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The inflationary bias of real uncertainty and the harmonic Fisher equation*

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Summary. We argue that real uncertainty itself causes long-run nominal inflation. Consider an infinite horizon cash-in-advance economy with a representative agent and real uncertainty, modeled by independent, identically distributed endowments. Suppose the central bank fixes the nominal rate of interest. We show that the equilibrium long-run rate of inflation is *strictly higher*, on almost every path of endowment realizations, than it would be if the endowments were constant.

Indeed, we present an explicit formula for the long-run rate of inflation, based on the famous Fisher equation. The Fisher equation says the short-run rate of inflation should equal the nominal rate of interest less the real rate of interest. The long-run Fisher equation for our stochastic economy is similar, but with the rate of inflation replaced by the *harmonic mean* of the growth rate of money.

Keywords and Phrases: Inflation, Equilibrium, Control, Interest rate, Central bank, Harmonic Fisher equation.

JEL Classification Numbers: C7, C73, D81, E41, E58.

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1 Introduction

Our goal is to understand the behavior of prices and money in a simple, stationary economy with exogenous production subject to independent and identically distributed shocks. We show that there is a unique, neutral stationary equilibrium, both for the case when the economy has no loan market and also when there is a central bank that sets an interest rate ρ, the same for both savers and borrowers. If the economy has no loan market, then the money supply remains constant and, in equilibrium, prices are independent and identically distributed.

For the economy with a central bank, inflation (or deflation) is possible. We calculate exactly the long-run rate of inflation as a function of the interest rate ρ and the distribution of the random shocks. Surprisingly, we find that larger productivity shocks lead to higher long-run inflation. We couch our analysis in terms of a representative agent with an arbitrary concave utility function $u(\cdot)$, a single good, and independent, identically distributed endowments. We prove that there is a unique neutral, stationary equilibrium, and derive explicit formulae for it.

According to the famous Fisher equation, the rate of inflation should depend on the monetary rate of interest and on the time-preference of the agents, and on nothing else. In a nonstochastic, stationary economy, this is precisely the case:

$$\frac{p_{n+1}}{p_n} = \frac{m_{n+1}}{m_n} = \beta(1+\rho). \tag{1.1}$$

Here β denotes the discount rate of the agents (its reciprocal is the rate of time-preference), $1+\rho$ denotes the gross monetary rate of interest, p_n is the price on day n, and m_n is the money supply on day n. If the central bank sets the rate of interest (the same for borrowers and for depositors) equal to the rate of time-preference for agents, the equilibrium rate of inflation and of monetary growth will be zero.

In a stochastic stationary economy with well-informed agents, who know the value of their endowment before deciding on expenditures, no simple formula like (1.1) can hold at each moment in time. Indeed, denoting consumption at time n by Y_n, we must have that

$$\frac{u'(Y_n)}{p_n} = (1+\rho)\beta \cdot E\left[\frac{u'(Y_{n+1})}{p_{n+1}}\right],$$

and since Y_{n+1} and p_{n+1} are not independent, one cannot hope for a clean expression for $E[p_{n+1}/p_n]$. However, in our stochastic economy we do derive a precise formula for the long-run rate of inflation $\lim_{r\to\infty}[\sqrt[r]{p_{n+r}/p_n}]$. A Fisher-like equation still holds, but with the one-period rate of inflation replaced by the harmonic mean of the one-period money growth rate, giving an inflationary bias:

$$\lim_{r\to\infty} \sqrt[r]{\frac{p_{n+r}}{p_n}} > \left\{E\left[\frac{m_n}{m_{n+1}}\right]\right\}^{-1} = \beta(1+\rho) \quad \text{almost surely.} \tag{1.2}$$

In case $\beta(1+\rho) = 1$, the harmonic mean of m_{n+1}/m_n, the gross rate of monetary growth, is one. As long as there is any random variation in the rate of monetary growth, its geometric mean must therefore be greater than one. With independent draws, the law of large numbers guarantees that the long-run (geometric) growth rate of the money supply is greater than one, on almost every path. Since the long-run rate of inflation must be equal to the long-run rate of growth of the money supply, this shows that when $\beta(1+\rho) = 1$, the slightest bit of monetary fluctuation typically[1] creates an inflationary bias, irrespective of the utility function of the representative agent.

There is some historical evidence that periods of stable output are associated with low inflation. Our analysis linking inflation with real uncertainty provides a modest reason for that empirical association. On the other hand, it can easily be shown in our model that money cannot grow faster than the nominal interest rate, and hence the long-run inflation is bounded from above by $1 + \rho$. The inflationary bias is therefore at most $(1-\beta)(1+\rho)$, which should be small, since β is ordinarily close to 1. We give a precise formula in Section 4. A bigger inflationary bias from real uncertainty might be generated in a model in which the central bank adjusts the interest rate, but we do not pursue that question here.

Suppose that physical endowments Y_n are independent and identically distributed (iid) and that $Yu'(Y)$ is not constant, where $u'(Y)$ is the marginal utility of consumption. We show that no matter what fixed interest rate ρ the central bank maintains, monetary stocks m_n and prices p_n must fluctuate unboundedly. Even if Y takes on just two values, prices p_n will eventually become unboundedly large or small, or both. Only by active management, making ρ a function of the physical endowment Y, can the central bank ensure that prices will stay bounded. Even such an active bank, however, cannot maintain absolutely fixed price levels $p_n = p$ for all n.

We show that if agents *do not know their endowment* before they are called upon to commit themselves to expenditures, then the original Fisher equation is restored irrespectively of the agents' utility function, and setting the rate of interest equal to the rate of time preference will result in an expected rate of inflation equal to zero.

Our model is in the spirit of the representative agent approach of Lucas (1978); we use dynamic programming methods in a microeconomic model of money, in the tradition of Shubik (1972), Shubik and Whitt (1973), Lucas (1980, 1990), Lucas and Stokey (1983), Stokey and Lucas (1989), Woodford (1994), and Karatzas, Shubik and Sudderth (1994). The microeconomic tradition of analyzing policy and money in a market-clearing model is vast; see, for example, Phelps (1967, 1970, 1973), Kydland and Prescott (1977), Barro (1990), Chari et al. (1991), Mankiw (1992), Sargent (1987, 1999), Alvarez, Lucas and Weber (2001), and Dubey and Geanakoplos (1992, 2003). To the best of our knowledge, however, the connection between real uncertainty and long-run inflation addressed in this paper seem to be treated here for the first time. The model of Lucas (1990), for instance, is extremely close to ours, but analyzes the case where the central bank interest rate is random and output is fixed. Models of Mehra and Prescott (1985), Weil (1992) and others

[1] The sole exception occurs when there is no variation in the rate of growth of the money supply. We show that this can only happen if $yu'(y)$ is constant over all random endowments y. Thus logarithmic utility turns out to be the exception to our inflationary bias paradigm, rather than the archetypical example it often is in other contexts.

examine the real interest rate, that is, the interest rate on bonds that pay one unit of good in each period. They find that real uncertainty might increase or decrease the real rate of interest, depending on the third derivative of utility.

There are precedents for our conclusion that prices must wander to either infinity or to zero no matter what fixed nominal interest rate ρ the central bank fixes. For example, Matsuyama (1991) showed that even for a nonstochastic economy, cyclical or chaotically fluctuating prices are possible. (But that is in a nonstationary equilibrium.)

An alternative generalization of the Fisher equation to stochastic economies is derived for models with exogenous prices by Benninga and Protopapadakis (1983) and Sarte (1998). These authors express the one-period deviation for the classic equation in terms of the covariance between the ratio of marginal utilities $u'(Y_{n+1})/u'(Y_n)$, and p_{n+1}/p_n.

We derive an explicit formula for the long-run rate of inflation, for arbitrary utility u, iid endowments $\{Y_n\}$, and nominal interest rate ρ, without any approximation. Over the last decade it has become fashionable to investigate the properties of monetary economies by using log-linear approximations around the riskless steady state economy (see, for example, Woodford, 2004). We have been able to avoid the need for such shortcuts by confining our attention to a representative agent economy. We do not know how to compute explicit formulas for heterogeneous agent economies. We also confine our attention to (what we prove) is the unique neutral stationary equilibrium, ignoring sunspot equilibria and nonstationary equilibria. These latter are studied in Woodford (1994) in an economy without real uncertainty.

Lastly, we note that our interest rate pertains to the trading period; an agent who wishes to sell in order to raise the revenue to make a simultaneous purchase must borrow the money at rate ρ. This embodies a genuine cash-in-advance constraint.

1.1 Preview

The derivation of the harmonic Fisher equation (1.2) will be undertaken in a completely specified general equilibrium model with a representative agent. It may be instructive to see briefly how to derive the harmonic Fisher equation in a reduced-form model based on two premises, stationarity and money-neutrality. First, we suppose that prices at time n are proportional to the supply of money at time n, namely

$$p_n = \mathbf{p}(Y_n) m_n \tag{1.3}$$

for an appropriate function $\mathbf{p}(\cdot)$ to be determined (see equation (4.6′)), where Y_n is the random endowment of the perishable good at time n. Secondly, we suppose that the money supply at time $n+1$ is proportional to the supply at time n, with a proportionality constant that depends only on the random endowment at time n, namely

$$m_{n+1} = \tau(Y_n) m_n \tag{1.4}$$

for an appropriate function $\tau(\cdot)$, to be determined (see equation (4.8)). In equilibrium, the agent is indifferent between spending a dollar on consumption and depositing it in the bank with interest to consume during the next period:

$$\frac{u'(y)}{p_n} = \beta(1+\rho) \cdot E_n \left[\frac{u'(Y_{n+1})}{p_{n+1}} \right], \quad \text{on } \{Y_n = y\}.$$

Here and in the sequel, $E_n[\cdot] = E[\cdot|\mathcal{F}_n]$ denotes conditional expectation with respect to the information $\mathcal{F}_n$ available to agents at time n; this information includes Y_n and m_n. Substituting for p_n, p_{n+1} and m_{n+1}, gives

$$\begin{aligned} \frac{u'(y)}{\mathbf{p}(y)m_n} &= \beta(1+\rho) \cdot E_n \left[\frac{u'(Y_{n+1})}{\mathbf{p}(Y_{n+1})m_{n+1}} \right] = \beta(1+\rho) \cdot E_n \left[\frac{u'(Y_{n+1})}{\mathbf{p}(Y_{n+1})\tau(y)m_n} \right] \\ &= \frac{\beta(1+\rho)}{\tau(y)m_n} \cdot E_n \left[\frac{u'(Y_{n+1})}{\mathbf{p}(Y_{n+1})} \right], \quad \text{on } \{Y_n = y\}. \end{aligned}$$

Let $z(y) \triangleq u'(y)/\mathbf{p}(y)$. Cancelling m_n from both sides, bringing $\tau(y)$ to the left, and then inverting both sides, gives

$$\frac{1}{\tau(y)} = \frac{1}{\beta(1+\rho)} \frac{z(y)}{E_n[z(Y_{n+1})]}, \quad \text{on } \{Y_n = y\}.$$

Assuming that the Y_{n+1} is independent of $\mathcal{F}_n$ for all $n \geq 1$, and taking expectations, we obtain

$$E\left[\frac{m_n}{m_{n+1}}\right] = E\left[\frac{1}{\tau(Y_n)}\right] = \frac{1}{\beta(1+\rho)} \frac{E[z(Y_n)]}{E[z(Y_{n+1})]} = \frac{1}{\beta(1+\rho)}, \tag{1.5}$$

because Y_n and Y_{n+1} have the same distribution. This is the harmonic Fisher equation of (1.2).

2 Equilibrium

2.1 The model

We consider a representative agent model extending over days or time-periods $n = 1, 2, \ldots$. On each day the agent receives a random endowment $Y_n(\omega)$ of a single perishable commodity, where Y_n is a random variable on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and ω is an element of Ω. The random variables $Y_1, Y_2, \ldots$, corresponding to the successive random endowments of the agent, are assumed to be independent with a common distribution λ. We often use Y with no subscript to denote a generic random variable with this distribution. We further assume that the support $\mathcal{Y}$ of the endowment variables is bounded away from both zero and infinity.

The total payoff to the agent in state ω from consumption $(x_1(\omega), x_2(\omega), \ldots)$ is

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n(\omega)),$$

where $\beta \in (0,1)$ is the discount factor and the utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is concave, strictly increasing, and differentiable on $(0,\infty)$.

The agent in the economy must sell his entire endowment $Y_n(\omega)$ for money in each period n at price $p_n(\omega)$, thereby receiving $p_n(\omega)Y_n(\omega)$ units of fiat money. He consumes only by purchasing commodities with the money he already has on hand (cash in advance). The monetary prices of the commodity are random variables $(p_1(\omega), p_2(\omega), ...)$. The agent regards himself as so small as to be unable to affect these prices by his actions.

In period 1 the agent begins with a quantity of fiat money m_1 (his "liquid wealth"). A governmental central bank stands ready to loan or borrow money at a given interest rate $\rho \geq 0$. The agent can lend an amount up to m_1 or borrow up to a limit $L_1(\omega)$ that we shall specify. Having chosen to borrow or lend $\tilde{m}_1(\omega)$ with $-m_1 \leq \tilde{m}_1(\omega) \leq L_1(\omega)$, the agent spends $b_1(\omega) = m_1 + \tilde{m}_1(\omega)$ on commodities and consumes the amount $x_1(\omega) = b_1(\omega)/p_1(\omega)$. At every subsequent period $n = 2, 3, ...$, the agent begins with liquid wealth

$$m_n(\omega) = (1+\rho)[m_{n-1}(\omega) - b_{n-1}(\omega)] + p_{n-1}(\omega)Y_{n-1}(\omega) \qquad (2.1)$$

and by choosing to borrow or lend $\tilde{m}_n(\omega)$, with $-m_n(\omega) \leq \tilde{m}_n(\omega) \leq L_n(\omega)$, the agent spends $b_n(\omega) = m_n(\omega) + \tilde{m}_n(\omega)$ and consumes $x_n(\omega) = b_n(\omega)/p_n(\omega)$. For notational simplicity we shall avoid using $\tilde{m}_n$ and x_n, since they are determined at every stage by b_n and p_n.

2.2 Equilibrium

At the beginning of period n, the agent knows the interest rate ρ, the past and present values of liquid wealth $m_1, m_2, ..., m_{n-1}, m_n$; prices $p_1, p_2, ..., p_{n-1}, p_n$; and endowments $Y_1, Y_2, ..., Y_{n-1}, Y_n$. (Notice that the agent is assumed to know p_n and his endowment Y_n for period n at the *beginning* of the period. Eventually we shall consider a model in which the agent does not know either p_n or Y_n when he chooses his bid in the period.) The agent must choose his bid b_n in period n to be a function of these variables, or, equivalently, to be measurable with respect to the sigma-field $\mathcal{F}_n$ generated by them.

The agent's budget set is

$$B(m_1, p, L, \rho) = \{b_n(\omega) : 0 \leq b_n(\omega) \leq m_n(\omega) + L_n(\omega) \text{ for almost all } \omega \in \Omega \text{ and } b_n \text{ is } \mathcal{F}_n\text{-measurable, for all } n \geq 1\}, \qquad (2.2)$$

where m_n is determined by $(m_1, \rho, p_1(\omega), ..., p_{n-1}(\omega); b_1(\omega), ..., b_{n-1}(\omega); Y_1(\omega), ..., Y_{n-1}(\omega))$ as in (2.1) above. Thus, the random interval $[0, m_n(\omega) + L_n(\omega)]$ corresponds to the set of actions available to the agent in period n.

The economy is in *equilibrium* at $\{p_n(\omega) : n \geq 1,\ \omega \in \Omega\}$, if the representative agent is optimizing in his budget set while consuming $Y_n(\omega)$, for all $n \geq 1$. More precisely, letting

$$b_n^*(\omega) \equiv p_n(\omega)Y_n(\omega) \quad \text{for all} \quad n \geq 1,\ \omega \in \Omega, \qquad (2.3)$$

the sequence $\{b_n^*(\omega) : n \geq 1, \omega \in \Omega\}$ attains the maximum of

$$E\left[\sum_{n=1}^{\infty} \beta^{n-1} u(b_n(\omega)/p_n(\omega))\right]$$

over the set $B(m_1, p, L, \rho)$ of (2.2).

We distinguish two cases for the borrowing limits $L_n(\omega)$. Setting

$$L_n(\omega) = \frac{p_n(\omega) Y_n(\omega)}{1+\rho}, \tag{2.4}$$

we get an economy with a bank that permits the agent to borrow up to the amount he is sure to receive in income. By setting $L_n(\omega) = 0$ and $\rho = 0$, we effectively obtain an economy without a central bank.

In the following two sections we construct equilibria for each of these two cases. Afterwards we shall consider a model in which the bank chooses an interest rate $\rho_n(\omega)$ in period n that need not be constant. We shall also construct equilibria with and without a bank, for the case in which the agent does not know his endowment $Y_n(\omega)$ or the price $p_n(\omega)$ when he is called upon to borrow and bid at time n.

We are primarily concerned with the behavior of the prices $p_n(\omega)$. For the models with a bank we shall give conditions on the interest rate ρ that result in inflation, deflation, or neither. Consumption in our models is trivial in equilibrium, with the agent consuming his endowment $Y_n(\omega)$ in each period n.

2.3 Neutral, stationary equilibrium (NSE)

We shall focus our attention on *stationary equilibria* in which prices $p_n(\omega)$ and bids on expenditures $b_n(\omega)$ at time n can all be expressed in terms of functions $\mathcal{P} : \mathbb{R}_{++} \times \mathcal{Y} \to \mathbb{R}$ and $\mathcal{B} : \mathbb{R}_{++} \times \mathcal{Y} \to \mathbb{R}$ that depend only on the liquid wealth $m_n(\omega)$ and the endowment $Y_n(\omega)$ in the period, namely

$$\begin{aligned} p_n(\omega) &= \mathcal{P}(m_n(\omega), Y_n(\omega)) \\ b_n(\omega) &= \mathcal{B}(m_n(w), Y_n(\omega)). \end{aligned} \tag{2.5}$$

The liquidity constraints $L_n(\omega)$ at time n can be expressed as a function

$$L_n(\omega) = \mathcal{L}(Y_n(\omega), p_n(\omega)) \tag{2.5$'$}$$

of current endowment $Y_n(\omega)$ and price $p_n(\omega)$, where $\mathcal{L} : \mathcal{Y} \times \mathbb{R}_{++} \to \mathbb{R}$ is given by

$$\mathcal{L}(y, p) = \left\{ \begin{array}{ll} \dfrac{p \cdot y}{1+\rho} & \text{, if there is a bank} \\ 0 & \text{, if not} \end{array} \right\}. \tag{2.6}$$

In fact, we shall construct *stationary* equilibria in which fiat money is *neutral* (doubling the liquid wealth doubles prices and bids without changing consumption):

$$\begin{aligned} p_n(\omega) &= m_n(\omega)\, \mathbf{p}(Y_n(\omega)) \\ b_n(\omega) &= m_n(\omega)\, \mathbf{b}(Y_n(\omega)) \end{aligned} \tag{2.7}$$

where $\mathbf{p} : \mathcal{Y} \to \mathbb{R}_{++}$ and $\mathbf{b} : \mathcal{Y} \to \mathbb{R}_{++}$ are given functions. Clearly in equilibrium we must have

$$\mathbf{b}(y) = \mathbf{p}(y)y, \quad \text{for all} \quad y \in \mathcal{Y}, \tag{2.3$'$}$$

from (2.3). It is evident from the monotonicity of u that there can be no equilibrium in which $p_n(\omega) = 0$ happens with positive probability for some $n \geq 1$. Hence in *neutral stationary* equilibrium we must have $\mathbf{b}(y) > 0$ and $\mathbf{p}(y) > 0$ for all $y \in \mathcal{Y}$, and $m_n(\omega) > 0$ almost surely, for all $n \geq 1$.

2.4 Dynamic programming

In neutral, stationary equilibrium the *macroscopic* variables (m, y, p) follow a stationary Markov process defined by the price function $\mathcal{P}(m, y) = m\mathbf{p}(y)$. Let Γ be the graph of the function $\mathcal{P}$, namely $\Gamma = \{(m, y, p) \in \mathbb{R}_{++} \times \mathcal{Y} \times \mathbb{R}_{++} : \mathcal{P}(m, y) = p\}$. Then the law of motion of the macroscopic variables is given by the transition mechanism $\Gamma \to \Delta(\Gamma)$ indicated below for each $(m, y, p) \in \Gamma$:

$$\begin{aligned} m' &= (m - py)(1 + \rho) + py \\ &= m(1 + \rho) - \rho py \\ &= m(1 + \rho - \rho\mathbf{b}(y)) \\ y' &\in \mathcal{Y}, \text{ distributed as } Y \\ p' &= \mathcal{P}(m', y') = m'\mathbf{p}(y'). \end{aligned} \tag{2.8}$$

Note that m' is determined without any uncertainty by (m, y, p), whereas y' and p' are stochastic. Note also that in order to have $m' > 0$, we must impose

$$\mathbf{b}(y) = \mathcal{P}(m, y)y/m < (1 + \rho)/\rho, \quad \text{for all } y \in \mathcal{Y}.$$

Facing this dynamic system, the representative agent with liquid wealth s must choose an expenditure rule at each macroscopic state (m, y, p) to maximize his utility. In stationary equilibrium, when $s = m$, the expenditure rule $(s, m, y, p) \mapsto py$ must be optimal.

In this case there must be a value function $V : \mathbb{R}_+ \times \Gamma \to \mathbb{R}$ satisfying the Dynamic Programming Equation

$$V(s, m, y, p) = \max_{0 \leq b \leq s + \mathcal{L}(y,p)} \left[u\left(\frac{b}{p}\right) + \beta \cdot E_Y V((s - b)(1 + \rho) + py, m', Y, p') \right] \tag{2.9}$$

and such that, for all $(m, y, p) \in \Gamma$ and $s = m$, we have

$$p \cdot y \in \underset{0 \le b \le m + \mathcal{L}(y,p)}{\arg\max} \left[u\left(\frac{b}{p}\right) + \beta \cdot E_Y V((m-b)(1+\rho) + py, m', Y, p') \right] \tag{2.10}$$

i.e.,

$$V(m, m, y, p) = u(y) + \beta \cdot E_Y[V(m', m', Y, p')]. \tag{2.11}$$

3 Constructing neutral, stationary equilibrium without a bank

In the absence of a bank we have $\mathcal{L}(y, p) = 0$ and the money supply does not change, so the representative agent will begin each period with the same liquid wealth $m = m_1$. We can see that $m' = m$ by taking $\rho = 0$ in the law of motion (2.8).

The representative agent always has the choice of spending his money at time n, or saving it to spend in the future. In neutral stationary equilibrium, $0 < py \le m$ for all $(m, y, p) \in \Gamma$. Hence the agent could spend ε less, saving the money until next period, and spending it there on consumption. In equilibrium it is optimal for him *not* to do so, giving in the notation of (2.8):

$$\frac{u'(y)}{p} \ge \beta \cdot E\left[\frac{u'(Y)}{p'}\right] \quad \text{for all } y \in \mathcal{Y}. \tag{3.1}$$

If $py < m$, the agent can spend ε more (and then ε less next period), giving the opposite inequality. Succinctly combining these observations gives

$$[1 - y\mathbf{p}(y)] \wedge \left[\frac{u'(y)}{\mathbf{p}(y)} - \beta \cdot E\left[\frac{u'(Y)}{\mathbf{p}(Y)}\right]\right] = 0, \quad \text{for all } y \in \mathcal{Y}. \tag{3.2}$$

(The notation $a \wedge b$ is used here and below for the minimum of a and b.)

We must find a function $\mathbf{p}$ (or equivalently $\mathbf{b}$) such that (3.2) is satisfied.

One natural (but generally insufficient) guess is that the agent will spend all his money in every period, namely $\mathcal{B}(m, y) = m$, $\mathbf{b}(y) = 1$, $\mathcal{P}(m, y) = m/y$, $\mathbf{p}(y) = 1/y$. Plugging this candidate $\mathbf{p}(y) = 1/y$ into (3.2) gives

$$yu'(y) \ge \beta \cdot E[Yu'(Y)] \tag{3.3}$$

for all $y \in \mathcal{Y}$. This indeed is an equilibrium if $yu'(y)$ is a constant (as it will be for $u(y) = \log(y)$). But if there is enough variation in $yu'(y)$ over $y \in \mathcal{Y}$, then (3.3) will fail for some y, and $\mathbf{p}(y) = 1/y$ will not give rise to an equilibrium.

Another plausible (but insufficient) guess, is to set price always proportional to marginal utility, namely $\mathbf{p}(y) = \frac{1}{a}u'(y)$ for some constant $a > 0$, for all $y \in \mathcal{Y}$. But plugging this into (3.2) gives a strictly positive second term on the left-hand side, for all $y \in \mathcal{Y}$, implying that $\mathbf{p}(y) = 1/y$, $\mathbf{b}(y) = 1$ for all $y \in \mathcal{Y}$. Hence we must have then $1/y = u'(y)/a$ or $yu'(y) = a$ for all $y \in \mathcal{Y}$, which can happen if u is logarithmic or if there is no endowment uncertainty. In general, however, $\mathbf{p}(y) = u'(y)/a$ is also insufficient.

Let $a \equiv \beta E[u'(Y)/\mathbf{p}(Y)]$. Any solution $\mathbf{p}(y)$ to (3.2) must make either the first term or the second term on the left equal to zero, for all $y \in \mathcal{Y}$. Hence $\mathbf{p}(y)$ can take on only two possible values, $1/y$ or $u'(y)/a$. If $yu'(y) < a$, then clearly $\mathbf{p}(y) = 1/y$ will not work (since the second term would be negative), hence $\mathbf{p}(y) = u'(y)/a$.

Suppose now that $yu'(y) > a$. Since the first term on the left is nonnegative, $\mathbf{p}(y) \leq 1/y$. Hence $u'(y)/\mathbf{p}(y) \geq yu'(y) > a$ making the second term positive. Thus we must have $\mathbf{p}(y) = 1/y$.

If $yu'(y) = a$, then both choices for $\mathbf{p}(y)$ are the same. In summary

$$\mathbf{p}(y) = \begin{Bmatrix} 1/y & , \text{ for } yu'(y) \geq a \\ \dfrac{u'(y)}{a} & , \text{ for } yu'(y) < a \end{Bmatrix} = \left(\frac{1}{y} \wedge \frac{u'(y)}{a} \right). \tag{3.4}$$

Observe that $\mathbf{p}(y)$ depends on a, namely, is of the form $\mathbf{p}_a(y) = \min\{1/y, u'(y)/a\}$. Plugging this into the definition of a, we get

$$a = \beta \cdot E[\max\{a, Yu'(Y)\}]. \tag{3.5}$$

It is evident that at $a = 0$, the right-hand side of (3.5) dominates the left, since $Yu'(Y) > 0$ for all $Y \in \mathcal{Y}$, while for $a > 0$ very large the left-hand side of (3.5) dominates the right since $\beta < 1$ and $E[Yu'(Y)] < \infty$. Hence there must be an $a \in (0, \infty)$ solving (3.5). Indeed, it is evident that near any a solving (3.5), the right-hand side increases less quickly in a than the left, so there is a unique $a \in (0, \infty)$ solving (3.5).

Theorem 3.1. *With $a \in (0, \infty)$ defined as in* (3.5)*, the function* $\mathbf{p}$ *of* (3.4) *gives rise to a neutral stationary equilibrium for the economy without a bank. This is the unique neutral stationary equilibrium.*

Proof. This candidate equilibrium uniquely satisfies (3.2) by construction. We leave to the Appendix the proof that the implied bidding strategy $b_n^*(\omega) \equiv m_1 \mathbf{p}(Y_n(\omega))Y_n(\omega)$ is optimal in the representative agent's budget set. □

In the equilibrium just constructed, liquid wealth remains constant at m_1, bids are given as

$$b_n(\omega) = \mathbf{b}(Y_n(\omega)) \cdot m_1, \quad \text{where} \quad \mathbf{b}(y) \equiv y\mathbf{p}(y) = \min\left(1, \frac{yu'(y)}{a}\right) \tag{3.6}$$

and prices fluctuate independently and identically across periods, according to the rule

$$p_n(\omega) = \mathcal{P}(m_1, Y_n(\omega)) = \min\left\{ \frac{1}{Y_n(\omega)}, \frac{u'(Y_n(\omega))}{a} \right\} m_1. \tag{3.7}$$

No matter what the discount rate β, there is no inflation. If the endowment random variable Y is bounded away from zero and infinity, so are the prices.

In the special case where there is no real uncertainty, $Y_n(\omega) = \bar{y}$ for all ω and $yu'(y) = \bar{y}u'(\bar{y})$ for all $y \in \mathcal{Y}$, (3.3) shows that in the unique equilibrium $p_n(\omega) = m_1/\bar{y}$ for all ω, and $b_n(\omega) = m_1$ for all ω. The agent always spends all his liquid wealth every period on goods. He wishes he could spend more, but since he cannot borrow, he is unable to do so.

Even when there is uncertainty, if utility u is logarithmic, (3.3) shows that $p_n(\omega) = m_1/Y_n(\omega)$ for all ω and again the agent spends all his money each period: $b_n(\omega) = m_1$ for all ω.

In general, we see that with positive probability $p_n(\omega) = m_1/Y_n(\omega)$; the agent spends all his money and wishes he could borrow (at zero interest) and spend more.

4 Constructing neutral, stationary equilibrium with a bank

In the presence of a central bank, there may well be inflation. We wish to study how inflation depends on the bank interest rate ρ and on the real economy.

In the presence of a central bank that maintains a constant interest rate, the money supply is endogenous and the representative agent may well begin each period with different liquid wealth. We suppose that the bank stands ready to accept deposits or give loans at an exogenous interest rate $\rho > 0$. We shall construct a neutral stationary equilibrium $(\mathbf{b}, \mathbf{p})$. From the law of motion (2.8) we have

$$\begin{aligned} m' &= \tau(y)m \\ p' &= m'\mathbf{p}(y') = \frac{m'\mathbf{b}(y')}{y'} \end{aligned} \tag{4.1}$$

where

$$\tau(y) \equiv 1 + \rho - \rho\mathbf{b}(y) \leq 1 + \rho \quad \text{for all } y > 0. \tag{4.2}$$

As noted earlier, neutrality implies that the growth rate τ of liquid wealth between periods $n-1$ and n depends only on Y_{n-1}, and therefore that m_n is $\mathcal{F}_{n-1}$-measurable.

Note that as long as $\mathbf{b}(y) \leq (1+\rho)/\rho$, the agent's borrowing constraint is satisfied; and as long as $\mathbf{b}(y) < (1+\rho)/\rho$, his future liquid wealth stays strictly positive, which is necessary in neutral, stationary equilibria.

In neutral, stationary equilibrium, if there is one, the agent could always have borrowed (deposited) a little more or a little less. Hence we must have

$$\frac{u'(y)}{p} = (1+\rho)\beta \cdot E\left[\frac{u'(Y)}{p'}\right], \quad \text{for all } y \in \mathcal{Y}. \tag{4.3}$$

(This is the analogue to equation (3.1) when there was no bank.) Plugging in our formulas for p and p', and recalling $m\mathbf{b}(y) = py$, gives

$$\begin{aligned} \frac{yu'(y)}{m\mathbf{b}(y)} &= (1+\rho)\beta \cdot E\left[\frac{Yu'(Y)}{m'\mathbf{b}(Y)}\right] \\ &= \frac{(1+\rho)\beta}{\tau(y)m} \cdot E\left[\frac{Yu'(Y)}{\mathbf{b}(Y)}\right], \quad \text{for all } y \in \mathcal{Y}. \end{aligned} \tag{4.4}$$

Cancelling m from both sides, and using (4.2), we have

$$\frac{yu'(y)}{\mathbf{b}(y)} = \frac{(1+\rho)\beta}{1+\rho-\rho\mathbf{b}(y)} \cdot E\left[\frac{Yu'(Y)}{\mathbf{b}(Y)}\right], \quad \text{for all } y \in \mathcal{Y}. \tag{4.5}$$

Note that $\rho > 0$ is now necessary for the existence of neutral, stationary equilibrium with a bank. Indeed, if $\rho = 0$, then by (4.1), $\tau(y) = 1$ for all y, giving $yu'(y)/\mathbf{b}(y) = \beta E[Yu'(Y)/\mathbf{b}(Y)]$ for all $y \in \mathcal{Y}$, by (4.4). Integrating with respect to the distribution λ of Y on both sides we obtain a contradiction to $\beta < 1$.

Lemma 4.1. *If $\rho > 0$, there is a unique function $\mathbf{b} : \mathcal{Y} \to \mathbb{R}_+$ that simultaneously satisfies the optimality condition* (4.4) *and the condition $0 < \mathbf{b}(y) < (1+\rho)/\rho$ for all $y \in \mathcal{Y}$:*

$$\mathbf{b}(y) = \frac{1+\rho}{\rho}\left[\frac{(1-\beta)yu'(y)}{(1-\beta)yu'(y)+\beta E[Yu'(Y)]}\right], \quad y \in \mathcal{Y}. \tag{4.6}$$

Theorem 4.2. *Defining $\mathbf{b}(y)$ as in* (4.6)*, and letting $\mathcal{P}(m,y) = m\mathbf{b}(y)/y$ and $\mathcal{B}(m,y) = m\mathbf{b}(y)$, gives the unique neutral, stationary equilibrium for the economy with a bank that sets the interest rate equal to ρ in every period. In particular*

$$\mathbf{p}(y) = \frac{\mathbf{b}(y)}{y} = \frac{1+\rho}{\rho}\frac{(1-\beta)u'(y)}{(1-\beta)\cdot yu'(y)+\beta\cdot E[Yu'(Y)]}, \quad y \in \mathcal{Y}. \tag{4.6$'$}$$

The proofs are given in the Appendix.

Having described the unique neutral, stationary equilibrium, we now proceed to study the growth rate of liquid wealth and price. When there is no uncertainty, and $\mathcal{Y} = \{\bar{y}\}$, formulas (4.6), (4.6′), and (4.2) become

$$\begin{aligned}
\mathbf{b}(\bar{y}) &= \frac{(1+\rho)(1-\beta)}{\rho} \\
\mathbf{p}(\bar{y}) &= \frac{(1+\rho)(1-\beta)}{\rho\bar{y}} \\
\tau(\bar{y}) &= 1+\rho-\rho\mathbf{b}(\bar{y}) = \beta(1+\rho).
\end{aligned}$$

It follows that

$$p_n(\omega) = m_n\mathbf{p}(\bar{y}) = m_1[\tau(\bar{y})]^{n-1}\mathbf{p}(\bar{y}) = \beta^{n-1}(1-\beta)\frac{m_1}{\bar{y}}\frac{(1+\rho)^n}{\rho}.$$

The rate of inflation is then

$$\frac{p_{n+1}(\omega)}{p_n(\omega)} = \beta(1+\rho),$$

which is the classical Fisher equation.

Clearly, increasing the interest rate ρ gives a higher growth rate to prices. An interesting point is that by choosing a higher ρ the central bank will start the economy at a *lower* initial price $p_1 = [(1-\beta)(1+\rho)/\rho\bar{y}]m_1$, but eventually

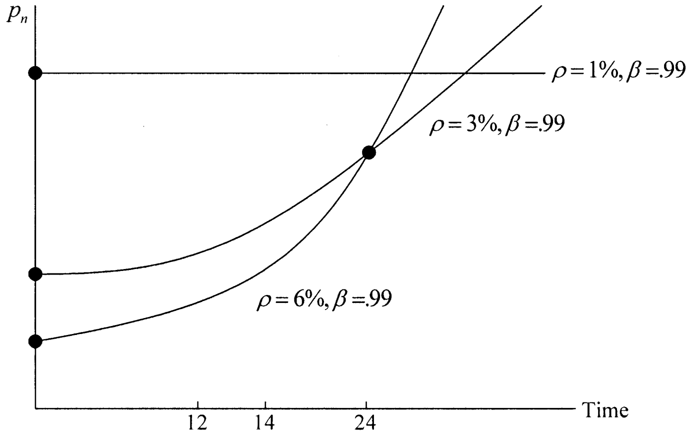

Figure 1. Certainty case

the price level will become higher. If $\rho' > \rho$, then letting p'_n denote the price level with interest rate ρ', we get

$$\frac{p'_n(\omega)}{p_n(\omega)} = \frac{(1+\rho')^n}{(1+\rho)^n}\frac{\rho}{\rho'}.$$

(As we observe in Corollary 4.8 below, this formula continues to hold when we do have real uncertainty.) Going from $\rho = 3\%$ to $\rho' = 6\%$ will cut the initial price level almost in half. If $\beta = .99$, then after about 14 periods the price level will be back to where it would have been at time 1, and after 24 periods the price level with $\rho' = 6\%$ will have caught up to where it would be with $\rho = 3\%$.

The "natural" choice for the central bank interest rate is $1+\rho = 1/\beta$. When there is no uncertainty, the standard Fisher equation shows that this natural ρ generates zero inflation.

We also can look at expected growth rates of money and prices when there is real uncertainty. It turns out that the *harmonic mean* of money growth is $\beta(1+\rho)$, as it was in the certainty case. Since along a path the growth rate is measured by the geometric mean, which is higher than the harmonic mean, it follows that uncertainty introduces an inflationary trend into observed price changes.

Theorem 4.3 (Harmonic Fisher Equation). *In neutral, stationary equilibrium the expected gross rate of future money growth has harmonic mean $\beta(1+\rho)$:*

$$E\left[\frac{1}{\tau(Y)}\right] = \frac{1}{\beta(1+\rho)} = E_n\left[\frac{m_k}{m_{k+1}}\right](\omega), \tag{4.7}$$

for all $k > n$ and almost all $\omega \in \Omega$. In particular, if $(1+\rho)^{-1} = \beta$, the harmonic mean of gross future money growth is one.

Proof. An argument for this result was presented in the Introduction. (Notice that (1.5) corresponds to the conclusion of the theorem. The argument for (1.5) in Section 1 was based on (1.3) and (1.4), which correspond to (4.1).)

Here is an alternative argument, based on the formula

$$\tau(y) = \beta(1+\rho)\frac{E[Yu'(Y)]}{(1-\beta)yu'(y) + \beta E[Yu'(Y)]} \tag{4.8}$$

which follows from (4.2) and (4.6) by trivial algebra. Invert both sides of (4.8) to get a formula for $1/\tau(y)$. Then integrate with respect to the distribution of Y to obtain (4.7). □

The standard Fisher equation relates the rate of inflation to the short-run nominal and real interest rates (which in a risky steady state are the same as the long-run real and nominal interest rates). With uncertainty, the short-run real rate of interest in any period n will fluctuate, depending on the realization of Y_n. The real rate of interest $\bar{\rho}_r$ in period r will similarly fluctuate, solving

$$u'(Y_n) = (1+\bar{\rho}_r)^r \beta^r E[u'(Y_{n+r})].$$

Since the random variable $\{Y_n\}$ are iid and bounded from above and away from zero, $\lim_{r\to\infty}(1+\bar{\rho}_r) = 1/\beta$ will hold a.s. Thus, our harmonic Fisher equation relates the growth rate of the money supply to the long-run nominal and real interest rates.

We just saw that when Y is almost surely constant (so $\mathcal{Y}$ consists of a single point), the classical Fisher equation holds. Surprisingly, the same formula is restored even with real uncertainty, provided that utility is logarithmic. We get:

$$\tau(y) = \beta(1+\rho), \quad \forall y \in \mathcal{Y}$$

from (4.8), if $y \mapsto yu'(y)$ is constant.

Corollary 4.4. *If $y \mapsto yu'(y)$ is constant, then the unique neutral, stationary equilibrium with a bank satisfies*

$$m_n(\omega) = m_1[\beta(1+\rho)]^{n-1}, \quad p_n(\omega) = \frac{(1-\beta)(1+\rho)m_1}{Y_n(\omega)}[\beta(1+\rho)]^{n-1}$$

for all n and almost all $\omega \in \Omega$.

In particular, provided that $\beta(1+\rho) = 1$, there is no inflationary trend. In that case, liquid wealth stays constant and prices $p_n(\omega)$ are iid *and bounded away from both zero and infinity.*

But the next two corollaries show that the logarithmic case is very exceptional. In every other case, there will be inflation with probability one, if the central bank sets $\rho = (1/\beta) - 1$.

Corollary 4.5. *Liquid wealth grows at the (continuously compounded) rate $E[\log\tau(Y)]$ along almost every path. This is faster than it would grow in the riskless economy obtained by replacing $Y_n(\omega)$ with $\bar{Y}$ for all n and ω, provided that $Yu'(Y)$ is not almost surely constant. In the "natural" case, i.e., where the central bank sets $\rho = (1/\beta) - 1$, liquid wealth grows at a strictly positive rate and $\lim_{n\to\infty} m_n = \infty$ almost surely.*

Proof. By repeated application of (4.1), we obtain

$$m_n(\omega) = m_1 \prod_{k=1}^{n-1} \tau(Y_k(\omega)), \tag{4.9}$$

thus also

$$\frac{\log m_n(\omega) - \log m_1}{n-1} = \frac{1}{n-1} \sum_{k=1}^{n-1} \log \tau(Y_k(\omega)).$$

By the law of large numbers, $(\log m_n - \log m_1)/(n-1)$ converges almost surely to $E[\log \tau(Y)]$. Now we use the fact that the geometric mean is greater than the harmonic mean whenever there is nontrivial randomness. Indeed, Jensen's inequality with Theorem 4.3 implies

$$e^{-E[\log \tau(Y)]} < E[e^{-\log \tau(Y)}] = E\left[\frac{1}{\tau(Y)}\right] = \frac{1}{\beta(1+\rho)},$$

and so $E[\log \tau(Y)] > -\log E[1/\tau(Y)] = \log \beta(1+\rho)$. When $\beta(1+\rho) = 1$, we get $E[\log \tau(Y)] > 0$ and the growth rate of liquid wealth must be positive. Hence $\log m_n \to \infty$ almost surely. □

The random sequence $\{p_n\}$ of prices is more complex than $\{m_n\}$, the random sequence of liquid wealth, since p_n is a function of Y_n while m_n is not. Knowing $Y_n(\omega)$, the rate of inflation between periods n and $n+1$ depends on $Y_{n+1}(\omega)$ and is not determined simply by $\tau(Y_n(\omega))$. But since Y is assumed to be bounded away from both zero and infinity, we can prove that prices grow at the same rate as liquid wealth *in the long run.*

Corollary 4.6 (Inflationary bias). *Assume that $Yu'(Y)$ is not almost surely constant. Then on almost every path we will observe a long-run inflation rate higher than we would have in the same economy without uncertainty (in which Y is replaced by its expectation $\bar{y} \triangleq E(Y) = \int y\lambda(dy)$). In particular, we have*

$$\log(1+\rho) \geq \lim_{r\to\infty} \frac{\log[p_{n+r}(\omega)] - \log p_n(\omega)}{r} = E[\log(\tau(Y))] > \log(\beta(1+\rho))$$

for almost all ω, and

$$1+\rho \geq \lim_{r\to\infty} \sqrt[r]{\frac{p_{n+r}(\omega)}{p_n(\omega)}} = e^{E[\log \tau(Y)]} > \beta(1+\rho)$$

for almost all $\omega \in \Omega$.

In particular, when $\beta(1+\rho) = 1$, we have: $\lim_{n\to\infty} p_n(\omega) = \infty$ *almost surely.*

Proof. From (2.7) and (4.9), we obtain

$$p_n(\omega) = \mathbf{p}(Y_n(\omega))m_n(\omega) = \mathbf{p}(Y_n(\omega))m_1 \cdot \prod_{\kappa=1}^{n-1} \tau(Y_k(\omega)). \tag{4.10}$$

Replacing n with $n + r$, we get

$$\frac{p_{n+r}(\omega)}{p_n(\omega)} = \frac{\mathbf{p}(Y_{n+r}(\omega))}{\mathbf{p}(Y_n(\omega))} \cdot \prod_{k=n}^{n+r-1} \tau(Y_k(\omega)), \ \ \textit{for } n \geq 1.$$

Take the logarithm of both sides and divide by $r - 1$. The strong law of large numbers, the boundedness of $\mathbf{p}(Y)$, and the independence of $\tau(Y_1), \tau(Y_2), \ldots$ gives the equality. The final inequality follows from the geometric-harmonic comparison, as in the proof of Corollary 4.5. The first inequality is immediate from (4.2). □

In view of Corollary 4.6, one would expect that an increase in real uncertainty will lead to increased long-run inflation. This is the case if by increased uncertainty we mean that the random variable $Z = Yu'(Y)$ is replaced by $\tilde{Z} = \tilde{Y}u'(\tilde{Y})$, where the transformation from Z to $\tilde{Z}$ is a mean-preserving spread: $E(Z) = E(\tilde{Z})$.

Corollary 4.7. *Suppose that the random variable $\tilde{Z} = \tilde{Y}u'(\tilde{Y})$ can be written in the form $\tilde{Z} = Z + \varepsilon$, where $Z = Yu'(Y)$ and ε is a random variable such that the conditional distribution of ε given Z is nontrivial and has mean zero almost surely. Then $E[\log \tau(Y)] < E[\log \tau(\tilde{Y})]$.*

Proof. By Jensen's inequality,

$$E[\log \tau(Y)] - E[\log \tau(\tilde{Y})] = E\left[\log\left(\frac{\tau(Y)}{\tau(\tilde{Y})}\right)\right] < \log E\left[\frac{\tau(Y)}{\tau(\tilde{Y})}\right].$$

But

$$\begin{aligned} E\left[\frac{\tau(Y)}{\tau(\tilde{Y})}\right] &= E\left[\frac{(1-\beta)(Z+\varepsilon)+\beta\mu}{(1-\beta)Z+\beta\mu}\right] \\ &= E\left[1 + \frac{(1-\beta)\varepsilon}{(1-\beta)Z+\beta\mu}\right] \\ &= 1 + E\left[\frac{1-\beta}{(1-\beta)Z+\beta\mu} \cdot E(\varepsilon|Z)\right] \\ &= 1, \end{aligned}$$

where $\mu = E[Yu'(Y)] = E[\tilde{Y}u'(\tilde{Y})]$. □

Corollary 4.8. *Under the same hypotheses as Corollary* 4.6, *there exists* $0 < \rho^* < 1/\beta - 1$ *such that:*

(i) *for* $\rho > \rho^*$, $\lim_{n\to\infty} p_n(\omega) = \infty$ *almost surely, and*
(ii) *for* $\rho < \rho^*$, $\lim_{n\to\infty} p_n(\omega) = 0$ *almost surely.*

Furthermore, when $\rho = \rho^*$, *we have* $\limsup_{n\to\infty} p_n(\omega) = \infty$ *almost surely* and $\liminf_{n\to\infty} p_n(\omega) = 0$ *almost surely. Thus, no constant interest rate ρ can keep prices bounded away from both zero and infinity.*

Proof. Let us denote by $\tau_\rho(y)$ the function of (4.8), to make explicit its dependence on the parameter ρ. Clearly then, $E[\log \tau_\rho(Y)]$ is continuous and strictly increasing in ρ. When $\rho = 0$ we have $\tau_\rho(y) < 1$ (thus $\log \tau_\rho(y) < 0$) for all y, and when

$\rho = 1/\beta - 1$ we have $E[\log(\tau_\rho(y))] > 0$, as shown in Corollary 4.5. Hence there is a unique ρ^* with

$$E[\log(\tau_{\rho^*}(Y))] = 0.$$

By Corollary 4.6, $\lim_{n\to\infty} p_n(\omega) = 0$ or ∞, according to weather $\rho < \rho^*$ or $\rho > \rho^*$, respectively. But if $\rho = \rho^*$, then by (4.9), $\log m_n$ is a random walk without drift, and so $\limsup_{n\to\infty} m_n(\omega) = \infty$ and $\liminf_{n\to\infty} m_n(\omega) = 0$ almost surely. By Corollary 4.6, the same must be true of p_n. □

It is interesting to note that when ρ is low enough, so that we have both $E[\tau_\rho(y)] = 1$ and $E[\log(\tau_\rho(y))] < 0$, then $p_n(\omega) \to 0$ almost surely. But $\mathrm{Var}(p_n) \to \infty$, so with smaller and smaller probability a price path might shoot to a higher and higher level, before eventually falling to zero.[2]

When the central bank changes the interest rates, it affects prices but not real consumption (since that is always $Y_n(\omega)$). It is somewhat surprising to note that though prices depend on $Y_n(\omega)$, the price *change* resulting from different interest rates does not.

Corollary 4.9. *Consider two interest rates ρ and ρ' with corresponding equilibrium prices $p_n(\omega)$ and $p'_n(\omega)$. Then*

$$\frac{p'_n(\omega)}{p_n(\omega)} = \left[\frac{(1+\rho')}{(1+\rho)}\right]^n \frac{\rho}{\rho'}$$

for all ω and all n.

Proof. This follows immediately from (4.10), (4.8), and (4.6′) and their interaction. □

5 Model with an active bank

Suppose that we are in exactly the same situation as the one studied in Section 4, and under exactly the same assumptions, except that now the bank sets an interest rate $\rho(y) \in [0, \infty)$ in each period based on the observed value y of the endowment variable Y in the period, where $\rho(Y) > 0$ holds with positive probability. As in the previous section, we assume that the agent's expenditure

$$\mathcal{B}(y, m) = \mathbf{b}(y)m$$

[2] To see that $\mathrm{Var}(m_n) \to \infty$ when $E[\tau_\rho(y)] = 1$, observe that

$$\mathrm{Var}(m_n) = E(m_n^2) - (E(m_n))^2 = E(m_n^2) - m_1^2$$

and

$$E(m_n^2) = m_1^2 \prod_{k=1}^{n-1} E(\tau(Y_n)^2) = m_1^2[E(\tau(Y)^2)]^{n-1}$$

where

$$E(\tau(Y)^2) > [E\tau(Y)]^2 = 1.$$

is proportional to his liquid wealth m, when the observed endowment value is y. The old calculation (4.1) shows that

$$m_{n+1}(\omega) = \tau(Y_n(\omega))\, m_n(\omega)$$

where now the rate of growth of liquid wealth is

$$\tau(y) \triangleq 1 + \rho(y) - \rho(y)\mathbf{b}(y). \tag{5.1}$$

We shall construct an equilibrium for this model, which generalizes that of Theorem 4.2. Then we shall consider the question of whether the bank can select the interest rates $\rho(y)$ in such a way that prices are bounded away from zero and infinity for all n. (In the next section, we shall consider the more difficult problem, where the bank strives to maintain *constant* prices. We shall conclude that this is typically impossible to achieve in our models.)

The optimality condition (4.5) is replaced in this section by

$$\frac{yu'(y)}{\mathbf{b}(y)} = \frac{\beta(1+\rho(y))}{1+\rho(y)-\rho(y)\mathbf{b}(y)} \cdot E\left[\frac{Yu'(Y)}{\mathbf{b}(Y)}\right], \quad \forall\, y \in \mathcal{Y}. \tag{5.2}$$

Theorem 5.1. *Given the interest rate function $\rho(\cdot)$, with $\lambda\{y \in \mathcal{Y} : \rho(y) > 0\} > 0$, there is a unique function $\mathbf{b}(\cdot)$ such that equation (5.2) holds, namely*

$$\frac{1}{\mathbf{b}(y)} = \frac{\rho(y)}{1+\rho(y)} + \frac{\beta}{1-\beta} \cdot \frac{1}{yu'(y)} \cdot E\left(\frac{\rho(Y)}{1+\rho(Y)} \cdot Yu'(Y)\right), \; y \in \mathcal{Y}. \tag{5.3}$$

Defining $\mathcal{B}(m,y) = m\mathbf{b}(y)$ and $\mathcal{P}(m,y) = m\mathbf{b}(y)/y$ gives the unique neutral, stationary equilibrium.

Proof. The proof is similar to the proofs of Lemma 4.1 and Theorem 4.2, which can be found in the Appendix. □

5.1 Stabilizing prices

Is there some interest-rate policy $\rho(\cdot) = \{\rho(y)\}_{y\in\mathcal{Y}}$ for the central bank, that stabilizes prices, keeping them bounded away from both zero and infinity forever? If the interest rate ρ is fixed for all time, and if $\tau(y)$ is not identically 1 (almost surely), then the arguments of Corollaries 4.6–4.7 show that equilibrium prices cannot remain bounded away from both zero and infinity forever. But in the next theorem we show that by taking $\rho(y) = 0$ or $\rho(y)$ just high enough that $\mathbf{b}(y) = 1$, the active central bank can stabilize prices. It does so effectively by putting itself out of business, reducing the equilibrium to the no-banking equilibrium of Section 3.

Theorem 5.2. *Suppose Y is bounded away from both zero and infinity, almost surely. Then there is indeed an active interest-rate policy $\rho(\cdot)$ for the central bank, that stabilizes prices $\{p_n\}_{n\in\mathbb{N}}$: there exists $K > 0$ such that in the unique neutral stationary equilibrium corresponding to $\rho(\cdot)$*, we have

$$0 < \frac{1}{K} < p_n(\omega) < K < \infty \quad \textit{for all } n \geq 1 \textit{ and almost all } \omega \in \Omega.$$

Proof. Consider the equilibrium without a bank given in Section 3; in particular, with $\mathbf{b}(y) = 1 \wedge (yu'(y)/a)$. Define

$$\rho(y) = \left\{ \begin{array}{ll} \dfrac{yu'(y)}{a} - 1 & \text{, if } yu'(y) \geq a \\ 0 & \text{, if } yu'(y) < a \end{array} \right\}$$

where $a \equiv \beta E[u'(Y)/\mathbf{p}(Y)]$. Recalling from (3.6) that $\mathbf{b}(y) = 1$ if $yu'(y) \geqq a$, this gives $\tau(y) = 1$ for all $y \in \mathcal{Y}$. Recalling that $1/\mathbf{p}(y) = y/\mathbf{b}(y)$, we see that the optimality condition (5.2) is also satisfied. By Theorem 5.1, this is an equilibrium. (The reader can check that the set $\{y \in \mathcal{Y} : yu'(y) > a\}$ has positive λ-measure, hence so does the set $\{y \in \mathcal{Y} : \rho(y) > 0\}$, and then the explicit equilibrium constructed here corresponds to the equilibrium given by Theorem 5.1.) By (3.7) these prices are indeed bounded away from zero and infinity. □

5.2 Fixed prices

Another possible goal for an active bank might be to hold prices *exactly* constant, rather than holding prices within finite bounds as in Theorem 5.2. However, if the endowment variable Y is not itself constant, *it is typically impossible for the bank actively to adjust interest rates so as to hold prices constant*. To avoid unenlightening technicalities, we shall give a proof only for the special case where the endowment variable Y takes three values a, b, and c with positive probabilities where $0 < a < b < c$. We assume this special structure for the rest of the subsection. Our proof applies to all stationary equilibria, not just to neutral stationary equilibria.

Suppose that we want the price p_n to be the same in every period n, say $p_n \equiv 1$. Thus, for each value $y \in \mathcal{Y} \equiv \{a, b, c\}$, if $Y_n = y$, we require

$$p_n = b_n/y = 1,$$

thus

$$b_n = y,$$

for all $y \in \{a, b, c\}$. The optimality condition (5.2) takes the form:

$$\frac{u'(y)}{1} = \beta(1 + \rho(y)) \cdot E\left[\frac{u'(Y)}{1}\right], \quad y \in \{a, b, c\},$$

or equivalently

$$1 + \rho(y) = \frac{u'(y)}{\beta \cdot E[u'(Y)]}, \quad y \in \{a, b, c\}.$$

Suppose $u'(c) < u'(b) < u'(a)$. If $\rho(c) \geq 0$, then $\rho(a) > \rho(b) > 0$.

Next, we look at the behavior of the liquid wealth of the agent. The law of motion gives

$$m_{n+1} = (1 + \rho(Y_n)) \cdot (m_n - Y_n) + Y_n.$$

An easy proof by induction shows that, if $Y_1 = Y_2 = \cdots = Y_n = y$, then,

$$m_n = (1 + \rho(y))^{n-1} \cdot (m_1 - y) + y.$$

Now consider possible values for the initial money-supply m_1. If $m_1 < a$ and $Y_1 = Y_2 = \cdots = Y_n = a$, then for very large n we have

$$m_n = (1 + \rho(a))^n \cdot (m_1 - a) + a < 0,$$

a contradiction. On the other hand, if $m_1 > a$, and $Y_1 = Y_1 = \cdots = Y_n = a$, then $m_n = (1 + \rho(a))^n (m_1 - a) + a > c$ for sufficiently large n. But then following the rule $b(y) = y$, the agent will never spend more than $c < m_k$ for all $k \geq n$, which cannot be optimal, another contradiction. Thus there is no equilibrium unless $m_1 = a$. Applying the same argument, there is no equilibrium unless $m_1 = b$, a contradiction.

6 Equilibrium with low information

Suppose now that the agent does not know $Y_n(\omega)$ or $p_n(\omega)$ when he visits the bank or decides on his expenditure $b_n(\omega)$ at time n. We shall show that the classical Fisher equation is restored in this case.

Formally, the only change we need to make in the model is to leave Y_n and p_n out of the set of random variables generating the σ-field $\mathcal{F}_n$. Hence the agent's bid must be

$$\mathcal{B}(m, y) = m\mathbf{b}, \tag{6.1}$$

where $\mathbf{b} > 0$ is now constant. Equilibrium prices $\mathcal{P}(m, y)$ will now satisfy

$$\mathcal{P}(m, y) = \mathcal{B}(m, y)/y = m\mathbf{b}/y. \tag{6.2}$$

Since $\mathbf{b}$ is now a constant, so is the rate of growth of liquid wealth, namely:

$$\tau = 1 + \rho - \rho\mathbf{b}. \tag{6.3}$$

6.1 Low information equilibrium without a bank

First, we study the case without a bank. The first-order condition (3.2) now becomes

$$(1 - \mathbf{b}) \wedge \left(E\left[\frac{Yu'(Y)}{\mathbf{b}}\right] - \beta E\left[\frac{Yu'(Y)}{\mathbf{b}}\right]\right) = 0. \tag{6.4}$$

It follows immediately that $\mathbf{b} = 1$ in neutral, stationary equilibrium. The next theorem implies that $\mathcal{B}(m, y) = m$ and $\mathcal{P}(m, y) = m/y$ is indeed an equilibrium.

Theorem 6.1. *In the low information model without a bank, there is a unique neutral, stationary equilibrium $\mathcal{P}$, in which*

$$\mathcal{P}(m, y) = m/y.$$

Proof. We only need verify that $\mathcal{B}(m, y) \equiv m$ is optimal for the agent, given the prices above. The proof is short so we provide it here, rather than deferring it to the Appendix.

Consider more generally a single agent with initial wealth $s \geq 0$, but who will receive m as revenue each period. The agent can bid any amount $b \in [0, s]$, receive $u(b/\mathcal{P}(m, Y))$ in utility, and then move to $s - b + Y\mathcal{P}(m, Y) = s - b + m$ at the next stage. Let $V(s)$ be the optimal reward for this agent. Then the function $V(\cdot)$ satisfies the Bellman equation

$$\begin{aligned} V(s) &= \sup_{0 \leq b \leq s} \left[E\left(u\left(\frac{b}{\mathcal{P}(m, Y)} \right) \right) + \beta \cdot V(s - b + m) \right] \\ &= \sup_{0 \leq b \leq s} [\tilde{u}(b) + \beta \cdot V(s - b + m)], \end{aligned} \tag{6.5}$$

where

$$\tilde{u}(b) \equiv E\left[u\left(\frac{b}{\mathcal{P}(m, Y)} \right) \right] = E\left[u\left(\frac{bY}{m} \right) \right], \quad b \in [0, \infty)$$

is concave, and can be regarded as another utility function. Standard arguments show that $V(\cdot)$ inherits from $u(\cdot)$ the properties of continuity, concavity and strict increase. Consequently,

$$\psi(b; s) \equiv \tilde{u}(b) + \beta \cdot V(s - b + m), \quad 0 \leq b \leq s,$$

has a point of maximum, namely

$$c(s) \in \arg\max \psi(\cdot; s).$$

We need to show that $c(m) = m$. Of course, $c(m) \leq m$, by the rules of the game. Suppose, by way of contradiction, that $c(m) < m$. Now

$$V(m) = \tilde{u}(c(m)) + \beta \cdot V(2m - c(m)),$$

and clearly,

$$V(c(m)) \geq \tilde{u}(c(m)) + \beta \cdot V(c(m) - c(m) + m) = \tilde{u}(c(m)) + \beta \cdot V(m),$$

from (6.5). Subtracting the expression for $V(c(m))$ from $V(m)$ gives the first inequality below, and the strict increase of $V(\cdot)$ and the assumption $c(m) < m$ then imply

$$V(m) - V(c(m)) \leq \beta[V(2m - c(m)) - V(m)] < V(2m - c(m)) - V(m),$$

contradicting the concavity of $V(\cdot)$. □

In the equilibrium of Theorem 6.1, the money supply stays fixed and the successive prices are $m_1/Y_1, m_1/Y_2, \ldots$. Although they fluctuate randomly, these prices have the same distribution, and are bounded away from zero and infinity. There is no inflation or deflation in this economy.

6.2 A low-information model with a bank

Again, agents must bid without knowledge of their endowment in each period. However, they are now permitted to borrow or make deposits in a bank. The bank charges borrowers and pays depositors at a fixed rate of interest $\rho \in (0, \infty)$.

The first-order condition (4.5) now becomes

$$E\left[\frac{Yu'(Y)}{\mathbf{b}}\right] = \beta(1+\rho) \cdot E\left[\frac{Yu'(Y)}{\tau\mathbf{b}}\right]. \tag{6.6}$$

This can only be satisfied if

$$\tau = \beta(1+\rho), \tag{6.7}$$

restoring the old Fisher equation. Combining (6.3) and (6.7) gives

$$\mathbf{b} = (1-\beta)(1+\rho)/\rho \tag{6.8}$$

Theorem 6.2. *The low-information economy with a bank has an equilibrium* $(\mathbf{p}, \mathbf{b})$ *at which*

$$\mathcal{B}(m, y) = = \frac{(1-\beta)(1+\rho)}{\rho} m,$$
$$\mathcal{P}(m, y) = \frac{(1-\beta)(1+\rho)}{\rho} \frac{m}{y}.$$

Proof. See Appendix.

7 Extensions

7.1 Production

We considered a simple representative agent economy in which equilibrium consumption was always the same, $x_n(\omega) = Y_n(\omega)$, independent of central bank policy. It would be interesting to consider a model with heterogeneous agents, or with a representative agent with production, in which the central bank had to balance the twin goals of price stabilization and efficient consumption.

7.2 Heterogeneous agents

The harmonic Fisher equation was derived from the premise that

$$m_{n+1} = f(Y_n) m_n, \tag{7.1}$$

where Y_n represented the endowment variable at time n. In a heterogeneous agent economy, we could take Y_n to represent the real shocks to all the agents. But equation (7.1) would not hold in general, because the distribution of liquid wealth would also matter. It would be very interesting to work out whether there is an analogue nevertheless for the Fisher equation.

7.3 Differential information

We present in this subsection a simple example involving two types of agents, who differ only in the information they receive about their respective endowments. As the example illustrates, differences in information may result in differences in wealth and consumption.

Example 7.1. Assume that there is no bank, that every agent is risk-neutral with utility function $u(x) \equiv x$, and that the endowment variable Y takes on the values 1 and 5 with probability 1/2 each. Let the discount factor be $\beta = 1/2$ and let the supply of money held by the agents be $m = 1$. Finally, suppose that half of the agents, called type 1, have low information in that they have no knowledge of the endowment variable Y before bidding in each period; and that the other half of the agents, called type 2, have high information in that they do know Y before bidding.

Then there is an equilibrium with two wealth states: In the first, type 1 agents have wealth $s = 1$, and type 2 agents have the same wealth $\tilde{s} = 1$; in the second type 1 agents have wealth $s = 3/5$ and type 2 agents have wealth $\tilde{s} = 7/5$. It can be shown that, in equilibrium, an optimal strategy for type 1 agents is always to bid their entire wealth, and an optimal strategy for type 2 agents is to bid all if $Y = 5$, but to bid 1/5 if $Y = 1$ and $\tilde{s} = 1$ and to bid 3/5 if $Y = 5$ and $\tilde{s} = 7/5$. The price depends on the value of Y. For example, if $s = \tilde{s} = 1$ and $Y = 1$, then the total bid is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{5} = \frac{3}{5}$$

and the price is

$$p_1 = \frac{3/5}{1} = 3/5.$$

The law of motion gives the new wealth values for the two types as

$$s_1 = 1 - 1 + \frac{3}{5} \cdot 1 = \frac{3}{5}, \; \tilde{s}_1 = 1 - \frac{1}{5} + \frac{3}{5} \cdot 1 = \frac{7}{5}.$$

If $s = \tilde{s} = 1$ and $Y = 5$, then the price is

$$p_2 = \frac{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1}{5} = \frac{1}{5},$$

and the new wealth values are

$$s_1 = \tilde{s}_1 = 1 - 1 + \frac{1}{5} \cdot 5 = 1.$$

Similar calculations show that for $s = 3/5$, $\tilde{s} = 7/5$, and $Y = 1$, the price is $p_1 = 3/5$ and the next wealth values are $s_1 = 3/5$, $\tilde{s}_1 = 7/5$; while for $s = 3/5$, $\tilde{s} = 7/5$, $Y = 5$, the price is $p_2 = 1/5$ and $s_1 = \tilde{s}_1 = 1$. If the economy is equally likely to start in either of the two wealth states (1,1) and (3/5,7/5), then another easy calculation shows that the average daily utility earned by type 1 agents is 8/3 and that earned by the better-informed type 2 agents is 10/3.

8 Appendix: The proofs

Here we supply the proofs that were omitted from the main body of the paper.

8.1 Proof of Theorem 3.1

We must show that, if prices are given by (3.4) and (3.5), then it is optimal for an agent with wealth m and endowment y to bid $\mathcal{B}(m,y) = y\mathcal{P}(m,y)$ in any given period. To show that this is so, consider more generally an agent with initial wealth $s \geq 0$ and endowment y. The agent can bid any amount $b \in [0,s]$, and begin the next period with wealth $s - b + y\mathcal{P}(m,y) = s - b + \mathcal{B}(m,y)$. By assumption, the agent is so small that his actions do not affect the price or the total wealth m, which remains constant in this economy with no bank. Thus, for simplicity of notation, we write

$$p(y) \equiv \mathcal{P}(m,y) = m \cdot \min\left[\frac{1}{y},\ \frac{u'(y)}{a}\right] \text{ and}$$
$$b(y) \equiv \mathcal{B}(m,y) = y \cdot \mathcal{P}(m,y) = m \cdot \min\left[1,\ \frac{yu'(y)}{a}\right]$$

throughout this subsection.

The agent with wealth s and endowment y faces a dynamic programming problem with optimal reward function $V(s,y)$ satisfying the Bellman equation:

$$V(s,y) = \sup_{0 \leq b \leq s}\left[u\left(\frac{b}{p(y)}\right) + \beta \cdot E[V(s - b + b(y), Y)]\right],\quad s \geq 0,\ y \in \mathcal{Y}.$$

Notice that this dynamic programming problem has state space $[0,\infty) \times \mathcal{Y}$, action sets $\mathcal{A}(s,y) = [0,s]$, law of motion

$$(s,y) \mapsto (s - b + b(y), Y)$$

under action b, and daily reward $r((s,y),b) = u(b/p(y))$. It suffices to show that the optimal bid b at state (m,y) is $b(y)$, for every $y \in \mathcal{Y}$.

To prove that this is so, we introduce *another* dynamic programming problem with the same states (s,y) and the same law of motion, but with larger action sets $\tilde{\mathcal{A}}(s,y) = [-m,s]$ and with daily reward for taking action b at (s,y) equal to

$$\tilde{r}((s,y),b) = u_y(b) \equiv A_y + \lambda_y b,\quad -m \leq b < \infty,$$

where

$$\lambda_y \equiv \frac{u'(y)}{p(y)} \equiv \frac{1}{m} \cdot \max\{a, yu'(y)\},\quad A_y \equiv u(y) - \lambda_y b(y). \tag{8.1}$$

Notice that

$$u_y(b(y)) = u(y) = u\left(\frac{b(y)}{p(y)}\right),$$
$$u'_y(b(y)) = \lambda_y = \frac{1}{p(y)}\ u'(y) = \frac{1}{p(y)}\ u'\left(\frac{b(y)}{p(y)}\right).$$

Thus, the affine function $u_y(\cdot)$ is the tangent line to the graph of the concave function $b \mapsto u(b/p(y))$ at the point $b = b(y)$. In particular, $u_y(b) \geq u(b/p(y))$ for all $b \in [0, s]$. Consequently, the expected return from any strategy π, which is available in both problems, will be at least as large in the modified problem as it was in the original problem.

Let π^* be the strategy that, at each state (s, y), uses action

$$\mathcal{B}^*(s, y) \equiv \left\{ \begin{array}{ll} s & ;\text{if } \ yu'(y) \geq a \\ b(y) + s - m & ;\text{if } \ yu'(y) < a \end{array} \right\} \in \widetilde{\mathcal{A}}(s, y). \tag{8.2}$$

Notice that, for every $y \in \mathcal{Y}$, we have

$$\mathcal{B}^*(m, y) \equiv \left\{ \begin{array}{ll} m & ;\text{if } \ yu'(y) \geq a \\ b(y) & ;\text{if } \ yu'(y) < a \end{array} \right\} = b(y),$$

and that under the law of motion

$$(m, y) \mapsto (m - \mathcal{B}^*(m, y) + b(y), Y) = (m, y).$$

Thus, for an initial state (m, y), the return from π^* is the same in both problems; namely,

$$u(y) + \frac{\beta}{1-\beta}\ E[u(Y)].$$

It now suffices to show that the strategy π^* is optimal in the modified problem, for it must then be optimal at states (m, y) in the original problem as well.

Let $W(s, y)$ be the optimal reward function in the modified problem. Then W satisfies the Bellman equation

$$W(s, y) = (TW)(s, y),$$

where T is the operator

$$(T\Phi)(s, y) \equiv \sup_{-m \leq b \leq s} [u_y(b) + \beta \cdot E\Phi(s - b + b(y), Y)], \tag{8.3}$$

whose domain is the collection of functions $\Phi : [0, \infty) \times \mathcal{Y} \to \mathbb{R}$ for which the right-hand side of (8.3) is well-defined.

Define $Q(s, y)$ to be the expected return in the modified problem from the strategy π^* at the initial state (s, y), and let

$$v(y) \equiv Q(m, y) = u(y) + \frac{\beta}{1-\beta}\ E[u(Y)]. \tag{8.4}$$

Clearly $Q(s,y) \le W(s,y)$, and $E[v(Y)] = (1-\beta)^{-1}E[u(Y)]$, so

$$v(y) = u(y) + \beta \cdot E[v(Y)]. \tag{8.5}$$

Lemma 8.1. *For every initial state* (s,y)*, we have:* (i) $Q(s,y) = v(y) + \lambda_y(s-m)$, *and* (ii) $(TQ)(s,y) = Q(s,y)$.

Proof. From (8.2), we have $s - \mathcal{B}^*(s,y) + b(y) = m$. Hence,

$$\begin{aligned} Q(s,y) &= u_y(\mathcal{B}^*(s,y)) + \beta \cdot E[Q(m,Y)] \\ &= A_y + \lambda_y \mathcal{B}^*(s,y) + \beta \cdot E[v(Y)] \\ &= A_y + \lambda_y b(y) + \lambda_y(s-m) + \beta \cdot E[v(Y)] \\ &= u(y) + \lambda_y(s-m) + \beta \cdot E[v(Y)] \\ &= v(y) + \lambda_y(s-m), \end{aligned} \tag{8.6}$$

thanks to (8.4), (8.5), and (i) is verified. To verify (ii), let

$$\begin{aligned} \psi_y(b) &\equiv u_y(b) + \beta \cdot E[Q(s-b+b(y),Y)] \\ &= A_y + \lambda_y b + \beta \cdot E[v(Y) + \lambda_Y(s-b+b(y)-m)]. \end{aligned} \tag{8.7}$$

The coefficient of b in this expression is

$$\lambda_y - \beta \cdot E[\lambda_Y] \;=\; \frac{u'(y)}{p(y)} - \beta \cdot E\left[\frac{u'(Y)}{p(Y)}\right].$$

By (3.2) and (3.4), this coefficient is positive for $\lambda_y > a$, and the maximum of $\psi_y(\cdot)$ on the interval $[-m,s]$ is then attained at $\mathcal{B}^*(s,y) = s$; whereas for $\lambda_y = a$, the coefficient is zero and in this case every point of the interval, including $\mathcal{B}^*(s,y)$, attains the maximum. In either case, we have:

$$\begin{aligned} (TQ)(s,y) &= \max_{0 \le b \le s} \psi_y(b) = \psi_y(\mathcal{B}^*(s,y)) \\ &= u_y(\mathcal{B}^*(s,y)) + \beta \cdot E[Q(m,Y)] = Q(s,y)\,. \end{aligned}$$

□

Lemma 8.2. *There is a real number* k *such that* $u_y(b) \ge -k > -\infty$ *for all* $y \in \mathcal{Y}$ *and* $b \ge -m$*.*

Proof. This is a simple calculation based on (A.1) and (3.4):

$$\begin{aligned} u_y(b) &= A_y + b\lambda_y \ge A_y - m\lambda_y \\ &= u(y) - [m + b(y)]\frac{u'(y)}{p(y)} \\ &\ge u(y) - 2m\frac{u'(y)}{p(y)} \\ &= \left\{ \begin{array}{ll} u(y) - 2yu'(y) & \text{, for } \; yu'(y) \ge a \\ u(y) - 2m & \text{, for } \; yu'(y) < a \end{array} \right\}. \end{aligned}$$

By assumption, y is bounded away from zero and infinity. Hence, $u(y)$ and $-yu'(y)$ are bounded away from negative infinity. □

To complete the proof of Theorem 3.1, first notice that, by adding the constant k from Lemma 8.2 to the daily reward $\tilde{r}((s,y),b) = u_y(b)$, we obtain an equivalent problem with positive daily rewards. Indeed, by adding k to the daily reward, we merely add $k/(1-\beta)$ to the total discounted reward. For a dynamic programming problem with positive daily rewards, a theorem of Blackwell (1966) states that the optimal reward function W is the least nonnegative fixed point of the operator T of (8.3). By Lemma 8.1, Q, is such a fixed point and, being the expected reward from π^*, $Q \leq W$. Hence, $Q = W$ and π^* is optimal. The proof of Theorem 3.1 is now complete.

8.2 Proof of Lemma 4.1

Set $\theta \equiv E[Yu'(Y)/\mathbf{b}(Y)]$ and rewrite (4.5) as

$$\frac{yu'(y)}{\mathbf{b}(y)} = \frac{(1+\rho)\beta\theta}{\tau(y)} = \frac{(1+\rho)\beta\theta}{1+\rho-\rho\mathbf{b}(y)}$$

or, equivalently,

$$\frac{yu'(y)}{\mathbf{b}(y)} = \frac{\rho}{1+\rho} \cdot yu'(y) + \beta\theta. \tag{8.8}$$

Integrate with respect to the distribution of the random variable Y, to get

$$\theta = \frac{\rho}{1+\rho} E[Yu'(Y)] + \beta\theta$$

or equivalently

$$\theta = \frac{\rho}{(1-\beta)(1+\rho)} E[Yu'(Y)].$$

Substituting for θ in (8.8) and dividing by $yu'(y)$, we get

$$\begin{aligned}\frac{1}{\mathbf{b}(y)} &= \frac{\rho}{1+\rho}\left[1 + \frac{\beta}{1-\beta} \cdot \frac{E[Yu'(Y)]}{yu'(y)}\right] \\ &= \frac{\rho}{1+\rho}\left[\frac{(1-\beta)yu'(y) + \beta E[Yu'(Y)]}{(1-\beta)yu'(y)}\right].\end{aligned}$$

Take the reciprocal to obtain (4.6).

We have shown that bids $\mathbf{b}(y)$ that satisfy (4.5) must be given by formula (4.6). The argument reverses to show that, if we define $\mathbf{b}(\cdot)$ by (4.6), then (4.5) holds. Thus (4.6) gives the unique solution to the functional equation (4.5).

8.3 Proof of Theorem 4.2

The proof is similar to that of Theorem 3.1. Let $\mathbf{b}(y)$ be defined by (4.6) and suppose prices satisfy $\mathcal{P}(m,y) = m\mathbf{b}(y)/y$. We must show that $\mathcal{B}(m,y) = m\mathbf{b}(y)$ is the optimal bid for an agent with wealth m and endowment y. For the proof we consider the more general situation of an agent with wealth $s \geq 0$ and endowment $y \in \mathcal{Y}$. The agent can bid any amount $b \in [0, s + \mathcal{B}(m,y)/(1+\rho)]$, borrowing or lending the difference between s and b according as s is smaller or larger than b. The agent receives $u(b/\mathcal{P}(m,y))$ in utility and begins the next period with wealth $(1+\rho)(s-b) + y\mathcal{P}(m,y) = (1+\rho)(s-b) + \mathcal{B}(m,y)$. The total wealth in the economy becomes $\tau(y)m$ as in (4.1) and (4.2). Thus the agent faces a dynamic programming problem with optimal reward function $V(s,y,m)$, which satisfies the Bellman equation

$$V(s,y,m) = \sup_{0\leq b\leq s+\mathcal{B}(m,y)/(1+\rho)} \times \left[u\left(\frac{b}{\mathcal{P}(m,y)}\right) + \beta EV((1+\rho)(s-b) + \mathcal{B}(m,y), Y, \tau(y)m)\right]. \tag{8.9}$$

This dynamic programming problem has state space $[0,\infty) \times \mathcal{Y} \times [0,\infty)$, action sets $\mathcal{A}(s,y,m) = [0, s + \mathcal{B}(m,y)/(1+\rho)]$, law of motion

$$(s,y,m) \mapsto ((1+\rho)(s-b) + \mathcal{B}(m,y), Y, \tau(y)m)$$

under action b, and daily reward function $r((s,y,m),b) = u(b/\mathcal{B}(m,y))$. What must be shown is that an optimal bid b at states of the form (m,y,m) is $\mathcal{B}(m,y)$.

As in the proof of Theorem 3.1, we introduce a modified dynamic programming problem with the same states (s,y,m) and the same law of motion, but with larger action sets $\widetilde{\mathcal{A}}(s,y,m) = [-m, s + \mathcal{B}(m,y)/(1+\rho)$ and with daily reward

$$\tilde{r}((s,y,m),b) = u_{y,m}(b) = A_{y,m} + \lambda_{y,m} b,$$

where

$$\lambda_{y,m} \equiv \frac{u'(y)}{\mathcal{P}(m,y)} = \frac{yu'(y)}{m\mathbf{b}(y)}, \quad A_{y,m} = u(y) - \lambda_{y,m}\mathcal{B}(m,y). \tag{8.10}$$

The affine function $u_{y,m}(\cdot)$ is tangent to the concave function $b \mapsto u(b/\mathcal{P}(m,y))$ at the point $b = \mathcal{B}(m,y)$. Thus $u_{y,m}(y) \geq u(b/\mathcal{P}(m,y))$ for all b; so the return from any strategy available in both problems is at least as large in the modified problem as in the original problem.

Let π^* be the strategy for the modified problem that, at each state (s,y,m), uses the action

$$\mathcal{B}^*(s,y,m) = \mathcal{B}(m,y) + (s-m). \tag{8.11}$$

Since

$$\mathcal{B}^*(m,y,m) = \mathcal{B}(m,y),$$

and, under the law of motion,

$$\begin{aligned}(m, y, m) \mapsto &((1+\rho)(m - \mathcal{B}(m, y)) + \mathcal{B}(m, y), Y, \tau(y)m) \\ &= (\tau(y)m, Y, \tau(y)m),\end{aligned} \tag{8.12}$$

the strategy π^* chooses the same actions and has the same expected return in both problems, for an initial state of the form (m, y, m). Thus, if π^* is optimal in the modified problem, then it must be optimal in the original problem as well for initial states (m, y, m).

Therefore, it suffices to show π^* is optimal in the modified problem. To do so, let $W(s, y, m)$ be the optimal reward and let $Q(s, y, m)$ be the expected return from π^* for an initial state (s, y, m). The Bellman equation can be written as

$$W(s, y, m) = (TW)(s, y, m),$$

where

$$(T\Phi)(s, y, m) \equiv \sup_{-m \le b \le s + \mathcal{B}(m,y)/(1+\rho)} \Delta(b; s, y, m)$$

with

$$\Delta(b; s, y, m) \triangleq [u_{y,m}(b) + \beta E[\Phi((1+\rho)(s-b) + \mathcal{B}(m, y), Y, \tau(y)m)]$$

is an operator acting on function $\Phi : [0, \infty) \times \mathcal{Y} \times [0, \infty) \to \mathbb{R}$ for which the right-hand side of the equation above is well defined.

By analogy with (8.4), we also define

$$v(y) \equiv Q(m, y, m) = u(y) + \frac{\beta}{1-\beta} E[u(Y)] \tag{8.13}$$

and observe that

$$v(y) = u(y) + \beta E[v(Y)]. \tag{8.14}$$

Lemma 8.3. *For every initial state* (s, y, m)*, we have*

(i) $Q(s, y, m) = v(y) + \lambda_{y,m}(s - m)$,
(ii) $(TQ)(s, y, m) = Q(s, y, m)$.

Proof. (i) By (8.11),

$$(1+\rho)(s - \mathcal{B}^*(s, y, m)) + \mathcal{B}(m, y) = (1+\rho)(m - \mathcal{B}(m, y)) + \mathcal{B}(m, y)) = \tau(y)m.$$

Hence, by definition of Q, (8.12), (8.13), and (8.14),

$$\begin{aligned}Q(s, y, m) &= u_{y,m}(\mathcal{B}^*(s, y, m)) + \beta E Q(\tau(y)m, Y, \tau(y)m) \\ &= A_{y,m} + \lambda_{y,m}\mathcal{B}^*(s, y, m) + \beta E[v(Y)] \\ &= u(y) - \lambda_{y,m}\mathcal{B}(m, y) + \lambda_{y,m}(\mathcal{B}(m, y) + s - m) + \beta \cdot E[v(Y)] \\ &= v(y) + \lambda_{y,m}(s - m).\end{aligned}$$

(ii) Define

$$\psi(b) \equiv u_{y,m}(b) + \beta \cdot E[Q((1+\rho)(s-b) + \mathcal{B}(m,y), Y, \tau(y)m)].$$

Part (i) implies that

$$\psi(b) = A_{y,m} + \lambda_{y,m} b + \beta \cdot E[v(Y) + \lambda_{Y,\tau(y)m}((1+\rho)(s-b) + \mathcal{B}(m,y) - \tau(y)m)].$$

The coefficient of b on the right-hand side is

$$\begin{aligned}\lambda_{y,m} - \beta(1+\rho)E[\lambda_{Y,\tau(y)m}] &= \frac{u'(y)}{\mathcal{P}(m,y)} - \frac{\beta(1+\rho)}{\tau(y)} E\left[\frac{u'(Y)}{\mathcal{P}(m,Y)}\right] \\ &= \frac{1}{m}\left[\frac{yu'(y)}{\mathbf{b}(y)} - \frac{\beta(1+\rho)}{\tau(y)} E\left[\frac{Yu'(Y)}{\mathbf{b}(Y)}\right]\right] \\ &= 0,\end{aligned}$$

by (4.5). Thus ψ is a constant function and (ii) is a trivial consequence. □

Lemma 8.4. *There is a real number k such that $u_{y,m}(b) \geq -k > -\infty$ for all $y \in \mathcal{Y}$, $m \in [0,\infty)$, and $b \geq -m$.*

Proof. Calculate as follows:

$$\begin{aligned}u_{y,m}(b) &= A_{y,m} + \lambda_{y,m} b \geq A_{y,m} - \lambda_{y,m} m \\ &= u(y) - \lambda_{y,m}[m + \mathcal{B}(m,y)] = u(y) - m[1 + \mathbf{b}(y)]\frac{yu'(y)}{m\mathbf{b}(y)} \\ &\geq u(y) - \frac{yu'(y)}{\mathbf{b}(y)}\left[1 + \frac{1+\rho}{\rho}\right] \\ &= u(y) - \frac{1+2\rho}{1+\rho}\left[yu'(y) + \frac{\beta}{1-\beta}E[Yu'(Y)]\right].\end{aligned}$$

The inequality is by (4.6), and the last two equality by (4.5). The desired result follows because $u(Y)$ and $Yu'(Y)$ are each bounded away from negative infinity. □

The proof of Theorem 4.2 can now be completed by the same argument that was used to complete the proof of Theorem 3.1 following Lemma 8.2.

8.4 Proof of Theorem 6.2

Consider the situation of an agent with wealth $s \geq 0$ at the beginning of a period when the total wealth in the economy is $m > 0$. The agent can bid any amount $b \in [0, s + \mathbf{b}/(1+\rho)]$ where $\mathbf{b}$ is defined by (6.8). The agent does not know the value of his endowment Y or the price $\mathcal{P}(m,Y)$ prior to choosing his bid b, but can calculate the expected utility $E[u(b/\mathcal{P}(m,Y))]$ that he will receive from consumption. He also knows that he will begin the next period with wealth

$(1+\rho)(s-b)+Y\mathcal{P}(m,Y) = (1+\rho)(s-b)+\mathbf{b}m$. Thus the agent faces a dynamic programming problem with optimal reward function $V(s,m)$ satisfying:

$$V(s,m) = \sup_{0 \le b \le s+\mathbf{b}m/(1+\rho)} \Theta(b;s,m), \tag{8.15}$$

where

$$\Theta(b;s,m) \triangleq \left[Eu\left(\frac{b}{\mathcal{P}(m,Y)}\right) + \beta \cdot V((1+\rho)(s-b)+\mathbf{b}m, \tau m) \right].$$

It suffices to show that the optimal bid at states (m,m) is $b = \mathbf{b}m$.

This dynamic programming problem corresponds to a special case of the one studied above in Section 8.3. To obtain the special case, replace the utility function of (8.9) by

$$\tilde{u}(x) \equiv E[u(xY)]$$

and then replace Y by the constant variable $\tilde{Y} \equiv 1$. The Bellman equation (8.15) is then equivalent to (8.9). In particular, the optimal bid at states $(m,1,m)$ is $\mathcal{B}(m,1) = \mathbf{b}m$, where the $\mathbf{b}$ of (4.6) is the same as that of (6.8) because $\tilde{Y}$ is constant.

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