DETERMINACY WITH NOMINAL ASSETS AND OUTSIDE MONEY

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Economic Theory

Determinacy with nominal assets and outside money

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Summary. We build a finite horizon model with inside and outside money, in which interest rates, price levels and commodity allocations are determinate, even though asset markets are incomplete and asset deliveries are purely nominal.

Keywords and Phrases: Central bank, Inside money, Outside money, Incomplete assets, Monetary equilibrium, Determinacy.

JEL Classification Numbers: D50, E40, E50, E58.

1 Introduction

In standard general equilibrium theory, the price level is indeterminate. If prices p clear markets, then so will λp for any $\lambda > 0$. When there are many states of the world, the dimension of indeterminacy is much greater; the scaling can be done independently in each state. Thus it is impossible to think of equilibrium determining interest rates or inflation. For that we need a theory of money.

The need becomes more urgent when we consider nominal assets promising delivery in units of account, for then the indeterminacy pertains to real allocations as well.[1]

The general equilibrium theory of money has a long and rich history which it would be impossible to summarize here. However, no model of money has come to be regarded as standard, in the same way as the Arrow and Debreu model (GE) or the incomplete asset model (GEI). One explanation is that it is not immediately

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[1] If there are S uncertain states in the future, and if the nominal asset promises do not fully span $\mathbb{R}^S$, then generically there will be $S-1$ dimensions of indeterminacy in consumption. See Balasko and Cass (1989) and Geanakoplos and Mas-Colell (1989).

obvious how to precisely formalize a sensible monetary economy that is as general as GE or GEI, and in which fiat money always has a positive, determinate value, and which allows for real effects of monetary policy.

In Dubey and Geanakoplos (1992, 2003a,b), we proposed a model of inside and outside money general enough to encompass GE and GEI, and showed that monetary equilibrium (ME) always exists. This was at first glance surprising, since ME exists even when GEI does not. Here we continue the argument by showing that ME are determinate, even when GEI are not.

Consider first the one-period version of our model, introduced in Dubey and Geanakoplos (1992). Agents begin, as in GE, with endowments of multiple commodities. But they also have *outside* money m, which is owned free and clear. In addition they can borrow M from a central bank. We call M *inside* money because every dollar of it that enters the economy does so against an offsetting obligation that guarantees its departure.

Money is the sole medium of exchange: all commodity purchases must be paid for in money. To facilitate trade, central bank loans are available before commodity markets meet, and must be repaid afterwards. Agents are free to spend their endowed money m without visiting the bank. But some agents may wish to spend more, and so they voluntarily borrow from the bank, then sell commodities to obtain money to repay the loan. The commodity prices p and the bank interest rate r are formed endogenously in equilibrium to balance the supply and demand for commodities, and to generate demand for bank money equal to its exogenous supply M. This simple model captures the transactions role of money.

If $m = 0$ and all the money is inside money $M > 0$, then there is indeterminacy in the value of money, though the interest rate $r = 0$.[2]

What is striking is that with outside money $m > 0$, no matter how little, the indeterminacy completely disappears. The explanation is simple. In equilibrium, no optimizing agent will be left holding worthless cash at the end. The interest rate will therefore need to be just high enough so that the central bank ends up with all the outside money as profit on its lending, $m = rM$, or $r = m/M > 0$. But with a positive interest rate and only one period, rational agents will spend all the cash that they borrow or own, and so expenditures must be $m + M$, pinning down price levels.

Thinking of the central bank as a revenue source for the government, and the endowments of outside money as (previous) expenditures of the government, equilibrium requires the government budget to be balanced, $m = rM$.[3] But out of equilibrium, at the wrong values of (r, p), it may well be that the central bank's interest earnings are below m, leaving the government in deficit. Of course, if $m = 0$,

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[2] Any scalar multiple λp of GE market clearing prices p constitutes an equilibrium, provided that total expenditures are less than or equal to M. If less, the agents hoard some of their borrowed money, returning it unspent.

[3] Every expenditure by an agent is some other agent's sales revenue. Hence total expenditures are equal to total sales revenue. Since agents collectively begin with m that they can spend free and clear, it follows that in the transformation of sales revenue into expendable income, m must be lost. But this loss can only go to the central bank. Hence its profit must be m.

then the government has no expenditures and can never run a deficit, in or out of equilibrium.

The foregoing logic has been echoed in the subsequent "fiscal theory of the price level" literature (see, e.g., Woodford, 1994; Sims, 1994; Cochrane, 2001). Following Woodford (1996, 1999), a fiscal policy rule is called "Ricardian" if it guarantees that the government balances its budget in and out of equilibrium. If the budget often fails to balance out of equilibrium, the fiscal policy is called non-Ricardian. In various specialized contexts, it is shown that Ricardian policies lead to indeterminacy and non-Ricardian to determinacy. In this terminology, our model is non-Ricardian when there is outside money, and Ricardian without it.

Dreze and Polemarchakis (1999, 2000, 2001) considered a variant of our model for which they derived indeterminacy. They argued that the "failure to adopt Walras Law has led authors like Dubey and Geanakoplos (1992, 2003) to different conclusions on the multiplicity of equilibria." In a one-period setting, the Dreze-Polemarchakis model is tantamount to assuming that households own 100% of the central bank, *and* that $m = 0$.[4] They show that fixing $M > 0$ and *any* $r > 0$ (but not too high),[5] will yield equilibrium. How do they get this indeterminacy?

The answer is that in their model there is no outside money. Nonetheless $r > 0$ is possible in equilibrium because the central bank is privately owned. Agents can collectively return more money $(1 + r)M$ than the M they borrow because the bank credits them with its profit rM at repayment time (or equivalently gives them $rM/(1 + r)$ at loan time).

This raises two obvious questions. Is there outside money, and are *all* the profits of the central bank distributed to households?

We argue that outside money is a fact of life. Whenever the government prints money and purchases real assets, like labor, from the private sector, it creates outside money. (History is replete with examples, many of which are entertainingly recounted in Galbraith, 1975.) In American law, the Treasury cannot literally print money to buy real assets, but must borrow it from the Federal Reserve. But the Federal Reserve can print the money, giving it to the Treasury in exchange for an IOU note. By not redeeming the IOU note, or equivalently by rolling it over in perpetuity, the Treasury prints outside money by proxy. Injections of outside money may be frequent or infrequent depending on the profligacy or restraint of governments. But that they occur cannot be open to doubt. A proper theory of money must take outside money into account.

To put the point in a different way, suppose a counterfeiter introduces a single dollar bill into the economy. This is surely outside money. In our model, this tiny injection would move the equilibrium slightly. But the huge continuum of Dreze and Polemarchakis equilibria would *all* collapse. There would be no equilibrium, since after their credit for bank profits, the agents would always have $1 more in their pockets than what is owed.

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[4] In their model and in our model of 1992, agents $h \in H$ have endowments m^h that can be positive or negative. But for them $\sum_h m^h \equiv 0$, while for us $\sum_{h \in H} m^h \equiv m > 0$.

[5] In equilibrium r cannot be higher than the "gains-to-trade" available at the initial endowment, as defined in Dubey and Geanakoplos (1992).

As for the Dreze and Polemarchakis hypothesis that the central bank distributes *all* its profits to private households, according to shares $\{\theta^h : h \in H\}$, who knows his share θ^h in the Federal Reserve? And when does he get the money? Note that if households get anything less than 100% of the central bank's profits, then with no outside money equilibrium requires $r = 0$, while with outside money there is determinacy.[6]

Magill and Quinzii (1992) proposed yet another monetary theory, in which there is no outside money and yet equilibrium is determinate. They accomplished this, however, by assuming forced sales of goods. In their model, each agent *must* sell *all* his goods to the government, which has an exogenously fixed stock M to pay in exchange. They themselves say that their model "involves a certain amount of brute force." In reality agents cannot be forced to sell their goods to anyone, least of all to the government.[7]

In this paper we extend our one-period model to a finite horizon with uncertainty, incomplete markets, and nominal assets.[8] In this setting the many facets of money and monetary policy come to light. Yet we show that determinacy still obtains for interest rates, inflation, and commodity allocations. Monetary policy is not neutral, and its effects can in principle be tracked because ME are determinate.

Our model is like GEI, but with two additions: inside *and* outside money. There is a banking system which injects inside money into the economy by way of short- and long-term loans. In addition there is outside money, as before. All trade is voluntary, and mediated by competitive market prices as in general equilibrium.

The demand for money in our model stems both from its need in immediate transactions, as in the one-period economy, and because agents want to hold it into the future as a store-of-value or for precautionary or speculative motives. On account of the presence of both long and short loans, it is no longer obvious what interest rates will turn out to be. By the end of every path in the tree of chance moves, the central bank will earn profit equal to the stock of outside money. But that can be achieved via high interest rates early and low interest rates later, or the reverse. Thus our multiperiod model gives rise to a term structure of interest rates, determined by the interaction of the real and monetary sectors.

At the extreme, some interest rates r_s might robustly turn out to be zero. Agents might then spend less than $M_s + m_s$ on commodity purchases. Pumping further money into the system by increasing M_s, without changing any other (future) M_σ, would then have absolutely no affect on equilibrium prices or allocations. We call this a *liquidity trap*. We prove that no matter what the utilities and endowments of commodities and outside money, a liquidity trap must emerge if the central bank dramatically increases money M_0 today, without similarly escalating future stocks M_s. Thus if the government wishes to engineer an increase in price levels today, perhaps in order to reduce the real wealth of the holders of large endowments of

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6 In the United States, individuals own private banks, which in turn have shares in the Federal Reserve. But the Federal Reserve does not distribute all its revenue to these banks. Routinely the Federal Reserve turns over money to the Treasury.

7 Lucas and Stokey (1987) also forced agents to sell all their goods for money.

8 We focus on nominal assets because, in standard GEI, they create the most consequential indeterminacy. Were we to substitute real assets, we would still get determinacy.

money, it will be eventually thwarted by a liquidity trap. The government can only succeed today if it can commit to a monetary expansion into the distant future.

For better perspective, we examine two contrasting models. In the first, the central bank sets positive interest rate targets, and precommits to supplying whatever money or bonds are demanded at those rates. In our second model, the bank has positive quantity targets and precommits to the size of its lending, letting interest rates be determined endogenously in equilibrium. Both bank policies lead to determinacy. But there are some fundamental differences. In the interest rate targets model, the quantity theory of money prevails. With quantity targets, it fails. With interest rate targets, determinacy obtains provided only that *some* agent has outside money at the start of the economy. With quantity targets, it becomes needful to have frequent injections of outside money in order to get determinacy: every agent must have outside money in every state. This is because the quantity targets model is more intricate and makes room for the robust emergence of "liquidity traps" in which short-term interest rates are zero and agents hoard money.

Equilibrium in both models can be described as the solution to a finite set of equations with the same number of unknowns. We explicitly present these equations, and check their independence to demonstrate determinacy. These equations form a recipe for the *computation* of monetary equilibrium in a dynamic economy with genuinely heterogenous agents, and incomplete markets.[9]

2 The monetary economy

2.1 The underlying real economy[10]

The set of states of nature is $S^* = \{0, 1, \ldots, S\}$. State 0 occurs in period 0, and then nature moves and selects one of the states $s \in S = \{1, \ldots, S\}$ which occur in period 1.

The set of commodities is $L = \{1, \ldots, L\}$. Thus the commodity space may be viewed as $\mathbb{R}_+^{S^*L}$, whose axes are indexed by $\{0, 1, \ldots, S\} \times \{1, \ldots, L\}$. The pair $s\ell$ denotes commodity ℓ in state s. All commodities are perishable.

The set of agents is $H = \{1, \ldots, H\}$. Agent h has initial endowment of commodities $e^h \in \mathbb{R}_+^{S^*L}$ and utility of consumption $u^h : \mathbb{R}_+^{S^*L} \to \mathbb{R}$. We assume that no agent has the null endowment of commodities in any state, i.e., for $s \in S^*$ and $h \in H$:

$$e_s^h = (e_{s1}^h, \ldots, e_{sL}^h) \neq 0;$$

and, further, that each named commodity is actually present in the aggregate, i.e.,

$$\Sigma_{h \in H} e_s^h \gg 0.$$

We also assume that each u^h is concave, smooth[11] and strictly monotonic.

[9] To simplify the intricate equilibrium equations that arise with incomplete markets, we confine attention to "active" equilibria in which each agent chooses to buy something in every state.

[10] For concreteness, we focus on one of the models presented in Dubey and Geanakoplos (2003b). The other variant models of that paper are susceptible to a similar analysis.

[11] I.e., C^2 (second partial derivatives exist and are continuous).

2.2 Money

Our model is designed to capture the multiple facets of money. We will suppose that money is the stipulated medium of exchange. All commodities and assets (to be introduced shortly) are traded exclusively[12] for money, and all assets promise delivery exclusively in money.

Money is fiat; unlike commodities it gives utility to no agent. Also unlike commodities, it cannot be privately produced.[13] It is perfectly durable. Its value resides in the fact that it can be used for transactions, and as a store of value (by carrying it forward for future use). It enters the economy in two ways, as inside or outside money.

2.2.1 Outside money

Money may be present in the private endowments of agents. Let

$$m_s^h \equiv \text{private endowment of money of } h \text{ in state } s \in S^*.$$

We can interpret m_s^h as a government transfer to agent h or as h's private inheritance from the (unmodeled) past. The vector $(m_s^h)_{s \in S}^{h \in H}$, is called *outside money*, because it enters the system free and clear of any offsetting debts. We assume that

$$\sum_{h \in H} m_0^h > 0.$$

2.2.2 Inside money

A crucial ingredient of our model is a central (government) bank which stands ready to lend or borrow money by buying or selling bank bonds[14] from the agents. For simplicity, we allow only two kinds of bank bonds. A short-term bank bond of state s is traded at the beginning of that state and promises 1 dollar at its end. The long-term bank bond is traded at the beginning of state 0 and promises 1 dollar just before[15] commodity trade in every future state $s \in S$ in period 1. Let $n \in N \equiv \{\bar{0}, 0, 1, ..., S\}$ index the bank bonds, where $n = \bar{0}$ is for the long bond, and $n = s$ for the short bond in state $s \in S^*$. Let r_n denote the interest rate on bond n. Then $1/(1 + r_n)$ denotes the price of bond n in terms of money. Selling a bond amounts to taking out a bank loan. Thus an agent who borrows z dollars on the short-term bank loan in state s (or on the long-term bank loan in state 0) owes

[12] This is for simplicity of presentation (see Dubey and Geanakoplos, 2003b) for a more general model, which permits direct trade between prespecified pairs of commodities, or pairs of commodities and assets, or pairs of assets; and which, moreover, allows for some asset deliveries to be nominated in terms of real commodities.

[13] Private production of commodities could easily be incorporated in our model (as in Dubey and Geanakoplos, 2003b), but we suppress it for simplicity.

[14] We call them bank bonds because the bank trades them.

[15] The timing of deliveries does not affect the determinacy of equilibrium. To fix ideas we suppose that asset deliveries (including that on the long bond) are due prior to commodity trade in period 1. In Dubey and Geanakoplos (2003b) the long bond comes due *after* commodity trade.

$(1 + r_s)z$ (or $(1 + r_{\bar{0}})z$) dollars after trade in state s (before trade in every $t \in S$). Similarly, a deposit of z dollars yields $(1 + r_s)z$ or $(1 + r_{\bar{0}})z$ dollars later.

When the bank lends money via a purchase of bank bonds, it creates inside money. Inside money remains in the economy only until the bank bond comes due, when it leaves, taking additional (interest) money out with it. When the bank borrows money by selling bonds, it temporarily reduces the stock of money. But when the bond comes due, it returns the borrowed money, and in addition creates more outside money to pay the interest.

2.2.3 Bank policy

At one extreme we may suppose the bank has quantity targets and precommits to the size of its borrowing or lending, letting interest rates be determined endogenously at equilibrium. At the other extreme we may suppose that the bank has interest rate targets, and precommits to supplying whatever money or bonds are demanded at those rates. Both policies lead to determinacy.

2.3 Assets

The set of assets is $J = \{1, \ldots, J\}$. The seller of one unit of asset $j \in J$ must deliver a state contingent vector of money. Thus we may view asset j as a vector A^j in $\mathbb{R}^S_+$ whose sth component A^j_s specifies the amount of money due prior to trade in state $s \in S$

The collection of all assets $(A^1, \ldots, A^J)$ will be denoted A.

2.4 Markets

Let $I = L \cup \{m\} \cup N \cup J$ be the set of all instruments in the economy. (Here m denotes money.) A market $s\alpha\beta$ (equivalently, $s\beta\alpha$) always involves a bilateral exchange between a pair of instruments α and β in a particular state s. The set of markets in our model is given by:

$$\mathcal{M} \equiv \{0\bar{0}m, (ssm)_{s\in S^*}, (0jm)_{j\in J}, (s\ell m)_{s\ell\in S^*L}\}.$$

i.e., there are markets for trading the long bond, short bonds, assets and commodities – all versus money. Denote

$$\begin{aligned} 0 \leq q^h_{s\alpha\beta} &\equiv \text{quantity of instrument } \alpha \text{ sold by } h \text{ to purchase} \\ &\quad\ \text{instrument } \beta \text{ (at the market } s\alpha\beta) \\ 0 \leq Q_{s\alpha\beta} &\equiv \text{quantity of instrument } \alpha \text{ sold by the government} \\ &\quad\ \text{to purchase instrument } \beta \text{ (at the market } s\alpha\beta) \end{aligned}$$

We take $Q_{s\alpha\beta} = q^h_{s\alpha\beta} = 0$ if there is no market $s\alpha\beta$ between α and β at date-event s. In particular, $Q_{s\alpha\beta} = q^h_{s\alpha\beta} = 0$ unless one of α or β is m. We shall suppose that the government acts only through the central bank, by borrowing or lending on the markets $n \in N$, and nowhere else: $Q_{s\alpha\beta} = 0$ if α or β is in $L \cup J$. We shall also suppose that the central bank does not indulge in wash sales: $Q_{smn}Q_{snm} = 0$ for all $n \in N$. Let

$$0 < p_{s\alpha\beta} \equiv \text{price of instrument } \alpha \text{ in terms of instrument } \beta \text{ (at the market } s\alpha\beta)$$

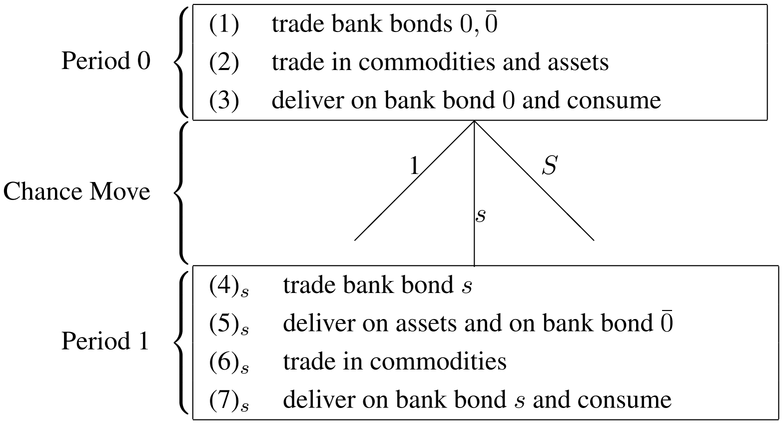
with $p_{s\alpha\beta} = p_{s\beta\alpha}^{-1}$. (We take $p_{s\alpha\beta} = 1$ if there is no market $s\alpha\beta$.) For convenience, we list prices as (r, p) where $r = (r_{\bar{0}}, r_0, r_1, \ldots, r_S)$ is the $(S^* + 1)$-dimensional vector of interest rates, and p is the $(S^*L + J)$-dimensional vector of commodity prices $p_{s\ell m}$ and asset prices p_{0jm}. (Thus $p_{snm} = 1/(1 + r_n)$ for $n \in N$ and $snm \in \mathcal{M}$.)

If the commodity or asset market $s\alpha\beta$ clears (as it will in equilibrium) and if there is positive trade at that market, then

$$p_{s\alpha\beta} = \frac{Q_{s\beta\alpha} + \Sigma_{h \in H} q^h_{s\beta\alpha}}{Q_{s\alpha\beta} + \Sigma_{h \in H} q^h_{s\alpha\beta}}.$$

3 The budget set

The agents regard (r, p) as fixed. Given (r, p), with $r \geq 0$ and $p \gg 0$, the *budget set* $B^h(r, p)$ available to agent h – specifying the sequence[16] of *market actions* $q^h_{s\alpha\beta} \geq 0$ and *consumptions* $x^h_{s\ell} \geq 0$ that are feasible for him – is depicted in the following diagram:



[16] The exact order in which the markets meet does not affect our determinacy results. For simplicity we have chosen to include only two consumption periods, suggesting a long period of time. In order to allow agents to borrow money in time to use it for purchases in the same period, bank loan markets need to meet before commodity markets. (Had we allowed for multiple periods, perhaps a nanosecond apart, we could have assumed that all markets meet simultaneously each period.) The timing of deliveries on the long loan and assets is arbitrary. We chose what seemed the simplest rule (the same as in Geanakoplos and Tsomocos, 2002). In Dubey and Geanakoplos (2003b) we put deliveries later.

We require, of course, that sales of commodities do not exceed endowments, i.e., $q^h_{s\ell m} \leq e^h_{s\ell}$; and that consumption be feasible, i.e., $x^h_{s\ell} \leq e^h_{s\ell} - q^h_{s\ell m} + q^h_{sm\ell}/p_{s\ell m}$ $(\forall s \in S^*, \ell \in L)$. The remaining constraints on the choices of agent h in $B^h(r,p)$ pertain to money. They require that the outflow of money at any time cannot exceed its stock on hand. Letting $\Delta(\nu)$ denote the difference between the right-hand-side and the left-hand-side of inequality (ν), we have:

Deposit: Money deposited $\leq$ money endowed:

$(1)^h$ $q^h_{0m\bar{0}} + q^h_{0m0} \leq m^h_0$

Spend: Money spent on purchases in state $0 \leq$ money unspent in $(1)^h$ plus money borrowed on long and short loans:

$(2)^h$ $\Sigma_{\ell \in L} q^h_{0m\ell} + \Sigma_{j \in J} q^h_{0mj} \leq \Delta(1)^h + \frac{q^h_{0\bar{0}m}}{1+r_{\bar{0}}} + \frac{q^h_{00m}}{1+r_0}.$

Repay and inventory: Money repaid (on net) on loan $0 \leq$ money unspent in $(2)^h$ plus money obtained from sales of commodities and assets:

$(3)^h$ $q^h_{00m} - (1+r_0)q^h_{0m0} \leq \Delta(2)^h + \Sigma_{\ell \in L} p_{0\ell m} q^h_{0\ell m} + \Sigma_{j \in J} p_{0jm} q^h_{0jm}$

Deposit: For every $s \in S$, money deposited on short loan $s \leq$ money unspent in $(3)^h$ and inventoried into state s, plus money endowed in state s:

$(4)^h_s$ $q^h_{sms} \leq \Delta(3)^h + m^h_s$

Deliver: For every $s \in S$, money spent on net asset deliveries (including net delivery on the long loan $\bar{0}$) $\leq$ money unspent in $(4)^h_s$ plus money borrowed on short loan s:

$(5)^h_s$ $(q^h_{0\bar{0}m} - (1+r_{\bar{0}})q^h_{0m\bar{0}}) + \Sigma_{j \in J} \left(q^h_{0jm} - \frac{q^h_{0mj}}{p_{0jm}} \right) A^j_s \leq \Delta(4)^h_s + \frac{q^h_{ssm}}{1+r_s}$

Spend: For every $s \in S$, money spent on purchases $\leq$ money unspent in $(5)^h_s$:

$(6)^h_s$ $\Sigma_{\ell \in L} q^h_{sm\ell} \leq \Delta(5)^h_s$

Repay: For every $s \in S$, money repaid (on net) on loan $s \leq$ money unspent in $(6)^h_s$ plus money received from commodity sales:

$(7)^h_s$ $q^h_{ssm} - (1+r_s)q^h_{sms} \leq \Delta(6)^h_s + \Sigma_{\ell \in L} p_{s\ell m} q^h_{s\ell m}$

Notice that $(5)^h_s$ implies that, for each agent, credits and debits are netted across all assets and the long loan. This is not essential. Variants of the model, in which there is no netting, or netting is confined within prespecified subgroups of assets, leave our results intact (see Dubey and Geanakoplos, 2003b).

4 Monetary equilibrium

We say that $(r, Q, p, q, x) \equiv ((r_n, Q_{snm}, Q_{smn})_{n\in N}, (p_{s\ell m})_{s\ell\in S^*L}, (p_{0jm})_{j\in J}, (q^h, x^h)_{h\in H})$ is a *monetary equilibrium* (and denote it ME) of the economy $E = ((u^h, e^h, m^h)_{h\in H}, A)$ if:

(8) $Q_{snm} + \Sigma_{h\in H} q^h_{snm} = (1 + r_n)(Q_{smn} + \sum_{h\in H} q^h_{smn})\ \forall n \in N$ with $smn \in \mathcal{M}$
(9) $\Sigma_{h\in H} q^h_{sm\ell} = p_{s\ell m} \Sigma_{h\in H} q^h_{s\ell m}, \forall s\ell \in S^*L$
(10) $\Sigma_{h\in H} q^h_{0mj} = p_{0jm} \Sigma_{h\in H} q^h_{0jm}, \forall j \in J$
(11) $(q^h, x^h) \in \arg\max_{(\tilde{q}, \tilde{x}^h)\in B^h(r,p)} u^h(\tilde{x}^h), \forall h \in H.$

In other words, at an ME, each agent must maximize on his budget set (11); and markets must clear for loans, commodities and assets ((8), (9), and (10)).

Given an ME (r, Q, p, q, x), denote $\mathcal{M}(Q, q) \equiv (s\alpha\beta \in \mathcal{M} : Q_{s\alpha\beta} + \Sigma_{h\in H} q^h_{s\alpha\beta} > 0)$. In other words, $\mathcal{M}(Q, q)$ is the set of markets in $\mathcal{M}$ in which there is positive trade at the ME, so that the markets are not *shut*. Then the *outcome* of the ME is defined to be $(\{p_{s\alpha\beta}\}_{s\alpha\beta\in\mathcal{M}(Q,q)}, \{x^h\}_{h\in H})$ and amounts to specifying the prices and trades at markets that are not shut at the ME.

5 Determinacy of monetary equilibrium

5.1 Interest rate targets

A subset of a Euclidian space is called *determinate* if it consists of isolated (i.e., locally unique) points. We shall show that almost all economies give rise to determinate "active" equilibrium outcomes.

We turn first to the case where the central bank targets (fixes) positive interest rates $r = (r_{\bar{0}}, r_0, r_1, ..., r_S) \gg 0$ with $r_{\bar{0}} > r_0$.

The space U^h of utilities of agent h consists of all linear perturbations of an arbitrary fixed utility $\bar{u}^h : \mathbb{R}^{S^*L}_+ \to \mathbb{R}$, i.e., all functions u^h of the form $u^h(x) = \bar{u}^h(x) + c \cdot x$, where $c \in N^h$ and N^h is a neighborhood of 0 in $\mathbb{R}^{S^*L}$. (Here $\cdot$ denotes dot product.) Put $U = \mathsf{X}_{h\in H} U^h$. (Thus U is identified with $\mathsf{X}_{h\in H} N^h$.) Given fixed interest rates $r \in \mathbb{R}^{S^*+1}_{++}$, fixed assets $A \in \mathbb{R}^{SJ}_+$, fixed commodity and money endowments $e \in \mathbb{R}^{S^*LH}_+$ and $m \in \mathbb{R}^{S^*H}_+$, the economy

$$u \equiv (u^1, \dots, u^H)$$

can be viewed as a point in U.

To state our result, we need one last notion. Let Λ be an open set in a Euclidean space. A subset $\tilde{\Lambda} \subset \Lambda$ will be called full in Λ if it is open and dense in Λ, and $\Lambda\backslash\tilde{\Lambda}$ has Lebesgue measure zero. A property will be said to hold for *generic* w in Λ if it holds for all w in a set that is full in Λ.

We call an ME *active* if every agent buys something in each state. We concentrate on active equilibria because the first-order conditions are radically simplified. We believe that determinacy obtains even over the set of all ME, but a proof would be much more involved. Incidentally, the same (simpler) point would apply if we assumed every agent sold something in every state (or just traded in every state). Many authors (like Lucas and Stokey, 1987) have assumed agents sell their whole endowments in every state. (Clearly there are open sets of economies which have no inactive ME.)

Theorem 1. *Let* $\tilde{1} \in \mathbb{R}^S$ *be the vector with all components equal to* 1. *Assume* $\tilde{1}, A^1, ..., A^J$ *are linearly independent. Then, in the interest rate targets model, the set of active* ME *outcomes is determinate for generic* u *in* U*; and is finite when there are no inactive* ME.

Our assumption implies that $J < S$. This is not a stringent requirement. Typically, the states in S are associated with shocks to the private fortunes (e.g., endowments) of the agents in the economy, and their number far exceeds that of assets (publicly traded on market). Moreover, in our model, taking out a long-term loan (or inventorying money) is a distinct asset, represented by $\tilde{1}$, which is not explicitly included in the J assets. Thus the assumption says that there is no "redundant" asset which can be obtained as a linear combination of the other assets. Clearly it holds for generic $(A^1, ..., A^J) \in \mathbb{R}_+^{SJ}$ when $J < S$.

We conjecture that Theorem 1 is true without assuming the linear independence of assets. With dependent assets, however, more complicated reductions may be needed to go to equivalent ME in which each agent acts on a linearly independent set of assets (though the economy may be using redundant assets). We have contented ourselves with the weaker Theorem 1, since it leads to a clean proof, and yet brings out the main ideas behind the determinacy phenomenon.

5.2 Quantity targets

We now turn to the situation in which the central bank targets the quantity of money in each state. This gives a somewhat more intricate model in which new phenomena appear. We can get a liquidity trap in which the initial short interest rate is zero and agents hoard money. The central bank is then powerless to change price levels by increasing M_0, unless it can commit itself to similar *future* increases in M_s, $s \in S$.

The robust possibility of zero interest rates also complicates the determinacy proof. By opening the door to hoarding, it destroys the "quantity theory of money" principle that all the money at hand will be spent, which seems at first glance to be responsible for pinning down equilibrium price levels. Nevertheless we are still able to prove generic determinacy of equilibrium. But now our conclusion only holds for almost all endowments of outside money (rather than for all endowments as in the interest rate target model). Thus we must suppose that every agent gets a fresh injection of outside money in every state in every period, whereas in Theorem 1 it sufficed that *some* agent had outside money at the start of the economy.

We take $M_n \equiv Q_{smn} > 0$ for $smn \in \mathcal{M}$ as exogenous (and all Q_{snm}=0). The vector $M \equiv (M_{\bar{0}}, M_0, M_1, ..., M_S) \in \mathbb{R}_{++}^{S^*+1}$ represents the quantity targets. Given an economy $E \equiv ((u^h, e^h, m^h)_{h \in H}, A)$, an ME $((r_n, M_n)_{n \in N}, (p_{0jm})_{j \in J}, (p_{s\ell m})_{s \in S^* L}, (q^h, x^h)_{h \in H})$ is defined for E exactly as before, with the interest rates $r \equiv (r_{\bar{0}}, r_0, r_1, ..., r_S)$ now thought of as endogenous.

Endowments $e \equiv (e^h)_{h \in H}$ and assets A and targets $M \in \mathbb{R}^{S^*+1}_{++}$ can be held fixed, but we will need to vary the money endowments $m \equiv (m^h)_{h \in H} \in \mathbb{R}^{S^*H}_{++}$. Thus we view an economy as a point (u, m) in $U \times \mathbb{R}^{S^*H}_{++}$.

Theorem 2. *Assume* $\tilde{1}, A^1, ..., A^J$ *are linearly independent. Then, in the quantity targets model, the set of active* ME *outcomes is determinate for generic* (u, m) *in* $U \times \mathbb{R}^{S^*H}_{++}$*; and is finite when there are no inactive* ME.

5.3 The liquidity trap

In order to illustrate the difference between the interest rate and quantity target models, it might help to describe a simple, robust scenario in which a liquidity trap must emerge. (See Dubey and Geanakoplos, 2003b, for other kinds of liquidity traps.)

Consider an economy $((u^h, e^h, m^h)_{h \in H}, A)$, with quantity targets $(M_{\bar{0}}, M_0, M_1, ..., M_S)$ in which $\sum_{h \in H} m_0^h > 0$ and the "γ_s-gains-to-trade hypothesis" holds, with $\gamma_s = \sum_{h \in H}(m_0^h + m_s^h)/M_s$ for all $s \in S$. As shown in Dubey and Geanakoplos (2003b), monetary equilibrium necessarily exists, and will continue to exist as the stocks of bank money are increased.

Now let $M_0 \to \infty$, holding all other M_n and the rest of the economy fixed.

Theorem 3. *In the above scenario, there exists a finite threshold* M_0^* *such that for all* $M_0 \geq M_0^*$*, the set of* ME *remains invariant and all* ME *have* $r_0 = 0$.

Thus once r_0 plunges to 0, the central bank cannot affect equilibrium prices by offering to loan more money on the short loan. Agents will simply hoard the extra money without spending it. Only by operating on the long loan, or by committing to *future* open market operations, can the central bank affect equilibrium.

The liquidity trap makes the proof of determinacy more difficult, because we can no longer presume that agents will spend all of M_0 (or any other M_s).

The proofs of Theorems 1–3 are developed through a series of lemmas in the rest of the paper.

6 Properties of monetary equilibrium

The lemmas below pertain to any ME (r, Q, p, q, x) of $E = ((u^h, e^h, m^h)_{h \in H}, A)$, in either the interest rate or quantity targets model.

Lemma 1. *At any* ME, $r_s \geq 0$ *for all* $s \in S^*$ *and* $r_{\bar{0}} \geq r_0$.

Proof. Let $r_s < 0$, the agents could infinitely arbitrage the central bank. If $r_{\bar{0}} < r_0$, then we must be in the quantity targets model, where $M_0 > 0$; but then nobody would borrow on the short loan and the M_0 market would not clear. □

In our second lemma we show that agents spend all the money at hand on purchases, when interest rates are strictly positive. The reason is that they can deposit any money they do not intend to spend (or else borrow less), receiving the money back with interest, before they face their next buying opportunity.

Lemma 2 (Quantity theory of money). *If $r \gg 0$ then each agent spends all the money at hand on purchases in every state, i.e., for all $h \in H$: $\Delta(2)^h = 0$ and $\Delta(6)^h_s = 0$ for all $s \in S$.*

Proof. Suppose some agent h does not spend all the money at hand on the purchase of assets and commodities in state 0, i.e., $\Delta(2)^h > 0$. Instead let h deposit $\varepsilon > 0$ more on $r_{\bar{0}}$ or borrow ε less on r_0 or $r_{\bar{0}}$. (Clearly one of these maneuvers is available to him.) The action on r_0 will leave him with εr_0 more money after trade in period 0, to inventory into period 1. The action on $r_{\bar{0}}$ will release $\varepsilon r_{\bar{0}}$ more money just before trade in period 1. In either event he can spend and consume more in each state $s \in S$, a contradiction.

Next consider $s \in S$. If $q^h_{ssm} > 0$, and if $\Delta(6)^h_s > 0$, then let h instead borrow $0 < \varepsilon < \Delta(6)^h_s/(1 + r_s)$ less on r_s, but spend $r_s\varepsilon$ more on commodity purchase in state s. Then h will carry instead $\Delta(6)^h_s - (1 + r_s)\varepsilon$ across trading time, but owe $(1 + r_s)\varepsilon$ less on r_s. Thus these new actions are budget feasible, yet leave h better off, a contradiction. Finally, if $q^h_{ssm} = 0$, there is no purpose in carrying money across trading time in state $s \in S$ (since there is no repayment due on r_s). So h would do better to spend this money on purchasing commodities. □

Lemma 3 (No worthless cash at end). *$\Delta(7)^h_s = 0$ $\forall s \in S$ and $\forall h \in H$.*

Proof. Suppose $\Delta(7)^h_s > 0$. Then h can borrow a little more on r_s, use the money to buy more commodities in state s (leaving all his other actions unchanged), without violating the inequality $(7)^h_s$, i.e., with enough money at hand to repay the extra loan. This improves his utility, a contradiction. □

We define two ME of an economy to be *equivalent* if they give rise to the same interest rates, prices and consumptions. (The equivalence requires somewhat more than identical outcomes, since interest rates and prices must coincide even at markets which are shut.)

Lemma 4 (No short deposits and no wash sales on the long bond). *If $r_{\bar{0}} > r_0 > 0$, there exists an equivalent* ME *such that no agent deposits money on short loans: $q^h_{sms} = 0$ for all $h \in H$ and $s \in S^*$; and no agent both borrows and deposits on the long loan: $q^h_{0\bar{0}m} q^h_{0m\bar{0}} = 0$ for all $h \in H$.*

Proof. Consider any $s \in S^*$, and any compatible $n \in N$. If both q^h_{snm} and q^h_{smn} are positive and $q^h_{snm}/(1+r_s) > q^h_{smn}$, set $q^h_{smn} = 0$ and reduce q^h_{snm} by $(1+r_s)q^h_{smn}$ to get the same ME; and do the reverse reduction if "$<$" holds; and set both equal to zero if "$=$" holds. Clearly we obtain an equivalent ME with no wash sales on loans.

Next suppose, for some $s \in S$, that some $q^h_{sms} > 0$ (and so, in view of the above reduction, $q^h_{ssm} = 0$). But then h will be left with worthless cash at the end of state s (i.e., with the return $(1 + r_s)q^h_{sms}$ on his deposit but no loan to repay), which cannot happen at an ME (see Lemma 3).

Finally suppose some $q^h_{0m0} > 0$. By our argument above he is not borrowing on r_0, nor is he depositing on r_s in *any* $s \in S$. Thus $(1 + r_0)q^h_{0m0}$ is inventoried for delivery/purchase in period 1. It then follows that he would do better to shift his deposit from 0 to $\bar{0}$, earning the interest rate $r_{\bar{0}} > r_0$, ending up with more money for consumption of commodities in every state in period 1, a contradiction. □

Lemma 5 (No wash sales on commodities). *Suppose* $r \gg 0$. *Then* $q^h_{sm\ell} q^h_{s\ell m} = 0$ *for all* $h \in H$, $s \in S^*$ *and* $\ell \in L$.

Proof. Suppose $q^h_{sm\ell} q^h_{s\ell m} > 0$ and $q^h_{ssm} > 0$ for some $s \in S^*$. (Note that if $q^h_{s\ell m} > 0$ and $s \in S$, then necessarily $q^h_{ssm} > 0$, for the only reason to sell in $s \in S$ is to pay off the loan r_s.) Let h borrow ε less on r_s (i.e., reduce q^h_{ssm} by $(1 + r_s)\varepsilon$), spend ε less on the purchase of $s\ell$, and sell $[(1 + r_s)\varepsilon]/p_{s\ell m}$ less of $s\ell$. This is clearly budget-feasible. But then h ends up with $r_s\varepsilon/p_{s\ell m}$ more of $s\ell$ for consumption, improving his utility, a contradiction.

Next if $q^h_{00m} = 0$, but $q^h_{0\bar{0}m} > 0$ and $q^h_{0m\ell} q^h_{0\ell m} > 0$, notice that h must be inventorying all his sales revenue from period 0 into period 1 (since there is no repayment due on r_0), and he must be using this money to repay the loan on $r_{\bar{0}}$ (since, as the proof of Lemma 4 shows, there is a reduction of h's strategy with the same actions on commodity markets, but with no short depositing). So again let him borrow ε less on $r_{\bar{0}}$, spend ε less on 0ℓ, reduce his (inventoried) sales revenue from 0ℓ by $(1 + r_{\bar{0}})\varepsilon$, and wind up consuming $r_{\bar{0}}\varepsilon/p_{0\ell m}$ more of 0ℓ, a contradiction.

Finally, if $q^h_{00m} = q^h_{0\bar{0}m} = 0$, then $q^h_{0m\bar{0}} < m^h_0$ since h is spending money on commodity purchase. Let him deposit ε more on $r_{\bar{0}}$, spend ε less on 0ℓ and sell $\varepsilon/p_{0\ell m}$ less of 0ℓ. Then in each $s \in S$ he can spend $r_{\bar{0}}\varepsilon$ more on purchases, a contradiction. □

Lemma 6 (No wash sales on assets). *Suppose* $r_{\bar{0}} > 0$ *and* $r_0 > 0$. *Then, there exists an equivalent* ME *in which* $q^h_{0mj} q^h_{0jm} = 0$ *for all* $h \in H$ *and* $j \in J$.

Proof. Same as Lemma 5. □

Lemma 7 (Inventorying implies fully depositing on the long loan but also borrowing on the short loan). *Suppose* $r_{\bar{0}} > r_0 > 0$. *Then there is an equivalent* ME *in which*[17]

$$\Delta(3)^h > 0 \Rightarrow q^h_{0\bar{0}m} = 0,\ q^h_{0m\bar{0}} = m^h_0, \text{ and } q^h_{00m} > 0.$$

Proof. If h is inventorying money, i.e., $\Delta(3)^h > 0$, he must already have obtained the money just after sales in period 0, that is, in time to use it to repay the short loan in period 0. Furthermore, since by Lemma 4 he is not depositing on the short loans in period 1, this inventoried money is also available for repayment on the loan $r_{\bar{0}}$.

Since $\Delta(3)^h > 0$, it would be a mistake to borrow long at rate $r_{\bar{0}}$ when he could instead borrow ε less on $r_{\bar{0}}$ and ε more on $r_0 < r_{\bar{0}}$ and repay out of the inventoried money, leaving a profit of $(r_{\bar{0}} - r_0)\varepsilon$. Thus $q^h_{0\bar{0}m} = 0$.

Similarly, if $q^h_{0m\bar{0}} < m^h_0$, h could instead deposit ε more on $r_{\bar{0}}$ and borrow ε more on r_0, repaying $(1 + r_0)\varepsilon$ out of the inventoried money and then receiving $(1 + r_{\bar{0}})\varepsilon$ in long-bond payments, thus gaining $(r_{\bar{0}} - r_0)\varepsilon$ for later use. Hence $q^h_{0m\bar{0}} = m^h_0$.

[17] If $m^h_0 > 0$ then, since $q^h_{0m\bar{0}} q^h_{0\bar{0}m\bar{0}} = 0$ (Lemma 3), $q^h_{0m\bar{0}} = m^h_0$ implies $q^h_{0\bar{0}m} = 0$ automatically. The conclusion $q^h_{0\bar{0}m} = 0$ has content in the case $m^h_0 = 0$.

Since h is depositing all his money m_0^h on the long bond, and not borrowing on the long bond, the only way he could be purchasing commodities at state 0 is by borrowing on the short loan, i.e., $q_{00m}^h > 0$. □

Lemma 8 (Sales imply short borrowing). *If* $0 < q_{s\ell m}^h$ *for some* $\ell \in L$, $s \in S$, *then* $q_{ssm}^h > 0$. *If* $r_{\bar{0}} > r_0 > 0$, *then there is an equivalent* ME *such that*

$$\left\{ \begin{array}{ll} 0 < q_{0jm}^h & \textit{for some } \; j \in J \\ \textit{or} & \\ 0 < q_{0\ell m}^h & \textit{for some } \; \ell \in L \end{array} \right\} \Rightarrow q_{00m}^h > 0.$$

Proof. Sales revenue for $s \in S$ are too late for anything except repayment of loans on r_s. Hence nobody would sell in $s \in S$ unless he had borrowed on r_s.

At state $s = 0$ an agent might sell in order to inventory the money. But from Lemma 7 we know that if he inventories he also borrows on the short loan. And if he does not inventory money, the sales revenue must go towards paying off $q_{00m}^h > 0$. □

Lemma 9 (Short borrowing implies sales). *Suppose* $r \gg 0$. *If* $q_{ssm}^h > 0$ *for any* $h \in H$ *and* $s \in S^*$, *then* $q_{s\ell m}^h > 0$ *for some* ℓ *if* $s \in S$, *and either* $q_{0\ell m}^h > 0$ *for some* $\ell \in L$ *or* $q_{0jm}^h > 0$ *for some* $j \in J$.

Proof. By Lemma 2, h is spending all his money on hand in purchases. It follows that he must have positive sales revenue if he took out a short loan, otherwise he would not be able to repay it. □

Lemma 10 (Interest rate wedge for buying and selling). *Suppose agent* h *buys good* $s1$ *in every state* $s \in S^*$. *Then if* $r_{\bar{0}} \geq r_0 \geq 0$,

(i) $0 < q_{s\ell m}^h < e_{s\ell}^h \Rightarrow \frac{\nabla_{s1}^h}{p_{s1m}} = (1 + r_s) \frac{\nabla_{s\ell}^h}{p_{s\ell m}}$

(ii) $0 < q_{0jm}^h \Rightarrow \frac{\nabla_{01}^h}{p_{01m}} = \frac{1+r_0}{p_{0jm}} \sum_{s \in S} A_s^j \frac{\nabla_{s1}^h}{p_{s1m}}$

(iii) $\left\{ \begin{array}{c} q_{0\bar{0}m}^h > 0 \\ \textit{or} \\ q_{0m\bar{0}}^h < m_0^h \end{array} \right\} \Rightarrow \frac{\nabla_{01}^h}{p_{01m}} = (1 + r_{\bar{0}}) \sum_{s \in S} \frac{\nabla_{s1}^h}{p_{s1m}}$

(iv) $\Delta(3)^h > 0 \Rightarrow \frac{\nabla_{01}^h}{p_{01m}} = (1 + r_0) \sum_{s \in S} \frac{\nabla_{s1}^h}{p_{s1m}}$

Proof. In (i), if LHS > RHS, then h could borrow ε more on r_s, buying ε / p_{s1m} more units of good $s1$, and repay the incremental loan by selling $(1 + r_s)\varepsilon / p_{s\ell m}$ more of $s\ell$, improving his utility, a contradiction. If LHS < RHS, h could spend ε less on $s1$, and sell $(1 + r_s)\varepsilon / p_{s\ell m}$ less of $s\ell$. The reduction ε in expenditure could be achieved by borrowing less on r_s, or by depositing ε more on r_s. This increases utility while preserving all his cash flows, a contradiction.

To argue (ii) and (iii), note that the money spent on commodity purchases $s1$ in every state $s \in S$ could have been used for deliveries on the long loan or assets. If it was obtained by borrowing on r_s, or inventorying from state 0, that occurs before deliveries. If the money was obtained as delivery, then on account of our assumption that deliveries are netted, it is by definition available to deliver.

In (ii), if LHS $>$ RHS, then h could borrow ε more on r_0, buy ε/p_{s1m} more of 01, defray the loan by selling $(1+r_0)\varepsilon/p_{0jm}$ of asset j, and deliver on the incremental sale of asset j by reducing his expenditures on $s1$ by $A_s^h(1+r_0)\varepsilon/p_{0jm}$ for each $s \in S$, raising his utility, a contradiction. If LHS $<$ RHS, the reverse argument holds, as in (i).

In (iii), if LHS $>$ RHS, the agent can borrow ε more on $r_{\bar{0}}$, spend the money on 01, and defray the loan by purchasing $(1+r_{\bar{0}})\varepsilon/p_{s1m}$ less of each good $s1$, for all $s \in S$, a contradiction. The argument reverses (either by borrowing ε less or depositing ε more on $r_{\bar{0}}$) to show that we cannot have LHS $<$ RHS, just as in (i).

In (iv), if LHS $>$ RHS, h can improve by borrowing ε more on r_0, spending ε more on good 01, defraying the loan by inventorying $(1+r_0)\varepsilon$ less, and reducing expenditure on every good $s1$, for $s \in S$, by $(1+r_0)\varepsilon$, a contradiction. If LHS $<$ RHS, we consider two cases. Suppose $r_{\bar{0}} > r_0$. By Lemma 7, $q^h_{00m} > 0$. Then h can borrow ε less on r_0, spend ε less on good 01, and inventory $(1+r_0)\varepsilon$ more into each state $s \in S$ to spend on $s1$. This improves his utility, a contradiction. If $r_{\bar{0}} = r_0$, and $q^h_{0m\bar{0}} < m^h_0$, then (iv) has been proved in (iii). If $q^h_{0m\bar{0}} = m^h_0$, then in order to purchase 01 he must be borrowing either on r_0 or on $r_{\bar{0}}$. If $q^h_{00m} > 0$, we have just proved (iv) by showing that h could improve. If $q^h_{0\bar{0}m} > 0$, then (iv) follows from (iii). □

Implications (iii) and (iv) appear contradictory when $r_{\bar{0}} > r_0$. But by Lemma 7, $\Delta(3)^h > 0$ implies $q^h_{0\bar{0}m} = 0$ and $q^h_{0m\bar{0}} = m^h_0$, so there is in fact no contradiction.

7 The equations of an ME

Given any market action $q \equiv (q^h)_{h\in H}$, define, for $h \in H$ and $s \in S^*$,

$$
\begin{aligned}
L^h_s(+) &= \{\ell \in L : q^h_{sm\ell} > 0\}\\
L^h_s(-) &= \{\ell \in L : 0 < q^h_{s\ell m} < e^h_{s\ell}\}\\
J^h(+) &= \{j \in J : q^h_{0mj} > 0\}\\
J^h(-) &= \{j \in J : q^h_{0jm} > 0\}\\
N^h(-) &= \{n \in N : q^h_{snm} > 0 \text{ if } n = s \in S^*;\ q^h_{0\bar{0}m} > 0 \text{ if } \ n = \bar{0}\}\\
N^h(+) &= \{\bar{0}\} \text{ if } m^h_0 > q^h_{0m\bar{0}} > 0,\ N^h(+) = \emptyset \text{ otherwise}
\end{aligned}
$$

Definition. The variables in $\cup_{s\in S^*}(L^h_s(+)\cup L^h_s(-))\cup J^h(+)\cup J^h(-)\cup N^h(-)\cup N^h(+)$ will be called *free variables for* h.

In view of Lemmas 4, 5 and 6, we restrict attention to q^h where no agent acts on both sides of any market. Thus the sets $L^h_s(+), L^h_s(-)$ are disjoint; so are $J^h(+), J^h(-)$; and if $\bar{0} \in N^h(-)$, then $N^h(+) = \emptyset$. Furthermore, in view of Lemma 1, we suppose $L^h_s(+) \neq \emptyset$ for q^h; and in view of Lemma 4, $q^h_{sms} = 0$ for all $s \in S^*$.

As before[18] we use the notation: $1 \in L_s^h(+)$ for $s \in S^*$, $L \in L_s^h(-)$ if $L_s^h(-)$ is nonempty; and $\bigtriangledown_{s\ell}^h \equiv (\partial u^h / \partial x_{s\ell})(x^h)$, where x^h is the final consumption of h at the ME under consideration.

$(12)_s^h$ For $s \in S^*$, the first-order conditions for commodities require

(a) $\frac{\bigtriangledown_{s1}^h}{p_{s1m}} - \frac{\bigtriangledown_{s\ell}^h}{p_{s\ell m}} = 0$ if $\ell \in L_s^h(+) \setminus \{1\}$;

and if $L \in L_s^h(-)$, i.e., $L_s^h(-) \neq \emptyset$

(b) $\frac{\bigtriangledown_{sL}^h}{p_{sLm}} - \frac{\bigtriangledown_{s\ell}^h}{p_{s\ell m}} = 0$ if $\ell \in L_s^h(-) \setminus \{L\}$

(c) $\frac{\bigtriangledown_{s1}^h}{p_{s1m}} - (1 + r_s)\frac{\bigtriangledown_{sL}^h}{p_{sLm}} = 0$

(Parts (a), (b) are obvious; (c) was proved in Lemma 10(i).)

For what follows, recall the budget set inequalities (1)–(7), and our notation: $\Delta(\nu) \equiv$ difference between the right and left hand sides of inequality ν.

$(13)^h$ Agent h spends all the money at hand on purchases in state 0, by Lemma 2:

$$\Delta(2)^h \equiv \Delta(1)^h + \frac{q_{0\bar{0}m}^h}{1 + r_{\bar{0}}} + \frac{q_{00m}^h}{1 + r_0} - (\Sigma_{\ell \in L_0^h(+)} q_{0m\ell}^h + \Sigma_{j \in J^h(+)} q_{0mj}^h) = 0$$

$(14)^h$ Either agent h does not inventory any money from period 0 into period 1, i.e.,

(a) $\Delta(3)^h \equiv \Delta(2)^h + \Sigma_{\ell \in L_0^h(-)} p_{0\ell m} q_{0\ell m}^h + \Sigma_{j \in J^h(-)} p_{0jm} q_{0jm}^h - q_{00m}^h = 0$

or, $\Delta(3)^h > 0$ and by Lemma 10(iv), we have the first-order condition

(b) $\frac{\bigtriangledown_{01}^h}{p_{01m}} - (1 + r_0)\tilde{1} \cdot \left(\frac{\bigtriangledown_{s1}^h}{p_{s1m}}\right)_{s \in S} = 0$

where '$\cdot$' denotes dot product and (recall) $\tilde{1}$ is the unit vector in $\mathbb{R}^S$.

$(15)_s^h$ Agent h spends all the money at hand on purchases in each state $s \in S$ by Lemma 1:

$$\Delta(3)^h + m_s^h + \frac{q_{ssm}^h}{1 + r_s} - \left((q_{0\bar{0}m}^h - (1 + r_{\bar{0}}) q_{0m\bar{0}}^h) + \Sigma_{j \in J} \left(q_{0jm}^h - \frac{q_{0mj}^h}{p_{0jm}} \right) A_s^j \right) - \Sigma_{\ell \in L_s^h(+)} q_{sm\ell}^h = 0.$$

$(16)^h$ If $q_{0\bar{0}m}^h > 0$ or if $0 < q_{0m\bar{0}}^h < m_0^h$, then by Lemma 10(iii), the first-order condition for the long bond requires

$$\frac{\bigtriangledown_{01}^h}{p_{01m}} - (1 + r_{\bar{0}})\tilde{1} \cdot \left(\frac{\bigtriangledown_{s1}^h}{p_{s1m}}\right)_{s \in S} = 0$$

[18] Though a commodity bought by h in state s is $s1$ in our notation, this is not to say that different agents are buying the same commodity in state s. We will not use $s1$ when we compare purchases across agents.

$(17)_s^h$ Agent h does not end up with any (worthless) cash in any state in S by Lemma 3:

$$\Delta(6)_s^h + \sum_{\ell \in L} p_{s\ell m} q_{s\ell m}^h - q_{ssm}^h = 0.$$

$(18)^h$ The first-order conditions for assets require

(a) $\frac{\nabla_{01}^h}{p_{01m}} - \frac{1}{p_{0jm}} A^j \cdot \left(\frac{\nabla_{s1}^h}{p_{s1m}}\right)_{s \in S} = 0$ if $j \in J^h(+)$

(b) $\frac{\nabla_{01}^h}{p_{01m}} - \frac{(1+r_0)}{p_{0jm}} A^j \cdot \left(\frac{\nabla_{s1}^h}{p_{s1m}}\right)_{s \in S} = 0$ if $j \in J^h(-)$

For $(18)^h$(b), we invoke Lemma 10(ii); and $(18)^h$(a) is obvious.

8 Determinacy with interest rate targets

In this section we shall prove Theorem 1.

The strategy of our proof is to represent monetary equilibrium as the solution to a system of simultaneous equations with the same number of unknowns, and then to apply the transversality theorem to prove that "generically" the solution to this system is a zero-dimensional manifold. Later we shall argue that it must be a finite set.

One minor complication is that equilibrium is in fact a solution of one of a finite number of systems of equations (each system defined over a different domain of variables) called regimes. But this is no problem, since the finite union of finite sets is still finite.

Our exogenous variables are $u = (u^h)_{h \in H}$, as we hold $((r_n)_{n \in N}$, $(A^j)_{j \in J}, (e^h, m^h)_{h \in H})$ fixed.

Our endogenous variables are $((Q_{snm}, Q_{smn})_{n \in N}, (p_{s\ell m})_{s\ell \in S^* L}$, $(p_{0jm})_{j \in J}, (q^h, x^h)_{h \in H})$. For each of a finite number of *regimes*, we shall partition the endogenous actions $(q^h)_{h \in H}$ into *free* variables and *fixed* variables. All other endogenous variables will be *forced*. The number of corresponding equations will be equal to the number of free variables.

8.1 Legitimate specifications of actions

Recall that $\mathcal{M}$ is our set of markets, where $s\alpha\beta$ is identified with $s\beta\alpha$. Now we want to distinguish $s\alpha\beta$ from $s\beta\alpha$, so let

$$\mathcal{M}^* \equiv \{s\alpha\beta \in S^* \times I \times I : s\alpha\beta \in \mathcal{M}\}.$$

For each agent h and each $s\alpha\beta \in \mathcal{M}^*$, agent h can take an action $q_{s\alpha\beta}^h \geq 0$, which is *a priori* bounded above by $e_{s\ell}^h$ (if $s\alpha\beta = s\ell m$) or by m_0^h (if $s\alpha\beta = 0m\bar{0}$). Let B be the finite collection of these bounds: $B = \{0\} \cup \{m_0^h : h \in H\} \cup \{e_{s\ell}^h : s \in S^*, \ell \in L, h \in H\}$. Intuitively, in equilibrium each action $q_{s\alpha\beta}^h$ could be interior (which we have called *free*), or coincident with one of its bounds (which we call fixed). We need to distinguish free from fixed, because the first-order conditions apply only to free variables.

To make all this precise, consider a *specification*

$$\sigma : H \times \mathcal{M}^* \to \{\textit{free}\} \cup B.$$

Note that there are only a finite number of specifications.

A specification σ is called *legitimate* iff all the following conditions (i)–(vii) hold:

(i) For commodity markets $(s\ell m)_{s\ell \in S^* \times L}$ the pair $(\sigma(hs\ell m), \sigma(hsm\ell))$ can take on only one of four specifications:

$$\{(0, \textit{free}), (\textit{free}, 0), (e^h_{s\ell}, 0), (0, 0)\}.$$

(ii) For asset markets $(0jm)_{j\in J}$ the pair $(\sigma(hsjm), \sigma(hsmj))$ can take on only one of three specifications:

$$\{(0, \textit{free}), (\textit{free}, 0), (0, 0)\}.$$

(iii) For the long-loan market $(0\bar{0}m)$ the pair $(\sigma(h0\bar{0}m), \sigma(h0m\bar{0}))$ can take on only one of four specifications:

$$\{(0, \textit{free}), (\textit{free}, 0), (m^h_0, 0), (0, 0)\}$$

(iv) For the short-loan markets $(ssm)_{s\in S^*}$, $(\sigma(hssm), \sigma(hsms))$ can take on only two specifications:

$$\{(\textit{free}, 0), (0, 0)\}.$$

(v) For all $s\alpha\beta \in \{s\ell m\}_{s\ell\in S^* L} \cup \{0jm\}_{j\in J}$, $[\sigma(hs\alpha\beta) = 0 \ \forall h \in H] \Rightarrow [\sigma(hs\beta\alpha) = 0$ for all $h \in H]$ (i.e., the market is shut).

(vi) $(\forall h \in H)$, $(\forall s \in S^*)$ $(\exists \ell = \ell(h, s)$ for which $\sigma(hsm\ell) = \textit{free})$.

(vii) $[\sigma(hssm) \neq 0] \Leftrightarrow [\sigma(hs\ell m) \neq 0$ for some $\ell \in L$ or $\sigma(hsjm) \neq 0$ for some $j \in J]$.

Let $\Omega = \{(q^h_{s\alpha\beta})^{h\in H}_{s\alpha\beta\in\mathcal{M}^*} \in \mathbb{R}^{H\times\mathcal{M}^*}_+ : q^h_{0m\bar{0}} \leq m^h_0$ and $q^h_{s\ell m} \leq e^h_{s\ell}$, for all $h \in H$, $s \in S^*$, $\ell \in L\}$. Given $q \in \Omega$, we define the *induced specification* $\tilde{\sigma} = \tilde{\sigma}(q)$ by

$$\tilde{\sigma}(hs\alpha\beta) = \begin{cases} \textit{free} \text{ if } q^h_{s\alpha\beta} > 0 \\ 0 \text{ if } q^h_{s\alpha\beta} = 0 \end{cases}$$

for all $hs\alpha\beta \in H \times \mathcal{M}^*$, *except* (for all $h \in H$, $s \in S^*$, $\ell \in L$)

$$\tilde{\sigma}(h0m\bar{0}) = m^h_0 \text{ if } q^h_{0m\bar{0}} = m^h_0$$

and

$$\tilde{\sigma}(hs\ell m) = e^h_{s\ell} \text{ if } q^h_{s\ell m} = e^h_{s\ell}.$$

Notice that if $q = (q^h)_{h\in H}$ gives the market actions at an active ME, then $q \in \Omega$ and the induced specification $\sigma^*(q)$ is legitimate. The reason is that (i), (ii), (iii), and (iv) follow from the no-wash-sales Lemmas 4, 5, 6; (v) follows from market clearing, (vi) holds at any active ME, and (vii) follows from Lemmas 8 and 9.

Any specification partitions each agent's actions $q^h_{s\alpha\beta}$ into free and fixed variables. The remaining variables $(p_{s\ell m}, p_{0jm}, x^h_{s\ell}, Q_{smn}, Q_{snm})$ are all called *forced* variables, since they are completely determined as smooth functions of the free and fixed variables by the requirement that markets clear and consumption is not wasteful:

$$p_{s\alpha\beta} = \frac{\sum_{h\in H} q^h_{s\beta\alpha}}{\sum_{h\in H} q^h_{s\alpha\beta}} \text{ if } s\alpha\beta \text{ is not shut by } q \text{ and if } \{\alpha,\beta\}\cap N = \phi$$

$$x^h_{s\ell} = \left\{ \begin{array}{ll} e^h_{s\ell} - q^h_{s\ell m} + q^h_{sm\ell}/p_{s\ell m} & \text{if } s\ell m \text{ is not shut} \\ e^h_{s\ell} & \text{otherwise} \end{array} \right\}$$

$$Q_{smn} = \max\left(\frac{1}{1+r_n}\sum_{h\in H} q^h_{snm} - \sum_{h\in H} q^h_{smn}, 0\right)$$

$$Q_{snm} = \max\left((1+r_n)\sum_{h\in H} q^h_{smn} - \sum_{h\in H} q^h_{snm}, 0\right).$$

The *regime* is defined by its free variables, by the values assigned to its fixed variables, and by the set of equations which must hold. These equations are necessary conditions for each agent h to be optimizing. (Market clearing has already been taken care of by the forced variables.) It is critical to observe that there will be precisely one equation for each free variable, given by the following table.

Variable	Equation	
$q^h_{sm\ell}, \ell \neq 1$	$(12)^h_s$(a)	
$q^h_{sm\ell}, \ell \neq L$	$(12)^h_s$(b)	
q^h_{sLm}	$(12)^h_s$(c)	
q^h_{0m1}	$(13)^h$	
q^h_{00m}	$(14)^h$(a)	or $(14)^h$(b)
q^h_{sm1}	$(15)^h_s$	
$q^h_{0\bar{0}m}$	$(16)^h$	
$q^h_{0m\bar{0}}$	$(16)^h$	
q^h_{ssm}	$(17)^h_s$	
q^h_{0mj}	$(18)^h$(a)	
q^h_{0jm}	$(18)^h$(b)	

The table associates one equation to each variable of a given agent h, except for q^h_{00m}, where either one of two equations might apply. For notational convenience we will continue to suppose that (as in Sect. 8) his designated purchase in each state is $\ell(h,s) = 1$, and that if he sells interior amounts of commodities in any state, one of them may be called L.

In any regime only some of these variables are free, and then only the corresponding equations are considered. Observe that at most one of the $q^h_{0\bar{0}m}$ and $q^h_{0m\bar{0}}$ is free, hence there is no problem in associating both with the same equation.

Whenever $(16)^h$ is operative (and hence either $q^h_{0\bar{0}m}$ or $q^h_{0m\bar{0}}$ is free), we assign $(14)^h$(a) to q^h_{00m}. But if both $q^h_{0\bar{0}m}$ and $q^h_{0m\bar{0}}$ are fixed (and hence $(16)^h$ is missing), and q^h_{00m} is free, then we need to consider two separate regimes: one with $(14)^h$(a) matched to q^h_{00m}, and the other with $(14)^h$(b) matched to it. By Lemma 7 an agent never inventories money between periods 0 and 1, while at the same time having a free variable on the long loan, i.e., either borrowing, or depositing an interior amount, on the long loan. Thus equation $(14)^h$(b), which comes from the first-order conditions for inventorying, is never invoked simultaneously with $(16)^h$, the first-order condition on the long loan.

8.2 Transversality of the equilibrium map

For each legitimate specification σ, define the domain

$$D(\sigma) \equiv D = \{q \in \Omega : \tilde{\sigma}(q) = \sigma\}.$$

This set is isomorphic to an open set in Euclidean space with dimension $d =$ the number of free variables specified by σ.

Consider a matching of the free variables in σ to equations, as in the above table. The specification σ, and the matching, together define a regime. For the regime we consider the map $\psi : \mathcal{E} \times D \to \mathbb{R}^d$ given by $\psi(u, q) =$ LHS of the d equations in the matching. (Note that the LHS are continuously differentiable functions of u and q.) It is evident that if $q \in D$ is an active ME of $u \in \mathcal{E}$, then $\psi(u, q) = 0$, i.e., $q \in \psi_u^{-1}(0)$ where $\psi_u(q) \equiv \psi(u, q)$. Since dimension $D = d$, $\psi_u^{-1}(0)$ will be a zero-dimensional manifold provided $\psi_u : D \to \mathbb{R}^d$ is transverse to 0, which in turn will follow for almost all $u \in U$ (by the transversality theorem) if we can show that each map ψ is transverse to 0. This is the task to which we now turn.

It will suffice to show how to unilaterally perturb each of the equations of agent h.

To perturb the first-order conditions for commodities, $(12)^h_s$(a) or $(12)^h_s$(b), adjust $\nabla^h_{s\ell}$, $\ell \notin \{1, L\}$. This does not disturb any of the other equations.

For $(12)^h_s$(c), adjust ∇^h_{sL}. This will not disturb any other equation except $(12)^h_s$(b), which we compensate via $\nabla^h_{s\ell}$ for $\ell \in L^h_s(-) \setminus \{L\}$.

Next consider the set of first-order conditions for inventorying or the long loan, and the assets: *one of* $(14)^h$(b) or $(16)^h$ (since, as we saw, they are never invoked together), along with $(18)^h$(a) and $(18)^h$(b). Recall that the vectors $\tilde{1}$, $A^1, \ldots, A^J$ are linearly independent. Therefore, by adjusting $(\nabla^h_{s1})_{s \in S}$, we can adjust $(\nabla^h_{s1}/p_{s1m})_{s \in S}$ in a direction perpendicular to all but one of these vectors, and thereby unilaterally perturb any one of the equations from this set. In the process $(12)^h_s$(a) and $(12)^h_s$(c) are disturbed (for $s \in S$). We restore these by adjusting $\nabla^h_{s\ell}$ for $\ell \in L^h_s(+) \setminus \{1\}$, and ∇^h_{sL} respectively. The latter further disturbs $(12)^h_s$(b) which is restored via $\nabla^h_{s\ell}$ for $\ell \in L^h_s(-) \setminus \{L\}$.

Now we perturb the no worthless cash conditions $(14)^h$(a), $(17)^h_s$ one at a time. Fix $s \in S^*$ and $h \in H$ with $q^h_{ssm} > 0$. Let h vary (say increase) q^h_{ssm} by an infinitesimal $\varepsilon > 0$, spending $\varepsilon/(1 + r_s)$ more on his designated purchase $\ell = \ell(h, s) = 1$. By (v) there must be sellers k of ℓ, and they now receive $\varepsilon^k = (q^k_{s\ell m}/\sum_{i\in H} q^i_{s\ell m})(\varepsilon/(1+r_s))$ more money. By property (vii), every seller of ℓ is a borrower on r_s. So their q^k_{ssm} can be varied infinitesimally in either direction. Let them each increase q^k_{ssm} by ε^k, spending $\varepsilon^k/(1 + r)$ on their designated commodities. That gives another set of sellers more money. Iterate the process infinitely often.

Summing all of the infinite changes, but relying on the telescoping property of geometric sums, we see that total bond sales $\sum_{k\in H} q^k_{ssm}$ increase by precisely $\sum_{t=0}^{\infty} \varepsilon/(1+r)^t = \varepsilon((1+r)/r)$. Thus total expenditures on commodities go up by ε/r. It follows that the forced variables, $p_{s\ell m}$ and $x^k_{s\ell m}$, also change infinitesimally, for all $\ell \in L$, $k \in H$. Perturbing utilities, we can restore the old ratios $\nabla^h_{s\ell}/p_{s\ell}$.

The important thing to note is that the whole infinite process does not affect the cash position of any agent $k \in H$ except for h. At every step of the iteration *after* h's initial purchase, each agent k (including possibly h himself) increases his bond sales q^k_{ssm} by precisely the amount of extra money he received in sales revenue in the previous iteration. Thus in the limit h decreases his cash balance at the end of state s by ε, and the cash balance of every $k \neq h$ remains unchanged.

Finally, it remains to perturb the "spend all the cash at hand" equations $(13)^h$ and $(15)^h_s$ one at a time. But this can be done exactly the same way we perturbed $(14)^h$(a) and $(17)^h_s$. Pick a state $s \in S^*$ and any $h \in H$. Let him vary (say increase) his spending on his designated purchase $\ell = \ell(h, s) = 1$, without borrowing the extra money. (If he decreases his spending, let him throw away the saved money.) This perturbs the corresponding equation that he spend exactly his cash on hand. The sellers of ℓ get more money, and they respond exactly in the same way as in the infinite process described earlier.

Thus we see that the map ψ is indeed transverse to 0, proving the determinacy of active ME outcomes.

8.3 Generic finiteness of ME

It is important to observe that we could have reduced the number of variables without subtracting equations and still proved ψ is transverse to 0. Indeed, fixing some $q^h_{s\alpha m} = 0$ or $q^h_{sm\alpha} = 0$ with $\alpha \in \{\bar{0}\} \cup J \cup L\backslash\{\ell(h, s)\}$ does not alter the transversality argument, since none of these variables is needed in the perturbation argument. Of course the consequence is that when there are more equations than free variables, the set of solutions is empty generically in U.

We proved in Section 8.2 that, generically, the set $\psi_u^{-1}(0)$ is a 0-dimensional manifold, and hence consists of isolated points in the open set D. Suppose that infinitely many of these points correspond to distinct active ME, and that the economy u has only active ME. If we could show that the q's of ME's are uniformly bounded, then we would have a convergent subsequence of points in $\psi_u^{-1}(0)$. By continuity, the limit point $\bar{q}$ would satisfy all the equations in ψ_u, and hence be an (active) ME. But since $\psi_u^{-1}(0)$ consists entirely of isolated points, we deduce that $\bar{q} \notin D$. Since $\bar{q}$ is bounded, this means that some components $\bar{q}^h_{s\alpha\beta}$ of $\bar{q}$ have hit their upper or lower bounds. A crucial observation is that one of these must be a commodity variable, or an asset variable, or the long-bond variable. (For recall that, by (vii), if short borrowing becomes zero for some agent in some state s, then some commodity or asset sale must simultaneously become zero in that state.) Thus we would get the situation described in the last paragraph, which generically cannot happen.

We can conclude our proof that generically there are only a finite number of ME (when all ME are active) by arguing that for any fixed (u, e, m, A, r), ME market actions q are uniformly bounded. The only potentially unbounded variables are asset sales and bond sales. But notice that the stock of outside money must be bounded at each $s \in S^*$, since the maximum deposits are equal to the initial stocks of endowment money, and the maximum bank interest rate $\bar{r} = \max_{n \in N}\{r_n\}$ is given. Since all inside money loaned at the banks must be returned with strictly positive interest at least equal to $\underline{r} \equiv \min_{n \in N}\{r_n\} > 0$, the total amount of inside money available to spend on the path in the tree terminating at s is at most

$$M(s) \leq \frac{\left(\left(\sum_{h \in H} m_0^h\right)(1+\bar{r}) + \sum_{h \in H} m_s^h\right)}{\underline{r}} < \infty.$$

Since there is no default, bank bond sales must be less than the sum of inside and outside money, i.e., less than $M(s) + (\sum_{h \in H} m_0^h)(1+\bar{r}) + \sum_{h \in H} m_s^h$, and hence are bounded. Finally, asset sales are also bounded, because their payoffs are linearly independent by hypothesis. If some asset sale converged to infinity, then even if asset purchases also converged to infinity, some agent would be required to deliver money converging to infinity in some state, which is impossible. □

9 Determinacy with quantity targets

We turn to the proof of Theorem 2. The difference from Theorem 1 is that now we must allow for the possibility that $r_s = 0$, and that $r_{\bar{0}} = r_0$. (Of course $r_{\bar{0}} \geq r_0$ in ME, for otherwise nobody would borrow on the short loan, and with $M_0 > 0$ that market would not clear.) It is evident that Lemmas 1, 3, and 10 remain intact. These are the most important lemmas, because they include all the first-order conditions for ME, the "no worthless cash at the end" condition, and the fact that every agent buys something in each state. The only equation to be lost is "spend all the money at hand," from Lemma 2 (the quantity theory of money). With $r_s = 0$, it might well be that an agent borrows money at zero interest and hoards it without spending it, returning it when due.

At the same time we seem to have many more free variables, because the no-wash-sales Lemmas 4–6 no longer hold. We also lose Lemmas 8 and 9, which allowed us to conclude that if short borrowing goes to zero, then so must the sale of some commodity. Finally, Lemma 7 no longer holds. Hence we must worry that, if $r_0 = r_{\bar{0}}$, an agent's action variables might be free on both the short and long loans in period 0, and that he may be simultaneously inventorying. These two free variables would then correspond to the inventory equation $(14)^h$(b) and the long bond first-order condition $(16)^h$, which (with $r_{\bar{0}} = r_0$) are the same equation. Previously these two equations were never invoked at the same time.

In summary, it now appears that for the proof of Theorem 2 we have fewer equations and more free variables than we had for the proof of Theorem 1. Nevertheless, determinacy can still be proved.

Lemma 11. *Every* ME *has an equivalent* ME *in which there are no wash sales on commodities, assets, or loans, and in which there are no short deposits.*

Proof. The first two paragraphs of the proof of Lemma 4 show that there is indeed an equivalent ME with no wash sales on any loans, and no deposits on short loans in period 1. (This part of the proof holds even if $r_s = 0$ for any $s \in S^*$.)

If $r_{\bar{0}} > r_0$, we already showed in Lemma 4 that there are no deposits on r_0. If $r_{\bar{0}} = r_0$, and some agent is depositing on r_0, then we may switch these deposits to $r_{\bar{0}}$. (This does not hurt the agent because he gets the same interest, though a bit later. However, in the time between, he was not depositing on the short loan in period 1.) Since every dollar deposited must be borrowed (for markets to clear in equilibrium) we can move an equivalent amount of borrowing from r_0 to $r_{\bar{0}}$. These borrowers clearly are not hurt by the shift, since they face the same interest and have longer to repay. Note that the loan markets continue to clear, at the same rates, after the shift. This move eliminates deposits on r_0. If it introduces wash sales on $r_{\bar{0}}$, these can be eliminated as before (in the first paragraph of the proof of Lemma 4).

As for commodities, we already showed there are no wash sales in state s if $r_s > 0$. If $r_s = 0$, let any agent doing wash sales reduce his purchases and sales until one hits zero, using the unspent money to replace the revenue he would have gotten from the eliminated sales. Asset wash sales can be reduced similarly. □

For our next lemma we need a definition. An *exceptional agent* at an ME is an $h \in H$ whose actions on r_0 and $r_{\bar{0}}$ are both free (i.e., not zero and not m_0^h if he is depositing on $r_{\bar{0}}$) and who is inventorying money from period 0 to period 1 (i.e., $\Delta(3)^h > 0$).

Lemma 12. *If an* ME *has exceptional agents, then $r_0 = r_{\bar{0}}$ and there is an equivalent* ME *with at most one exceptional agent.*

Proof. By Lemma 7, there cannot be an exceptional agent if $r_{\bar{0}} > r_0$. So suppose $r_0 = r_{\bar{0}}$. We may assume that the ME has been reduced to an equivalent ME as in Lemma 11. It will suffice to show that, given any two exceptional agents, there is an equivalent ME in which all the *other* agents' actions remain invariant, and at most one of these two is exceptional.

W.l.o.g. let agents 1 and 2 be exceptional. First suppose both are borrowing on $r_{\bar{0}}$. Let 2 borrow ε less on $r_{\bar{0}}$, ε more on r_0, return $(1 + r_0)\varepsilon$ dollars more on r_0, inventory $(1 + r_0)\varepsilon$ less into period 1, and return $(1 + r_0)\varepsilon$ less on $r_{\bar{0}}$ (in all the states $s \in S$). Let 1 do exactly the reverse. Further let all the other actions of 1 and 2, and all the actions of the other agents, remain the same. Agent 1 does not mind shifting his borrowing from $r_{\bar{0}}$ to r_0, even though this requires earlier repayment, as long as he is inventorying money.

It is evident that for small enough ε we remain at an ME without disturbing the outcome, and with both 1 and 2 still exceptional. Increase ε to the smallest level at which at least one of 1 or 2 ceases to be exceptional. Clearly such a level is reached.

Next suppose both 1 and 2 are borrowing on r_0 and 1 is borrowing on $r_{\bar{0}}$ while 2 is depositing on $r_{\bar{0}}$. Shift ε borrowing of 1 from $r_{\bar{0}}$ to to r_0 (with 1 repaying $(1 + r_0)\varepsilon$ earlier and inventorying $(1 + r_0)\varepsilon$ less) and also shift ε borrowing of 2 from r_0 to $r_{\bar{0}}$ (increasing 2's inventorying by $(1 + r_{\bar{0}})\varepsilon$), and keep doing so, until one of them stops being exceptional.

Finally suppose both 1 and 2 are borrowing on r_0 and depositing on $r_{\bar{0}}$. Let 1 deposit ε less on $r_{\bar{0}}$ and borrow ε less on r_0, and let 2 deposit ε more on $r_{\bar{0}}$ and borrow ε more on r_0 and inventory $\varepsilon(1 + r_0)$ less. Keep increasing ε until one of 1, 2 ceases to be exceptional. □

Lemma 13. *At any* ME *there is an equivalent* ME *in which all agents but one spend all their money at hand in every state.*

Proof. If $r_s > 0$, all agents spend all the money at hand in state s by Lemma 2. If $r_s = 0$, choose an agent $h^* \equiv h^*(s)$ arbitrarily to be the hoarder in state s, making sure that $q^{h^*}_{ssm} > 0$. If any agent h is not spending all the money at hand on purchases of commodities or assets, let him borrow ε less, or deposit ε more. In either case, let h^* borrow ε more at the same loan market, hoard and return it. By taking ε large enough, the lemma is proved. □

9.1 The transversality argument

We choose our set of *free* action variables $(q^h_{s\alpha\beta})$. In addition we specify whether there will be hoarders $h^*(s)$, $s \in S^*$, and whether there will be an exceptional agent h^{**}. If the specification excludes both, then we have exactly the same regimes as in Theorem 1. The only difference is that instead of treating $(r_{\bar{0}}, r_0, r_1, ..., r_S)$ as fixed, and Q_{smn}, Q_{snm} as forced, we do the reverse and take the r_n as forced:

$$1 + r_n = \frac{\sum\limits_{h \in H} q^h_{smn}}{M_n + \sum\limits_{h \in H} q^h_{smn}}.$$

If hoarding is specified, we must also specify states $s \in S^*$ in which we will fix $r_s = 0$ and designate the hoarders $h^*(s)$, with $q^{h^*(s)}_{ssm} > 0$ in those states. The "spend all the money at hand" equation for $h^*(s)$ in state s is dropped, but so is the free variable $q^{h^*(s)}_{ssm}$ (since it is now forced by the above equation and the requirement $r_s = 0$). We now associate the free variable $q^{h^*(s)}_{sm1}$ (which had been associated with the "spend all the money at hand" equation) with the equation $(14)^{h^*(s)}$(a $\setminus$ b) that had been earlier associated with $q^{h^*(s)}_{ssm}$.

In case an exceptional agent h^{**} is specified, we drop the redundant equation $(16)^{h^{**}}$. (For all non-exceptional agents $h \in H \setminus \{h^{**}\}$, clearly $(14)^h$(b) and $(16)^h$

are never invoked together, so $(16)^h$ is not redundant.) But we also drop the corresponding free variable $q^{h^{**}}_{0\bar{0}m}$ or $q^{h^{**}}_{0m\bar{0}}$ (by no wash sales, not both are free). The dropped free variable is now forced by the equation

$$1 + r_0 = \frac{\sum_{h\in H} q^h_{00m}}{M_0 + \sum_{h\in H} q^h_{0m0}} = \frac{q^{h^{**}}_{0\bar{0}m} + \sum_{h\in H\setminus h^{**}} q^h_{0\bar{0}m}}{M_{\bar{0}} + q^{h^{**}}_{0m\bar{0}} + \sum_{h\in H\setminus h^{**}} q^h_{0m\bar{0}}} = 1 + r_{\bar{0}}.$$

(Recall that there cannot be an exceptional agent unless $r_0 = r_{\bar{0}}$.)

The situation thus remains tight, i.e., # equations = # free variables. The perturbations of the first-order conditions for commodities, assets, the long loan, and inventorying can be done via utilities exactly as in Theorem 1. A difficulty arises in perturbing the "spend all the money at hand" equations, and also in perturbing the "no worthless cash at end" equations. In Theorem 1 we relied on $r_s > 0$ in order to construct an infinite chain that summed to a finite difference. In Theorem 2 we do not have that luxury, since we may well have $r_s = 0$. Thus we are forced to vary m^h_s, which explains the different hypotheses for the two theorems.

To perturb the "spend all the cash on hand" equations $(13)^h$ (or, $(15)^h_s$), vary m^h_0 (or, m^h_s). (Note that m^h_0 influences $(13)^h$ through $\Delta(1)^h$. Also note that when m^h_0 or m^h_s is increased, h in effect throws away money by not spending it; and when it is decreased, h creates money to spend it.) No other equation is affected.

To perturb the "no worthless cash at end" equations $(14)^h$(a) (or $(17)^h_s$), vary q^h_{00m} (or, q^h_{ssm}). This affects r_0 (or, r_s) when $r_0 > 0$ (or, $r_s > 0$) on account of the bond-market clearing condition, and thereby affects $(12)^k_0$(c), one of $(14)^k$(b) or $(16)^k$, and $(18)^k$(b) (or, $(12)^k_s$(c), $(15)^k_s$) for $k \in H$. (In the regime where r_0 (or, r_s) has been set to zero, the designated hoarder's $q^{h^*(s)}_{00m}$ (or, $q^{h^*(s)}_{ssm}$) is forced to vary so as to maintain $r_0 = 0$ (or, $r_s = 0$), and there is no chain-effect on the other equations.) First consider the repercussions of varying q^h_{00m}. We can compensate $(13)^k$ by varying m^k_0 (and thereby varying $\Delta(1)^k$). For $(12)^k_0$(c), vary ∇^k_{0L}; this disturbs $(12)^k_0$(b) which we can compensate via $\nabla^k_{0\ell}$ for $\ell \in L^h_0(-)\setminus\{L\}$. Next, since $\tilde{1}$ and $\{A^j\}_{j\in J}$ are linearly independent, the vector $(\nabla^k_{s1}/p_{s1m})_{s\in S}$ can be varied by adjusting $(\nabla^k_{s1})_{s\in S}$ in a direction that makes arbitrary prespecified dot products with $\tilde{1}$, $\{A^j\}_{j\in J}$ so as to undo the disturbance created in : one of $(14)^k$(b) or $(16)^k$ and $(18)^k$(b). The variation of $(\nabla^k_{s1})_{s\in S}$ disturbs equations $(12)^k_s$(a) and $(12)^k_s$(c) for $s \in S$, $k \in H$ but we can undo this via $\nabla^k_{s\ell}$ for $\ell \in L^k_s(+)\setminus\{1\}$ and ∇^k_{sL} respectively. The last now disturbs $(12)^k_s$(b) for $s \in S$, $k \in H$ but this is also undone by varying $\nabla^k_{s\ell}$ for $\ell \in L^k_s(-)\setminus\{L\}$. As for the effect (via r_s) of varying q^h_{ssm} (for $s \in S$) on $(12)^k_s$(c) and $(15)^k_s$, for $k \in H$, this is undone quite simply. Vary m^k_s to undo the disturbance in $(15)^k_s$. For $(12)^k_s$(c), vary ∇^k_{sL}; and, for the disturbance that spills over to $(12)^k_s$(b), vary $\nabla^k_{s\ell}$ for $\ell \in L^k_s(-)\setminus\{L\}$.

9.2 Generic finiteness of ME

It remains to check that the finiteness argument still goes through. First notice that if q^h_{ssm} is free then h is not hoarding money, since q^h_{ssm} is considered forced for the single agent who does everyone's hoarding. So if q^h_{ssm} is free, we may suppose that h spends all his cash. This implies, if $s \in S$, that $[q^h_{ssm} > 0$ and free$] \Leftrightarrow [q^h_{s\ell m} > 0$ for some $\ell \in L]$. (We have "$\Rightarrow$" because otherwise h will be unable to repay, and "$\Leftarrow$" because there is no use of sales revenue in period 1 other than to repay the short loan.) Hence if $q^h_{ssm} \to 0$ on some sequence of ME in $s \in S$ so must some $q^h_{s\ell m} \to 0$ giving a first-order condition in the limit without a corresponding free variable. As we explained in Theorem 1, generically this cannot happen.

Now consider the case where some free variable $q^h_{00m} \to 0$. Since h is spending all his money, he must be selling a commodity or an asset in period 0, otherwise he would be unable to repay the q^h_{00m}. Sales may not go to zero as $q^h_{00m} \to 0$, because h may be selling to inventory. But then the first condition for inventorying $(14)^h$(b), holds in the limit, while its associated variable q^h_{00m} ceases to be free, again leading to a situation which generically cannot happen. □

10 Existence of liquidity trap in the quantity targets model

Proof of Theorem 3. If $r_0 > 0$, then every agent will spend all the money at hand in state 0, as shown in Lemma 2 in Section 7. Thus total money spent is at least M_0. Sales of commodities are naturally bounded above by their endowments. Sales of assets must also be bounded (as we showed in the proof of Theorem 1) since asset payoffs are linearly independent. Hence some prices $p^h_{0\ell m}$ or p_{0jm} must go to infinity in state 0. But in each state $s \in S$ there is some commodity price $p_{s\ell(s)m}$ that does not go to infinity. This is so because by the gains-to-trade hypothesis the sale of some commodity in each state is bounded away from zero; and on the other hand, the stock of money to spend in state s is at most $M_s + \sum_{k\in H}(m^h_0 + m^h_s)$, since the central bank receives but does not pay interest ($Q_{snm} = 0$ by hypothesis). If $p_{0\ell m} \to \infty$, take h with $e^h_{0\ell} > 0$. He could sell $e^h_{0\ell}$, inventory the money, and buy an arbitrarily large quantity of some commodity $s\ell(s)$. But this eventually achieves higher utility than $u^h(\sum_{h\in H} e^k)$, which is more than he could be achieving in the monetary equilibrium, a contradiction.[19] If $p_{0jm} \to \infty$, any agent h could sell a sliver of p_{0jm}, inventory the money, using a tiny amount to deliver on the sliver, and using the rest to buy a huge amount of some good $s\ell(s)$, again a contradiction.

Thus, holding M_s constant for $s \in S$, expenditures at time 0 must stay bounded as $M_0 \to \infty$. Hence there is a threshold M^*_0 after which $r_0 = 0$ and the extra bank money $M_0 - M^*_0$ is simply boarded and returned, without any effect on prices or allocations.

[19] Since utilities are strictly monotonic, we may w.l.o.g. assume that $\exists D > 0$ such that

$$u^h(0, ..., 0, D, 0, ..., 0) > u^h\left(\sum_{k\in H} e^k\right)$$

for D in any component.

References

Balasko, Y., Cass, D.: The structure of financial equilibrium with exogenous yields: the case of restricted participation. Econometrica **57**, 135–162 (1989)

Cochrane, J.: Long-term debt and optimal policy in the fiscal theory of the price level. Econometrica **60**(1), 69–116 (2001)

Debreu, G.: The coefficient of resource utilization. Econometrica **19**, 273–291 (1951)

Dréze, J., Polemarchakis, H.: Money and monetary policy in general equilibrium. In Gérard-Varet, L.-A., Kirman, A.P., Ruggiero, M. (eds.) Economics, the next ten years. Oxford: Oxford University Press 1999

Dréze, J., Polemarchakis, H.: Intertemporal general equilibrium and monetary theory. In: Leijonhufvd, A. (ed.) Monetary theory as a basis for monetary policy. New York: Macmillan 2000

Dréze, J., Polemarchakis, H.: Monetary equilibrium. In: Debreu, G., Neuefeind, W., Trockel, W. (eds.) Economics essays – A Festschrift for Werner Hildenbrand. Berlin Heidelberg New York: Springer 2001

Dubey, P., Geanakoplos, J.: The value of money in a finite horizon economy: a role for banks. In: Dasgupta, P., Gale, D., Hart, O., Maskin, E. (eds.) Economic analysis of markets and games. Cambridge: MIT Press 1992

Dubey, P., Geanakoplos, J.: Inside and outside money, gains to trade, and IS-LM. Economic Theory **21**, 347–397 (2003a)

Dubey, P., Geanakoplos, J.: Monetary equilibrium with missing markets. Journal of Mathematical Economics **39**, 585–618 (2003b)

Galbraith, J.K: Money: whence it came, where it went. Boston: Houghton Mifflin Company 1975

Gale, .D.: Pure exchange equilibrium of dynamic economic models. Journal of Economic Theory **6**, 12–36 (1973)

Geanakoplos, J., Mas-Colell, A.: Real indeterminacy with financial assets. Journal of Economic Theory **47**, 22–38 (1989)

Geanakoplos, J., Tsomomos, D.: International finance in general equilibrium. Research in Economics **56**(1), 85–142 (2002)

Lucas, R.E., Stokey, N.L.: Money and rates of interest in a cash-in-advance economy. Econometrica **55**, 491–513 (1987)

Magill, M., Quinzii, M.: Real effects of money in general equilibrium. Journal of Mathematical Economics **21**, 301-342 (1992)

Samuelson, P.: An exact consumption loans model of interest with or without the social contrivance of money. Journal of Political Economy **LXVI**, 467–482 (1958)

Sims, C.: A simple model for the study of the determination of the price level and the interaction of monetary and fiscal policy. Economic Theory **4**, 381–399 (1994)

Woodford, M.: Monetary policy and price level determinacy in a cash-in-advance economy. Economic Theory **4**, 345–380 (1994)

Woodford, M.: Control of the public debt: a requirement for price stability? NBER Working Paper No. 5684 (1996)

Woodford, M.: Price-level determination under interest rate rules. Mimeo, Department of Economics, Princeton University (1999)