THREE BRIEF PROOFS OF ARROW'S IMPOSSIBILITY THEOREM

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Economic Theory

Exposita Notes

Three brief proofs of Arrow's Impossibility Theorem*

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Summary. Arrow's original proof of his impossibility theorem proceeded in two steps: showing the existence of a decisive voter, and then showing that a decisive voter is a dictator. Barbera replaced the decisive voter with the weaker notion of a pivotal voter, thereby shortening the first step, but complicating the second step. I give three brief proofs, all of which turn on replacing the decisive/pivotal voter with an extremely pivotal voter (a voter who by unilaterally changing his vote can move some alternative from the bottom of the social ranking to the top), thereby simplifying both steps in Arrow's proof. My first proof is the most straightforward, and the second uses Condorcet preferences (which are transformed into each other by moving the bottom alternative to the top). The third proof proceeds by reinterpreting Step 1 of the first proof as saying that all social decisions are made the same way (neutrality).

Keywords and Phrases: Arrow Impossibility Theorem, Pivotal, Neutrality.

JEL Classification Numbers: D7, D70, D71.

Let $\mathbf{A} = \{A, B, ..., C\}$ be a finite set of at least three alternatives. A transitive preference over $\mathbf{A}$ is a ranking of the alternatives in $\mathbf{A}$ from top to bottom, with ties allowed. We consider a society with N individuals, each of whom has a (potentially

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* I wish to thank Ken Arrow, Chris Avery, Don Brown, Ben Polak, Herb Scarf, Chris Shannon, Lin Zhou, and especially Eric Maskin for very helpful comments and advice. I was motivated to think of reproving Arrow's theorem when I undertook to teach it to George Zettler, a mathematician friend. After I presented this paper at MIT, a graduate student there named Luis Ubeda–Rives told me he had worked out the same neutrality argument as I give in my third proof while he was in Spain nine years ago. He said he was anxious to publish on his own and not jointly, so I encourage the reader to consult his forthcoming working paper. The proofs appearing here appeared in my 1996 CFDP working paper. Proofs 2 and 3 originally used May's notation, which I have dropped on the advice of Chris Avery.

different) transitive preference. A constitution is a function which associates with every N-tuple (or profile) of transitive preferences a transitive preference called the social preference.

A constitution respects *unanimity* if society puts alternative α strictly above β whenever every individual puts α strictly above β. The constitution respects *independence of irrelevant alternatives* if the social relative ranking (higher, lower, or indifferent) of two alternatives α and β depends only on their relative ranking by every individual. The constitution is a *dictatorship* by individual n if for every pair α and β, society strictly prefers α to β whenever n strictly prefers α to β.

Arrow's Theorem. *Any constitution that respects transitivity, independence of irrelevant alternatives, and unanimity is a dictatorship.*

The strategy in all three proofs is to find a limited dictator n^*, and then to prove that n^* must be a genuine dictator. We say that n^* is *pivotal* for alternatives α,β at a profile of preferences π if by putting α strictly above β, or the reverse, n^* induces the social preference to do the same, holding all other individual preferences fixed as in π. Barbera showed that there is a pivotal individual, who must then be a dictator.

We say that an individual n^* is *extremely pivotal* for an alternative β at a profile π if n^* is pivotal for all pairs α, β involving β at π. Such an individual can move β from the very bottom of the social profile to the very top. In the first proof, we show that there is an extremely pivotal individual who must be a dictator. The critical step in the proof is an extremal lemma asserting that if every individual preference ranks β at the very top or the very bottom, then so must the social preference, even if half the individuals put β at the top and the other half put β at the bottom.

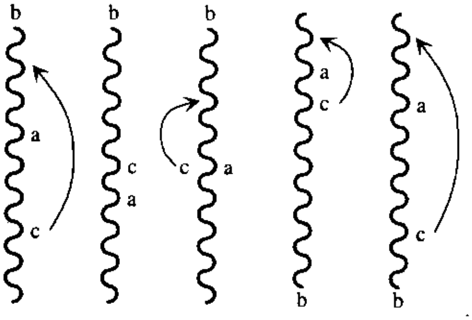
We say that an individual n^* is a *local dictator* at a profile π if n^* is pivotal for all pairs α, β at π. In our second proof, we show that there is a local dictator, who must then be a genuine dictator. The critical idea here is to look at Condorcet profiles, where each preference is a cyclical permutation of the others. Such a permutation is generated by moving the very top alternative to the very bottom. Arrow motivated his impossibility theorem by Condorcet's paradox.

In our last proof we begin with a lemma proving that Arrow's axioms imply *neutrality*, namely that every decision must be made the same way, independent of the names of the alternatives. We use it to show that any pivotal voter is a dictator. The neutrality lemma is equivalent to the extremal lemma, but for variety we give a different proof of it.

First Proof

Extremal Lemma. *Let alternative b be chosen arbitrarily. At any profile in which every voter puts alternative b at the very top or very bottom of his ranking of alternatives, society must as well (even if half the voters put b at the top and half put b at the bottom).*

Proof. Suppose to the contrary that for such a profile and for distinct a, b, c, the social preference put $a \geq b$ and $b \geq c$. By independence of irrelevant alternatives, this would continue to hold even if every individual moved c above a, because that could be arranged without disturbing any ab or cb votes (since b occupies an extreme position in each individual's ranking, as can be seen from the diagram). By transitivity the social ranking would then continue to put $a \geq c$, but by unanimity it would also put $c > a$, a contradiction, proving the lemma.



Next we argue that there is a voter $n^* = n(b)$ who is extremely pivotal in the sense that by changing his vote at some profile he can move b from the very bottom of the social ranking to the very top. To see this, let each voter put b at the very bottom of his (otherwise arbitrary) ranking of alternatives. By unanimity, society must as well. Now let the individuals from voter 1 to N successively move b from the bottom of their rankings to the very top, leaving the other relative rankings in place. Let n^* be the first voter whose change causes the social ranking of b to change. (By unanimity, a change must occur at the latest when $n^* = N$.) Denote by profile I the list of all voter rankings just before n^* moves b, and denote by profile II the list of all voter rankings just after n^* moves b to the top. Since in profile II b has moved off the bottom of the social ranking, we deduce from our first argument that the social preference corresponding to profile II must put b at the very top.

We argue third that $n^* = n(b)$ is a dictator over any pair ac not involving b. To see this, choose one element, say a, from the pair ac. Construct profile III from profile II by letting n^* move a above b, so that $a >_{n^*} > b >_{n^*} c$, and by letting all the agents $n \neq n^*$ arbitrarily rearrange their relative rankings of a and c while leaving b in its extreme position. By independence of irrelevant alternatives, the social preferences corresponding to profile III would necessarily put $a > b$ (since all individual ab votes are as in profile I where n^* put b at the bottom), and $b > c$ (since all individual bc votes are as in profile II where n^* put b at the top). By transitivity, society must put $a > c$. By independence of irrelevant alternatives, the social preference over ac must agree with n^* whenever $a >_{n^*} c$.

We conclude by arguing that n^* is also a dictator over every pair ab. Take a third distinct alternative c to put at the bottom in the construction of paragraph 2. From the third argument, there must be a voter $n(c)$ who is an $\alpha\beta$ dictator for any pair $\alpha\beta$ not involving c, such as ab. But agent n^* can affect society's ab ranking, namely at profiles I and II, hence this ab dictator $n(c)$ must actually be n^*. □

Second Proof

In a Condorcet profile each voter $n \in N$ is assigned one of the Condorcet preferences (cyclial permutations of the alphabetical order) described below:

$$\begin{array}{cccc}
\underline{\mathbb{C}_A} & \underline{\mathbb{C}_B} & \cdots & \underline{\mathbb{C}_C} \\
A & B & & C \\
B & C & & A \\
 & & & B \\
\vdots & \vdots & & \\
C & A & & \vdots
\end{array}$$

If all $n \in N$ are assigned to the first preference $\mathbb{C}_A$, then by unanimity, $\mathbb{C}_A$ must be the social preference. Among Condorcet profiles π such that the social preference is $\mathbb{C}_A$, find π_A that minimizes the number of voters with preferences $\mathbb{C}_A$. There must be at least one voter n^* in π_A with preferences $\mathbb{C}_A$, for otherwise C would be unanimously preferred to A. We shall show that n^* is a local dictator at π_A.

Suppose alternative β immediately follows α alphabetically. Suppose at π_A that n^* unilaterally switches to $\mathbb{C}_\beta$, giving the Condorcet profile π_β. By IIA, we still have $A >_{\pi_\beta} \cdots >_{\pi_\beta} \alpha$ and $\beta >_{\pi_\beta} \cdots >_{\pi_\beta} C$. Hence, for the social order to change, we must get $\alpha \leq_{\pi_\beta} \beta$. (Furthermore, if $\alpha =_{\pi_\beta} \beta$, then by transitivity and the fact $\#\mathbf{A} \geq 3$, $A >_{\pi_\beta} C$.)

Suppose instead that n^* switches to $-\mathbb{C}_A$, where $A < B < \cdots < C$, giving the non-Condorcet profile $\pi_{\bar{A}}$. Take two alphabetically consecutive alternatives α, β. Then $\mathbb{C}_\beta$ and $-\mathbb{C}_A$ agree on $\alpha\beta$ $(\beta > \alpha)$ and on AC $(C > A)$. Hence by IIA, $\alpha \leq_{\pi_{\bar{A}}} \beta$ since $\alpha \leq_{\pi_\beta} \beta$. Since α, β are arbitrary, this gives $A \leq_{\pi_{\bar{A}}} \cdots \leq_{\pi_{\bar{A}}} C$. Furthermore, if $\alpha =_{\pi_{\bar{A}}} \beta$, then by IIA, $\alpha =_{\pi_\beta} \beta$, and from the last paragraph, this would imply that $A >_{\pi_\beta} C$, and thus by IIA, $A >_{\pi_{\bar{A}}} C$, contradicting $A \leq_{\pi_{\bar{A}}} B \leq_{\pi_{\bar{A}}} \cdots \leq_{\pi_{\bar{A}}} C$. We conclude that $A <_{\pi_{\bar{A}}} B <_{\pi_{\bar{A}}} \cdots <_{\pi_{\bar{A}}} C$. Thus n^* is indeed a local dictator at π_A.

Let π be any nearly strict profile (where each individual preference has at most one pair of alternatives ranked equally) at which n^* is a local dictator. Change π to another nearly strict profile π' by letting a single voter $n \neq n^*$ raise some alternative half a step higher in his ranking in such a way that either he breaks a single tie $\alpha\beta$ or creates a single tie $\alpha\beta$ (but not both). We shall show that n^* is also a local dictator at π'. By IIA, this change by n cannot change the social ranking of any pair except possibly $\alpha\beta$. Modify π and π' by letting n^* rank $\alpha >_{n^*} \gamma >_{n^*} \beta$, for some third alternative γ. By hypothesis the social ranking at the modified π has $\alpha >_\pi \gamma >_\pi \beta$. Hence by IIA, $\alpha >_{\pi'} \gamma$ and $\gamma >_{\pi'} \beta$, so by transitivity $\alpha >_{\pi'} \beta$. Thus by IIA, the half-step move by n cannot change the power of n^* to enforce his strict preference over every pair at π'. Since π_A has no ties, for any pair $\alpha\beta$, a sequence of such half-moves can always be found to achieve arbitrary preferences for every voter $n \neq n^*$ over the given pair $\alpha\beta$. By IIA, n^* is a dictator. □

Third Proof

Strict Neutrality Lemma. *All binary social rankings are made the same way. Consider two pairs of alternatives ab and* $\alpha\beta$. *Suppose that in some profile* π *each voter strictly prefers a to b, or b to a, and suppose that in another profile* π' *each* *voter has the same relative ranking of* $\alpha\beta$ *as he does of* ab *in* π. *Then the social preference between* ab *in* π *is identical to the social preference between* $\alpha\beta$ *in* π' *and both social preferences are strict.*

Proof. Suppose WLOG that socially $a \geq b$ in π. Take $c \notin \{a, b\}$ and suppose first that $\alpha\beta = ac$ or $\alpha\beta = cb$ or $\alpha\beta = cd$ with $d \notin \{a, b, c\}$. Create a new profile π^* with α just above a for each voter n (if $\alpha \neq a$), and β just below b for each voter n (if $\beta \neq b$). Since all ab preferences are strict, this can be arranged while maintaining the same ab and $\alpha\beta$ preferences, as the diagram makes clear.

$$\begin{array}{ccccc} \alpha & \alpha & \alpha & b & b \\ a & a & a & \beta & \beta \\ b & b & b & \alpha & \alpha \\ \beta & \beta & \beta & a & a \end{array}$$

By unanimity, $\alpha >_{\pi^*} a$ (if $\alpha \neq a$) and $b >_{\pi^*} \beta$ (if $\beta \neq b$). By transitivity, $\alpha >_{\pi^*} \beta$. Reversing the roles of ab and $\alpha\beta$, we conclude by IIA that socially $a >_\pi b$ in the original profile. All that remains is to consider the situation where $\alpha\beta = ba$. This is handled by considering in turn profiles where the preferences are identical between the pairs (ab, ac), then (ac, bc), and then finally (bc, ba). This concludes the proof of the lemma.

Next, take two distinct alternatives a and b and start with a profile in which $b >_n a$ for all n. Beginning with $n = 1$, let each voter successively move a above b. By unanimity and the strict neutrality lemma, there will be a voter n^* who moves the social preference from $b > a$ to $a > b$ when he moves a up. Clearly n^* is pivotal. The situation is described below.

$$\begin{array}{ccccc} \underline{1} & & \underline{n^*} & & \underline{N} \\ a & a & a & b & b \\ b & b & b & a & a \end{array} \xrightarrow{} \begin{array}{c} \\ a \\ b \end{array} \qquad\qquad \begin{array}{ccccc} \underline{1} & & \underline{n^*} & & \underline{N} \\ a & a & b & b & b \\ b & b & a & a & a \end{array} \xrightarrow{} \begin{array}{c} \\ b \\ a \end{array}$$

We now show that n^* is a dictator. Take an arbitrary pair of alternatives α, β and let n^* rank $\alpha >_{n^*} \beta$. Let the $\alpha\beta$ rankings for $n \neq n^*$ be arbitrary. Take $c \notin \{\alpha, \beta\}$ and put c above everything for $1 \leq n < n^*$, c below everything for $n^* < n \leq N$, and $\alpha >_{n^*} c >_{n^*} \beta$. By IIA, neutrality, and the pivotal profile discovered above, socially $c > \beta$ and $\alpha > c$, and so by transitivity, $\alpha > \beta$.

$$\begin{array}{ccccc} \underline{1} & & \underline{n^*} & & \underline{N} \\ c & c & \alpha & \beta & \alpha \\ \beta & \alpha\beta & c & \alpha & \beta \\ \alpha & & \beta & c & c \end{array}$$

□

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