separated at $\tilde{E}$, then unreliable agents are more eager to sell most new assets, and so the separating equilibrium prevails, explaining why the separating equilibrium is the unique refined equilibrium in the Rothschild–Stiglitz model. This is demonstrated in detail in our companion paper Dubey and Geanakoplos (2002).

Center for Game Theory in Economics, SUNY at Stony Brook, Stony Brook, NY 11794-4384, U.S.A.; pradeepkdubey@yahoo.com; http://ms.cc.sunysb.edu/~pdubey/,

Cowles Foundation, Yale University, 30 Hillhouse Avenue, New Haven, CT 06511-8281, U.S.A.; john.geanakoplos@yale.edu; http://cowles.econ.yale.edu/faculty/geanakoplos.htm,

and

Cowles Foundation, Yale University, 30 Hillhouse Avenue, New Haven, CT 06511-8281, U.S.A.; martin.shubik@yale.edu; http://cowles.econ.yale.edu/faculty/shubik.htm.

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APPENDIX

PROOF OF THEOREM 1: Suppose first that penalties are finite, $\lambda \in \mathbb{R}_+^{HSJ}$. Fix a tremble $\varepsilon = (\varepsilon_j)_{j\in J} \gg 0$. We shall prove the existence of an ε-boosting equilibrium for small enough ε. For any small lower bound $b > 0$, define

$$\Delta_b = \left\{ (p,\pi) \in \mathbb{R}_+^{S^*\times L} \times \mathbb{R}_+^J : \sum_{\ell=1}^{L} p_{s\ell} = 1 \ \forall s \in S^*, \right.$$

$$\left. b \le p_{s\ell} \ \forall s\ell \in S^* \times L, \text{ and } 0 \le \pi_j \le \frac{1}{b} \ \forall j \in J \right\}.$$

Choose M large enough to ensure that: $\|x\|_\infty > M \Rightarrow u^h(x) > u^h(2\sum_{h'\in H} e^{h'})$ for all $h \in H$. (By assumption, $u^h(x) \to \infty$ as $\|x\| \to \infty$, so such an M exists.) Now define, for each $h \in H$,

$$\square^h = \left\{ (x,\theta,\varphi,D) \in \mathbb{R}_+^{S^*\times L} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{SLJ} : \right.$$

$$\left. \|x\|_\infty \le M, \theta_j \le 2\sum_{h'\in H} Q_j^{h'}, \varphi_j^h \le Q_j^h, \text{ and } \|D\|_\infty \le \|Q\|_\infty \|R\|_\infty \right\}.$$

Let $\square^H \equiv \mathsf{X}_{h\in H}\square^h$.

Denote $\eta \equiv (p,\pi,K,(x^h,\theta^h,\varphi^h,D^h)_{h\in H}) \in \Delta_b \times [0,1]^{S\times J} \times \square^H \equiv \Omega_b$.

Consider the map $\bar{K}_b : \Omega_b \to [0,1]^{S\times J}$ defined by

$$\bar{K}_{bsj}(\eta) = \begin{cases} \min\left\{ \dfrac{p_s \cdot R_{sj}\varepsilon_j + \sum_{h\in H} p_s \cdot D_{sj}^h}{p_s \cdot R_{sj}\varepsilon_j + \sum_{h\in H} p_s \cdot R_{sj}\varphi_j^h}, 1 \right\} & \text{if } R_{sj} \neq 0, \\ 1 & \text{if } R_{sj} = 0, \end{cases}$$

for each $s \in S$, $j \in J$. (Note that $R_{sj} \neq 0$ implies $p_s \cdot R_{sj} > 0$, since $R_s \geq 0$ and $p_s \gg 0$.) Clearly $\bar{K}_{hsj}$ is a continuous function.

Next, consider the correspondence $\psi_b^0 : \Omega_b \Rightarrow \Delta_b$ defined by

$$\psi_b^0(\eta) = \arg\max_{(p,\pi)\in\Delta_h} \left\{ p_0 \cdot \sum_{h\in H}(x_0^h - e_0^h) + \pi \cdot \sum_{h\in H}(\theta^h - \varphi^h) \right.$$
$$\left. + \sum_{s\in S} p_s \cdot \left[\sum_{h\in H}(x_s^h - e_s^h) - \sum_{j\in J}(1 - \bar{K}_{bsj}(\eta)) R_{sj}\varepsilon_j \right] \right\}.$$

Clearly this map is nonempty and convex-valued, and upper semi-continuous.

Finally for each $h \in H$, define the correspondence $\psi_b^h : \Omega_b \Rightarrow \square^h$ by

$$\psi_b^h(\eta) = \arg\max_{x,\theta,\varphi,D} \{ w^h(x, \theta, \varphi, D, p) : (x, \theta, \varphi, D) \in B^h(p, \pi, K) \cap \square^h \}.$$

Notice that ψ_b^h is nonempty valued and convex-valued, thanks to the continuity and concavity of w^h, for all $h \in H$. To check that $B^h(p, \pi, K) \cap \square^h$ is lower semi-continuous, let $p^n, \pi^n, K^n \xrightarrow{n} \bar{p}$, $\bar{\pi}$, $\bar{K}$ with $\bar{p} \gg 0$. Let $(\bar{x}, \bar{\theta}, \bar{\varphi}, \bar{D}) \in B^h(\bar{p}, \bar{\pi}, \bar{K})$. Fix $0 < \alpha < 1$. Then $(\alpha\bar{x}, \alpha\bar{\theta}, \alpha\bar{\varphi}, \alpha\bar{D}) \in B^h(p^n, \pi^n, K^n) \cap \square^h$ for sufficiently large n by the scaling property of the budget set, because $\bar{p}_s \cdot e_s^h > 0 \ \forall s \in S^*$. Since α was arbitrary, this shows that $B^h(p, \pi, K) \cap \square^h$ is lower semi-continuous in (p, π, K) whenever $p \gg 0$. Since $B^h(p, \pi, K) \cap \square^h$ is clearly upper semi-continuous, ψ_b^h is upper semi-continuous by the maximum principle.

Let $\psi_b : \Omega_b \Rightarrow \Omega_b$ be the correspondence defined by

$$\psi_b(\eta) = \psi_b^0(\eta) \times \{\bar{K}_b(\eta)\} \times \bigtimes_{h\in H} \psi_b^h(\eta).$$

By Kakutani's Theorem ψ_b has a fixed point $\eta^b \equiv (p^b, \pi^b, K^b, (x^h(b), \theta^h(b), \varphi^h(b), D^h(b))_{h\in H})$. To avoid notational clutter, we suppress the b.

Note that in state 0, $p_0 \cdot (\sum_h (x_0^h - e_0^h)) + \pi \cdot (\sum_h (\theta^h - \varphi^h)) = 0$ (since, given the monotonicity of each u^h, this equality holds for each h individually in his budget set). It follows that the "price player" could not make the value of excess demand (across commodities and assets) positive in period 0. Suppose for some $j \in J$, $\sum_{h\in H}(\theta_j^h - \varphi_j^h) > 0$. By taking $\tilde{\pi}_j = 1/b$ and $\tilde{\pi}_i = 0$ for $i \neq j$, it follows that

$$\frac{1}{b}\sum_h (\theta_j^h - \varphi_j^h) + \sum_{\ell\in L} \tilde{p}_{0\ell} \sum_{h\in H}(x_{0\ell}^h - e_{0\ell}^h) \leq 0,$$

for all $\tilde{p} \in \mathbb{P}_b \equiv \{q \in \mathbb{R}_+^L : q_\ell \geq b \ \forall \ell \in L, \ \sum_{\ell=1}^L q_\ell = 1\}$. Hence,

$$\sum_h (\theta_j^h - \varphi_j^h) \leq bL\|e_0\|_\infty.$$

Similarly, if $\sum_{h\in H}(x_{0\ell}^h - e_{0\ell}^h) > 0$ for some ℓ, then by taking all $\tilde{\pi}_j = 0$ and $\tilde{p}_{0\ell} = 1 - (L-1)b$ and $\tilde{p}_{0k} = b$ for all $k \neq \ell$, we get

$$\sum_{h\in H}(x_{0\ell}^h - e_{0\ell}^h) \leq \frac{(L-1)b\|e_0\|_\infty}{1 - (L-1)b}.$$

From the fact that $\bar{K}_b$ fixed K, and from the fact that agents have optimized so that $p_s \cdot D_{sj}^h \leq p_s \cdot R_{sj}\varphi_j^h$, whenever $p_s \cdot R_{sj} \neq 0$ we get

$$K_{sj} = \frac{p_s \cdot R_{sj}\varepsilon_j + \sum_{h\in H} p_s \cdot D_{sj}^h}{p_s \cdot R_{sj}\varepsilon_j + \sum_{h\in H} p_s \cdot R_{sj}\varphi_j^h} \leq 1.$$

Hence,

$$\sum_{h\in H} p_s \cdot D_{sj}^h = \sum_{h\in H} K_{sj} p_s \cdot R_{sj}\varphi_j^h - (1-K_{sj}) p_s \cdot R_{sj}\varepsilon_j.$$

From optimization of monotonic utilities in the budget set, we get

$$p_s \cdot (x_s^h - e_s^h) = \sum_{j\in J} K_{sj} p_s \cdot R_{sj}\theta_j^h - \sum_{j\in J} p_s \cdot D_{sj}^h.$$

Adding over agents $h \in H$, and substituting the above expression for $\sum_{h\in H} p_s \cdot D_{sj}^h$, we get

$$\begin{aligned} p_s \cdot \sum_{h\in H}(x_s^h - e_s^h) &= \sum_{j\in J}(1-K_{sj}) p_s \cdot R_{sj}\varepsilon_j + \sum_{j\in J}\sum_{h\in H} K_{sj} p_s \cdot R_{sj}(\theta_j^h - \varphi_j^h) \\ &\le \sum_{j\in J}(1-K_{sj}) p_s \cdot R_{sj}\varepsilon_j + J\|R\|_\infty bL\|e_0\|_\infty. \end{aligned}$$

Suppose $\sum_{h\in H}(x_{s\ell}^h - e_{s\ell}^h) - \sum_{j\in J}(1-K_{sj})R_{s\ell j}\varepsilon_j > 0$ for some $s \in S$. Since we are at a fixed point, the price player cannot increase the value of excess demand in state s by taking $\tilde{p}_{s\ell} = 1-(L-1)b$, and $\tilde{p}_{sk} = b$ for all $k \neq \ell$. Hence,

$$\begin{aligned} &\sum_{h\in H}(x_{s\ell}^h - e_{s\ell}^h) - \sum_{j\in J}(1-K_{sj})R_{s\ell j}\varepsilon_j \\ &\le \frac{1}{1-(L-1)b}\left\{(L-1)b\left[\|e_0\|_\infty + \|R\|_\infty \sum_{j\in J}\varepsilon_j\right] + J\|R\|_\infty bL\|e_0\|_\infty\right\}. \end{aligned}$$

Thus aggregate excess demand (including the external agent) goes to zero as $b \to 0$. Furthermore, $\sum_{h\in H} x^h \le 2\sum_{h\in H} e^h$ as $b \to 0$, provided the fixed $(\varepsilon_j)_{j\in J}$ were chosen small to begin with. If $p_{s\ell}/p_{sk}$ became unbounded as $b \to 0$, some agent with $e_{s\ell}^h > 0$ could have consumed M units of commodity sk, obtaining more utility than $u^h(2\sum_{h'\in H} e^{h'})$, for all small b; but since $x^h \le 2\sum_{h'\in H} e^{h'}$ for small enough b, this contradicts that h has optimized. We next argue that π_j must remain bounded as $b \to 0$. If $Q_j^h = 0\ \forall h$, then replace π_j with 1. Otherwise, if $\pi_j \to \infty$, any agent h with $Q_j^h > 0$ could replace his entire action by selling a tiny amount Δ of j, buying M $(\le \Delta\pi_j/L)$ units of each period 0 good. Since $e_s^h \neq 0$ for all s, and commodity price ratios are bounded in each state, agent h can do this without incurring any default. But this gives him utility that exceeds $u^h(2\sum_{h'} e^{h'})$, which is more than he can possibly be getting at the fixed point, a contradiction. Thus all asset prices are bounded.

Since all choices and all macro variables are uniformly bounded for small b, we can pass to convergent subsequences, obtaining $\bar{E} \equiv \langle \bar{p}, \bar{\pi}, \bar{K}, (\bar{x}^h, \bar{\theta}^h, \bar{\varphi}^h, \bar{D}^h)_{h\in H}\rangle$ as a limit point. Taking the limit of all inequalities derived above, we conclude that aggregate excess demand for commodities and assets is less than or equal to zero in E. Since price ratios $\bar{p}_{s\ell}/\bar{p}_{sk}$ are bounded in each state $s \in S^*$, the limiting $\bar{p} \gg 0$, and all agents have positive income in every state in $\bar{E}$. The bounds in $\square^h$ imposed on (x, θ, D) are not binding in $\bar{E}$. Hence, by concavity of w^h, individuals are optimizing in $\bar{E}$ on their actual budget sets.

Note finally that if all commodity prices are positive, there cannot be excess supply in any commodity in $\bar{E}$, otherwise the price player would be making negative profits. For the same reason there cannot be excess supply of any asset j in $\bar{E}$, unless $\pi_j = 0$. But then no agent would sell j unless $\lambda_{sj}^h R_{sj} = 0$ for all $s \in S$. Without loss of generality we may in this case take $\theta_j^h = \varphi_j^h = 0$ for all h.

Thus we have shown that $\bar{E}$ is an ε-boosting equilibrium. Letting $\varepsilon \to 0$ and taking limits we obtain a refined equilibrium. This proves the theorem for finite penalties λ.

If some penalties are infinite, we take limits of equilibria with increasing penalties. Since all actions must stay bounded along the sequence (because $Q_j^h < \infty$), any cluster point of these equilibria will serve as the desired refined equilibrium. *Q.E.D.*

PROOF OF THEOREM 2: Theorem 2 specializes the conditions of Theorem 1. Hence we have a $GE(R, \lambda, Q)$ equilibrium for all finite Q. Consider a sequence of equilibria, $\eta(Q) = (p(Q), \pi(Q), K(Q), (x^h(Q), \theta^h(Q), \varphi^h(Q), D^h(Q))_{h\in H})$, where $Q_j^h = Q \in \mathbb{N}$, for all $h \in H$, $j \in J$.

If there is a single Q with $\varphi_j^h(Q) < Q$, for all $h \in H$, $j \in J$, then by the concavity of each w^h, $\eta(Q)$ is a $GE(R, \lambda)$.

Passing to a convergent subsequence if necessary, we may suppose that for all $h \in H$ and $j \in J$,

$$\frac{\theta_j^h(Q)}{Q} \to \bar{\theta}_j^h, \quad \frac{\varphi_j^h(Q)}{Q} \to \bar{\varphi}_j^h.$$

Moreover, we might as well assume that for at least one j and some h and h', $\bar{\theta}_j^h \neq 0$ and $\bar{\varphi}_j^{h'} = 1$.

For notational convenience, we shall write R_{sj} and D_{sj}, instead of the more accurate R_{sLj} and D_{sLj}, and we shall suppose that real default in each state $s \in S$ is measured in terms of the commodity bundle $v_s = \mathbb{1}_L$, which is one in the Lth coordinate, and zero elsewhere. Since all assets are exclusively delivering in the Lth good, no harm results from these simplifications. Finally, w.l.o.g. take $p_{sL} = 1$ for all $s \in S$.

Observe that for any $h \in H$, $s \in S$, $j \in J$, the level of default

$$d_{sj}^h(Q) \equiv [R_{sj}\varphi_j^h(Q) - D_{sj}^h(Q)]^+ \leq \frac{1}{\lambda_{sj}^h}[u^h(e) - u^h(e^h)],$$

whenever $\lambda_{sj}^h > 0$, for otherwise agent h would have done better not trading at all. (At any $GE(R, \lambda, Q)$, $x^h \leq \sum_{h'} e^{h'} \equiv e$.) Hence if $\varphi_j^h(Q) \to \infty$,

$$\frac{[R_{sj}\varphi_j^h(Q) - D_{sj}^h(Q)]}{\varphi_j^h(Q)} = \frac{[R_{sj}\varphi_j^h(Q) - D_{sj}^h(Q)]^+}{\varphi_j^h(Q)} = \frac{d_{sj}^h(Q)}{\varphi_j^h(Q)} \to 0.$$

It follows that $K_{sj}(Q) \to 1$ for all $s \in S$ with $R_{sj} > 0$, provided that $\sum_{h\in H} \varphi_j^h(Q) = \sum_{h\in H} \theta_j^h(Q) \to \infty$.

Furthermore, since relative prices $p_{s\ell}(Q)/p_{sk}(Q)$ stay bounded,

$$\sum_{j\in J} K_{sj}(Q)R_{sj}\theta_j^h(Q) - \sum_{j\in J} D_{sj}^h(Q)$$

must stay bounded. Otherwise agent h would eventually be consuming a negative quantity in state s, or a quantity exceeding the aggregate endowment e_s, contradicting commodity market clearing.

Putting these last statements together, we must have that

$$\lim_{Q\to\infty} \frac{\sum_{j\in J} K_{sj}(Q)R_{sj}\theta_j^h(Q) - \sum_{j\in J} D_{sj}^h(Q)}{Q} = R_s(\bar{\theta}^h - \bar{\varphi}^h) = 0$$

for all $h \in H$, $s \in S$.

Since $\bar{\theta}^h$ promises exactly the same value of deliveries as the portfolio $\bar{\varphi}^h$, no matter what the relative prices p_s, we know that $\theta^h(Q) - \bar{\theta}^h$ reduces the promised receipts by exactly the same amount as $\varphi^h(Q) - \bar{\varphi}^h$ reduces promised deliveries, for any Q. (This would not be true if assets delivered multiple goods.)

Consider any h with $\bar{\varphi}^h \neq 0$, and hence $\bar{\theta}^h \neq 0$. For sufficiently large $Q \geq 1$,

$$\hat{\theta}^h = \theta^h(Q) - \bar{\theta}^h \geq 0,$$

$$\hat{\varphi}^h = \varphi^h(Q) - \bar{\varphi}^h \geq 0.$$

At any large Q, the agent could feasibly have chosen $\hat{\theta}^h$, $\hat{\varphi}^h$ and deliveries

$$\hat{D}^h_{sj} = D_{sj}(Q) - R_{sj}\bar{\varphi}^h_j \geq 0 \quad \text{for all} \quad j \in J.$$

With these choices he would pay exactly the same penalty as in the equilibrium $\eta(Q)$. He would receive exactly the same consumption at time 1 if $K_{sj}(Q) = 1$ for all j with $\bar{\theta}^h_j > 0$ (for then his receipts and deliveries both fall by $R_s \cdot \tilde{\theta}^h = R_s \cdot \bar{\varphi}^h$) and strictly more consumption otherwise (for then his receipts fall by $\sum_{j\in J} K_{sj}(Q)R_{sj}\bar{\theta}^h_j < \sum_{j\in J} R_{sj}\bar{\theta}^h_j \equiv R_s \cdot \bar{\theta}^h = R_s \cdot \bar{\varphi}^h$).

In order for him not to prefer this deviation, we must therefore have

$$\pi(Q)[\bar{\theta}^h - \bar{\varphi}^h] \leq 0 \quad \text{for all} \quad h \in H.$$

But since $\bar{\theta}^h$ and $\bar{\varphi}^h$ are limits of $GE(R, \lambda, Q)$ equilibrium portfolios,

$$\sum_{h\in H} \bar{\theta}^h = \sum_{h\in H} \bar{\varphi}^h;$$

hence we must have

$$\pi(Q)[\bar{\theta}^h - \bar{\varphi}^h] = 0 \quad \text{for all} \quad h \in H.$$

It now follows that household h would still prefer this deviation unless $\forall j \in J, \forall s \in S$,

$$[R_{sj} > 0, \text{ and } \bar{\theta}^h_j > 0 \text{ for any } h \in H] \Rightarrow [K_{sj}(Q) = 1].$$

Note finally that if $\bar{\varphi}^h_j > 0$, there must be some agent i with $\bar{\theta}^i_j > 0$, hence $K_{sj}(Q) = 1$ for all $s \in S$ with $R_{sj} > 0$ and either $\bar{\theta}^h_j > 0$ or $\bar{\varphi}^h_j > 0$.

Replacing $(p(Q), \pi(Q), K(Q), (x^h(Q), \theta^h(Q), \varphi^h(Q), D^h(Q))_{h\in H})$ with $(p(Q), \pi(Q), K(Q), (x^h(Q), \hat{\theta}^h, \hat{\varphi}^h, \hat{D}^h)_{h\in H})$, we get another $GE(R, \lambda, Q)$ with $\hat{\varphi}^h_j(Q) < Q$ for all h and j. (Notice that we are reducing sales and purchases only for assets with $K_{sj} = 1$, which therefore leaves the K unchanged.) *Q.E.D.*

On-the-Verge Equilibria

Solving equilibrium conditions (1)–(4) generally gives too many equilibria E and checking which of them are refined seems at first glance to be a daunting task. It requires constructing an infinite sequence of equilibria $E(\varepsilon) \to E$, as $\varepsilon \to 0$, satisfying (1*)–(4*).

Under certain circumstances, the task becomes dramatically simple. Suppose throughout this section that utilities u^h are differentiable, and that at a (possibly unrefined) equilibrium $(p, \pi, K, (x^h, \theta^h, \varphi^h, D^h)_{h\in H})$ we have[29] $p_s \cdot x^h_s > 0$ for all h and all s. We shall show that there are algebraic conditions on E that are easy to check and necessary for E to be refined. If it turns out that there is a unique E satisfying the algebraic conditions, then from our existence theorem we can conclude at once that E is a refined equilibrium without bothering with the sequence $E(\varepsilon)$. (In fact E will then be the unique refined equilibrium.)

We can define the marginal utility of money in state s to each agent h by $\mu^h_s = [\partial u^h(x^h)/\partial x_{s\ell}]/p_{s\ell}$ for any ℓ with $x^h_{s\ell} > 0$. The marginal utility to h of purchasing any asset j is then

$$MU^h_j = \sum_{s=1}^{S} \mu^h_s K_{sj} p_s \cdot R_{sj}$$

and the marginal disutility of selling asset j is

$$MDU^h_j = \sum_{s=1}^{S} p_{sj} \cdot R_{sj} \min\left\{\frac{\lambda^h_{sj}}{p_s \cdot v_s}, \mu^h_s\right\}.$$

[29]This will be implied by Debreu's smoothness condition of footnote 28.

An agent is said to be on the verge of buying (selling) asset j if he is not buying (selling) it, but would do so if the price π_j were ever so slightly lowered (raised):

$$\text{verge of buying:} \quad \pi_j = MU_j^h/\mu_0^h,$$

$$\text{verge of selling:} \quad \pi_j = MDU_j^h/\mu_0^h.$$

If, in the refined equilibrium, $p_s \cdot R_{sj} > 0$ and $K_{sj} < 1$, then we know that in the perturbation $E(\varepsilon)$, some agent h was actually selling j and not fully delivering in state s (otherwise $K_{sj} = 1$ on account of the external agent). It also follows that some agent was buying j (since markets clear in the perturbation and the external agent buys and sells the same amount of asset j). Passing to the limit, we conclude that at a refined equilibrium

$$\pi_j = \max_h\{MU_j^h/\mu_0^h\} = \min_h\{MDU_j^h/\mu_0^h\}$$

for all untraded assets j for which $K_{sj}p_s \cdot R_{sj} < p_s \cdot R_{sj}$ for some $s \geq 1$. We call this the *on-the-verge of trading* condition.

If $\mu_s^h > \lambda_{sj}^h/p_s \cdot v_s$ for some untraded asset j, we say that agent h is *strictly conscientious* for asset j in state s. In any perturbation of the equilibrium, such an agent will fully deliver on asset j in state s. Thus if it turns out that all agents h on-the-verge of selling an untraded asset j are strictly conscientious, then K_{sj} must be 1. Furthermore, if every agent is strictly conscientious in at least m states on asset j, then $K_{sj} \geq m/S$ for at least one state s. We will call these requirements "*on-the-verge boosting*."

Any equilibrium satisfying the on-the-verge of trading and boosting conditions is called an *on-the-verge equilibrium*. With smooth utilities, equilibrium conditions (1)–(4) can also be reduced to equations. Hence every on-the-verge equilibrium is the solution of one of a finite collection of systems of finite equations. As we have seen, every refined equilibrium is an on-the-verge equilibrium.

The on-the-verge of trading condition appears not to leave any gap between the marginal utility of buying and selling an asset j with default. If this were truly so, then one would generically find that there was positive trade in all assets. But as we have emphasized, and as we saw in our examples, equilibrium often involves inactive assets. The explanation of the paradox is that there is a gap, but it is filled by the external agent. In $E(\varepsilon)$ delivery rates $K(\varepsilon)$ are boosted above delivery rates $\bar{K}(\varepsilon)$ of the real agents. If the marginal utility of buying were computed using $\bar{K}(\varepsilon)$ instead of $K(\varepsilon)$, the gap would be visible.

PROOF OF THEOREM 4: We use the notation of Section 10.3. For each asset $j \in \mathcal{A} \setminus \tilde{\mathcal{A}}$, define $\pi_j = \min_{h\in H} MDU_j^h/\mu_0^j$. Let the set $\sigma_j \subset H$ of traders on-the-verge of selling j consist of those h who achieve the min just defined. For each $h \in \sigma_j$ and each $s \in S$, define $\hat{K}_{sj}^h = 0$ if $\lambda_{sj}^h \leq \tilde{p}_s \cdot v_s\mu_s^h$, and $\hat{K}_{sj}^h = 1$ otherwise.

Suppose $\tilde{E}$ is $\mathcal{A}$-efficient. For each $j \in \mathcal{A} \setminus \tilde{\mathcal{A}}$, consider a sequence $(\varphi_j^h(\varepsilon))_{h\in\sigma_j} \neq 0$ converging to 0 for which a central planner *cannot* find a Pareto improvement. Define $\bar{\varphi}_j^h = \lim_{\varepsilon\to 0}(\varphi_j^h(\varepsilon)/\sum_{i\in\sigma_j}\varphi_j^i(\varepsilon))$, for all $h \in \sigma_j$. Define $\bar{K}_j = \sum_{h\in\sigma_j}\bar{\varphi}_j^h\hat{K}_j^h$. We claim that at the macro variables $(\tilde{p}, (\tilde{\pi}_j)_{j\in\tilde{\mathcal{A}}}, (\pi_j)_{j\in\mathcal{A}\setminus\tilde{\mathcal{A}}}, (\bar{K}_j)_{j\in\tilde{\mathcal{A}}}, (\bar{K}_j)_{j\in\mathcal{A}\setminus\tilde{\mathcal{A}}})$, we cannot have $MU_j^{h^*}/\mu_0^{h^*} > \pi_j$ for any $(h^*, j) \in H \times (\mathcal{A} \setminus \tilde{\mathcal{A}})$. Otherwise the central planner could use asset j to Pareto improve on $\tilde{E}$ via the $(\varphi_j^h(\varepsilon))_{h\in\sigma_j}$. The planner could assign h^* to buy $\sum_{i\in\sigma_j H}\varphi_j^i(\varepsilon)$ units of asset j. Each seller h with $\hat{K}_{sj}^h = 1$ will deliver fully in state s (choosing a bundle D_{sj}^h with $\tilde{p}_s \cdot D_{sj}^h = \varphi_j^h(\varepsilon)\tilde{p}_s \cdot R_{sj}$). We conclude that the value of deliveries $\tilde{p}_s \cdot \sum_{h\in\sigma_j} D_{sj}^h \geq \tilde{p}_s \cdot \sum_{h\in\sigma_j}\varphi_j^h(\varepsilon)R_{sj}\hat{K}_{sj}^h = \sum_{i\in\sigma_j}\varphi_j^i(\varepsilon)\bar{K}_{sj}\tilde{p}_s \cdot R_{sj}$, no matter how the sellers h with $\hat{K}_{sj}^h = 0$ deliver. With smooth utilities consumption is interior, and the buyer h^* is almost indifferent to the exact delivery $\sum_{h\in\sigma_j} D_{sj}^h$ so long as $\tilde{p}_s \cdot \sum_{h\in\sigma_j} D_{sj}^h$ is unaffected. Hence h^* gains at least nearly $\sum_{i\in\sigma_j}\varphi_j^i(\varepsilon)(MU_j^{h^*}) > \sum_{i\in\sigma_j}\varphi_j^i(\varepsilon)(\mu_0^{h^*}\pi_j)$ in utility,

for small ε. Let the central planner also adjust consumption at date 0, taking away goods valued at $\pi_j \sum_{i\in\sigma_j} \varphi_j^i(\varepsilon) + \delta$, for very small $\delta > 0$, from agent h^*, and giving the fraction $\varphi_j^h(\varepsilon)/\sum_{i\in\sigma_j} \varphi_j^i(\varepsilon)$ of these goods to each seller $h \in \sigma_j$. Agent h^* and all agents $h \in \sigma_j$ are strictly better off, for small enough ε and δ, contradicting the hypothesis that $\tilde{E}$ is $\mathcal{A}$-efficient and confirming that $MU_j^h/\mu_0^h \le \pi_j$ for all $(h, j) \in H \times (\mathcal{A} \setminus \widetilde{\mathcal{A}})$. If $\max_{h\in H}\{MU_j^h/\mu_0^h\} = \pi_j$, set $K_j = \bar{K}_j$. Otherwise let $K_\alpha = (1-\alpha)\bar{K}_j + \alpha(1, 1, \ldots, 1)$. Take the smallest $\alpha \in [0, 1]$ for which $\max_{h\in H}\{MU_j^h/\mu_0\} = \pi_j$, if such an α exists. Otherwise take $\alpha = 1$. Set $K_j = K_\alpha$. Replacing $\bar{K}_j$ with K_j, it is evident that the on-the-verge conditions are satisfied.

Conversely, suppose $\tilde{E}$ is strongly $\mathcal{A}$-inefficient and let $j \in \mathcal{A} \setminus \widetilde{\mathcal{A}}$ be super beneficial at $\tilde{E}$. Let E be a refined equilibrium of the $\mathcal{A}$-economy, extending $\tilde{E}$, at which no asset in $\mathcal{A} \setminus \widetilde{\mathcal{A}}$ is traded. Let (π_j, K_j) be defined by E. Clearly $K_{sj} < 1$ for at least one s. Otherwise, even with full deliveries, the maximum marginal utility of buying j is less than or equal to π_j, which is less than or equal to the minimum marginal disutility of selling j. By separability, that would remain so after rearranging goods at time 0, contradicting the hypothesis that j is super beneficial. Consider the perturbations $E(\varepsilon)$ refining E, and defining sales $\varphi_j^h(\varepsilon)$, and deliveries $D_{sj}^h(\varepsilon)$. ($\sum_{h\in\sigma_j} \varphi_j^h(\varepsilon) \neq 0$ for small ε, for then $K_{sj} = 1$ for all $s \in S$.) Delivery rates in $E(\varepsilon)$ are always at least $\hat{K}_{sj}(\varepsilon) \equiv (\sum_{h\in\sigma_j} \varphi_j^h(\varepsilon)\hat{K}_{sj}^h)/\sum_{h\in\sigma_j} \varphi_j^h(\varepsilon)$, for small ε, since strictly conscientious agents will fully deliver. Hence $K_j \ge \hat{K}_j \equiv \lim_{\varepsilon\to 0} \hat{K}_j(\varepsilon)$.

Now define $\bar{\varphi}_j^h \equiv \lim_{\varepsilon\to 0} \varphi_j^h(\varepsilon)/\sum_{i\in\sigma_j} \varphi_j^h(\varepsilon)$ for $h \in \sigma_j$. We shall argue that a central planner cannot Pareto improve given sales $\varepsilon\bar{\varphi}_j^h$, using asset j with penalties $(1-\delta)\lambda_j$. If he did, delivery rates would be no more than $\hat{K}_j \le K_j$ (where we have used separability between time 0 and 1 and the continuity of marginal utility of consumption). But then there is no Pareto improving trade, since the marginal utilities of the potential buyer is less than or equal to the marginal utility of the seller. *Q.E.D.*

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