OVERLAPPING GENERATIONS MODEL OF GENERAL EQUILIBRIUM

BY

JOHN GEANAKOPLOS

COWLES FOUNDATION PAPER NO. 1275

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

2009

http://cowles.econ.yale.edu/

The New Palgrave Dictionary of Economics Online

overlapping generations model of general equilibrium

John Geanakoplos
From The New Palgrave Dictionary of Economics, Second Edition, 2008
Edited by Steven N. Durlauf and Lawrence E. Blume

Abstract

The OLG model of Allais and Samuelson retains the methodological assumptions of agent optimization and market clearing from the Arrow–Debreu model, yet its equilibrium set has different properties: Pareto inefficiency, multiplicity, positive valuation of money, and a golden rule equilibrium in which the rate of interest is equal to population growth (independent of impatience). These properties are shown to derive not from market incompleteness, but from lack of market clearing 'at infinity': they can be eliminated with land or uniform impatience. The OLG model is used to analyse bubbles, social security, demographic effects on stock returns, the foundations of monetary theory, Keynesian vs. real business cycle macromodels, and classical vs. neoclassical disputes.

Keywords

agent optimization; Allais, M.; animal spirits; Arrow–Debreu model of general equilibrium; backward induction; bubbles; Cobb–Douglas functions; comparative statics; consumption loan model; continuum of equilibria; cores; demography; double coincidence of wants; equilibrium; existence of equilibrium; expectations sensitivity hypothesis; impatience; incomplete markets; indeterminacy of equilibrium; infinite horizons; involuntary unemployment; Keynesianism; marginal utility of money; market clearing; money; multiple equilibria; new classical macroeconomics; numeraire; overlapping generations models of general equilibrium; Pareto efficiency; Pareto inefficiency; perfect foresight; price normalization; Samuelson, P. A.; sequential equilibrium; social security; Sraffa, P.; sunspots; uncertainty; uniform impatience; uniqueness of equilibrium

Article

The consumption loan model that Paul Samuelson introduced in 1958 to analyse the rate of interest, with or without the social contrivance of money, has developed into what is without doubt the most important and influential paradigm in neoclassical general equilibrium theory outside of the Arrow–Debreu economy. Earlier Maurice Allais (1947) had presented similar ideas which unfortunately did not then receive the attention they deserved. A vast literature in public finance and macroeconomics is based on the model, including studies of the national debt, social security, the incidence of taxation and bequests on the accumulation of capital, the Phillips curve, the business cycle, and the foundations of monetary theory. In this article I give a hint of these myriad applications only in so far as they illuminate the general theory. My main concern is with the relationship between the Samuelson model and the Arrow–Debreu model.

Allais's and Samuelson's innovation was in postulating a demographic structure in which generations overlap, indefinitely into the future; up until then it had been customary to regard all agents as contemporaneous. In the simplest possible example, in which each generation lives for two periods, endowed with a perishable commodity when young and nothing when old, Samuelson noticed a great surprise. Although each agent could be made better off if he gave half his youthful birthright to his predecessor, receiving in turn half from his successor, in the marketplace there would be no trade at all. A father can benefit from his son's resources, but has nothing to offer in return.

This failure of the market stirred a long and confused controversy. Samuelson himself attributed the suboptimality to a lack of double coincidence of wants. He suggested the social contrivance of money as a solution. Abba Lerner suggested changing the definition of optimality. Others, following Samuelson's hints about the financial intermediation role of money, sought to explain the consumption loan model by the incompleteness of markets. It has only gradually become clear that the 'Samuelson suboptimality paradox' has nothing to do with the absence of markets or financial intermediation. Exactly the same equilibrium allocation would be reached if all the agents, dead and unborn, met (in spirit) before the beginning of time and traded all consumption goods, dated from all time periods, simultaneously under the usual conditions of perfect intermediation. Indeed, in the early 20th century Irving Fisher (1907; 1930) implicitly argued that any sequential economy without uncertainty, but with a functioning loan market, could be equivalently described as if all markets met once with trade conducted at present value prices.

Over the years Samuelson's consumption loan example, infused with Arrow–Debreu methods, has been developed into a full-blown general equilibrium model with many agents, and multiple kinds of commodities and production. It is equally faithful to the neoclassical methodological assumptions of agent optimization, market clearing, price taking, and rational expectations as the Arrow–Debreu model. This more comprehensive version of Samuelson's original idea is known as the overlapping generations (OLG) model of general equilibrium.

Despite the methodological similarities between the OLG model and the Arrow–Debreu model, there is a profound difference in their equilibria. The OLG equilibria may be Pareto suboptimal. Money may have positive value. There are robust OLG economies with a continuum of equilibria. Indeed, the more commodities per period, the higher the dimension of multiplicity may be. Finally, the core of an OLG economy may be empty. None of this could happen in any Arrow–Debreu economy.

The puzzle is: why? One looks in vain for an externality, or one of the other conventional pathologies of an Arrow–Debreu economy. It is evident that the simple fact that generations overlap cannot be an explanation, since by judicious choice of utility functions one can build that into the Arrow–Debreu model. It cannot be simply that the time horizon is infinite, as we shall see, since there are classes of infinite horizon economies whose equilibria behave very much like Arrow–Debreu equilibria. It is the combination, that generations overlap indefinitely, which is somehow crucial. In Section 4 I explain how.

Note that in the Arrow–Debreu economy the number of commodities, and hence of time periods, is finite. One is tempted to think that, if the end of the world is put far enough off into the future, it could hardly matter to behaviour today. But recalling the extreme rationality hypotheses of the Arrow–Debreu model, it should not be surprising that such a cataclysmic event, no matter how long delayed, could exercise a strong influence on behaviour. Indeed, the OLG model proves that it does. One can think of other examples. Social security, based on the pay-as-you-go principle in the United States in which the young make payments directly to the old, depends crucially on people thinking that there might always be a future generation. Otherwise the last generation of young will not contribute; foreseeing that, neither will the second-to-last generation of young contribute, nor, working backward, will any generation contribute. Another similar example comes from game theory, in which cooperation depends on an infinite horizon. On the whole, it seems at least as realistic to suppose that everyone believes the world is immortal as to suppose that everyone believes in a definite date by which it will end. (In fact, it is enough that people believe, for every T, that there is positive probability the world lasts past T.)

In Section 1, I analyse a simple one-commodity OLG model from the present value general equilibrium perspective. This illustrates the paradoxical nature of OLG equilibria in the most orthodox setting. These paradoxical properties can hold equally for economies with many commodities, as pointed out in Section 4. Section 2 discusses the possibility of equilibrium cycles in a one-commodity, stationary, OLG economy. In Section 3, I describe OLG equilibria from a sequential markets point of view, and show that money can have positive value.

In the simple OLG economy of Section 1 there are two steady-state equilibria, and a continuum of non-stationary equilibria. Out of all of these, only one is Pareto efficient, and it has the property that the real rate of interest is always zero, just equal to the rate of population growth, independent of the impatience of the consumers or the distribution of endowments between youth and old age. This ‘golden rule’ equilibrium seems to violate Fisher's impatience theory of interest.

In Section 5 I add land to the one-commodity model of Section 1. It turns out that now there is a unique steady-state equilibrium that is Pareto efficient and that has a positive rate of interest, greater than the population growth rate, that increases if consumers become more impatient. Land restores Fisher's view of interest. In this setting it is also possible to analyse the effects of social security.

In Section 6 I briefly introduce variations in demography. It is well known that birth rates in the United States oscillated every 20 years over the 20th century. Stock prices have curiously moved in parallel, rising rapidly from 1945 to 1965, falling from 1965 to 1985, and rising ever since. One might therefore expect stock prices to fall as the post-war baby boom generation retires. But some authors have claimed that these parallel fluctuations of stock prices must be coincidental. Otherwise, since demographic changes are known long in advance, rational investors would have anticipated the price fluctuations and changed them. In Section 6 I allow the size of the generations to alternate and confirm that in OLG equilibrium land prices rise and fall with demography, even though the changes are perfectly anticipated.

In Section 7 I show that not just land but also uniform impatience restores the properties of infinite horizon economies to those found in finite Arrow–Debreu economies.

Section 8 takes up the question of comparative statics. If there is a multiplicity of OLG equilibria, what sense can be made of comparative statics? Section 8 summarizes the work showing that, for perfectly anticipated changes, there is only one equilibrium in the multiplicity that is ‘near’ an original ‘regular’ equilibrium. For unanticipated changes, there may be a multidimensional multiplicity. But it is parameterizable. Hence, by always fixing the same variables, a unique prediction can be made for changes in the equilibrium in response to perturbations. In Section 9 we see how this could be used to understand some of the New Classical–Keynesian disputes about macroeconomic policy. Different theories hold different variables fixed in making predictions.

Section 10 considers a neoclassical–classical controversy. Recall the classical economists’ conception of the economic process as a never-ending cycle of reproduction in which the state of physical commodities is always renewed, and in which the rate of interest is determined outside the system of supply and demand. Samuelson attempted to give a completely neoclassical explanation of the rate of interest in just such a setting. It now appears that the market forces of supply and demand are not sufficient to determine the rate of interest in the standard OLG model. In other infinite-horizon models they do.

Section 11 summarizes some work on sunspots in the OLG model. Uncertainty in dynamic models seems likely to be very important in the future.

An explanation of the puzzles of OLG equilibria without land is given in Section 4: lack of market clearing ‘at infinity’. By appealing to non-standard analysis, the mathematics of infinite and infinitesimal numbers, it can be shown that there is a ‘finite-like’ Arrow–Debreu economy whose ‘classical equilibria’, those price sequences which need not clear the markets in the last period, are isomorphic to the OLG equilibria. Lack of market clearing is also used to explain the suboptimality and the positive valuation of money.

1 Indeterminacy and suboptimality in a simple OLG model

In this section we analyse the equilibrium set of a one-commodity per period, overlapping generations (OLG) economy, assuming that all agents meet simultaneously in all markets before time begins, just as in the Arrow–Debreu model. Prices are all quoted in present value terms; that is, p_t is the price an agent would pay when the markets meet (at time $-\infty$) in order to receive one unit of the good at time t. Although this definition of equilibrium is firmly in the Walrasian tradition of agent optimization and market clearing, we discover three surprises. There are robust examples of OLG economies that possess an uncountable multiplicity of equilibria, that are not in the core, or even Pareto optimal. This lack of optimality (in a slightly different model, as we shall see) was pointed out by Samuelson in his seminal (1958) paper. The indeterminacy of equilibrium in the one-commodity case is usually associated first with Gale (1973). In later sections we shall show that these puzzles are robust to an extension of the model to multiple commodities and agents per period, and to a non-stationary environment. We shall add still another puzzle in Section 3, the positive valuation of money, which is also due to Samuelson.

A large part of this section is devoted to developing the notation and price normalization that we shall use throughout. In any Walrasian model the problem of price normalization (the ‘numeraire problem’) arises. Here the most convenient solution in the long run is not at first glance the most transparent.

Consider an overlapping generation (OLG) economy $E = E_{-\infty,\infty}$ in which discrete time periods t extend indefinitely into the past and into the future, t **Z**. Corresponding to each time period there is a single, perishable consumption good x_t. Suppose furthermore that at each date t one agent is ‘born’ and lives for two periods, with utility

$$u^t(\ldots, x_t, x_{t+1}, \ldots) = a^t \log x_t + (1 - a^t) \log x_{t+1}$$

defined over all vectors

$$x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in L = R_+^{\mathbf{Z}}.$$

Thus we identify the set of agents A with the time periods **Z**. Let each agent t A have endowment

$$e^t = (\ldots, e_t^t, e_{t+1}^t, \ldots) \in L$$

which is positive only during the two periods of his life. Note that

$$\sum_{t \in A} e_s^t = e_s^{s-1} + e_s^s \text{ for all } s \in \mathbf{Z}.$$

An equilibrium is defined as a (present value) price vector

$$p = (\ldots, p_{-1}, p_0, p_1, \ldots) \in L$$

and allocation

$$\bar{x} = [x^t = (\ldots, x_t^t, x_{t+1}^t, \ldots); t \in A]$$

satisfying $\bar{x}$ is feasible, that is,

$$\sum_{t \in A} x_s^t = \sum_{t \in A} e_s^t, \text{ for all } s \in \mathbf{Z} \quad (1)$$

and

$$\sum_{s \in \mathbf{Z}} p_s e_s^t < \infty \text{ for all } t \in A \quad (2)$$

and

$$x^t \in \arg\max_{x \in L} \left\{ u^t(x) \,\middle|\, \sum_{s \in \mathbf{Z}} p_s x_s \le \sum_{s \in \mathbf{Z}} p_s e_s^t \right\}. \quad (3)$$

The above definition of equilibrium is precisely in the Walrasian tradition, except that it allows for both an infinite number of traders and commodities. All prices are finite, and consumers treat them as parametric in calculating their budgets. The fact that the definition leads to robust examples with a continuum of Pareto-suboptimal equilibria calls for an explanation. We shall give two of them, one at the end of this section, and one in Section 4. Note that condition (2) becomes necessary only when we consider models in which agents have positive endowments in an infinite number of time periods.

As usual, the set of (present value) equilibrium price sequences displays a trivial dimension of multiplicity (indeterminacy), since, if p is an equilibrium, so is kp for all scalars $k>0$. We can remove this ambiguity by choosing a price normalization $q_t = p_{t+1} / p_t$, for all $t \in \mathbf{Z}$. The sequence $q = (\ldots, q_{-1}, q_0, \ldots)$ and allocations (x^t; $t \in A$) form an equilibrium if (1) above holds together with

$$x^t \in \arg\max_{x \in L} \left\{ u^t(x) \mid x_t + q_t x_{t+1} \le e_t^t + q_t e_{t+1}^t \right\}. \quad (4)$$

Notice that we have taken advantage of the finite lifetimes of the agents to combine (2) and (3) into a single condition (4). We could have normalized prices by choosing a numeraire commodity, and setting its price equal to one, say p_0=1. The normalization we have chosen instead has three advantages as compared with this more obvious system. First, the q system is time invariant. It does not single out a special period in which a price must be 1; if we relabelled calendar time, then the corresponding relabelling of the q_t would preserve the equilibrium. In the numeraire normalization, after the calendar shift, prices would have to be renormalized to maintain $p_0=1$. Second, on account of the monotonicity of preferences, we know that, if the preferences and endowments are uniformly bounded

$$0 < \underline{a} \le a^t \le \bar{a} < 1,\ \ 0 < \underline{e} \le e^t_t,\ \ e^t_{t+1} \le \bar{e} \le 1 \text{ for all } t \in A$$

then we can specify uniform a priori bounds $\underline{k}$ and $\bar{k}$ such that any equilibrium price vector q must satisfy $\underline{k} \le q_t \le \bar{k}$ for all $t \in$ **Z**. Third, it is sometimes convenient to note that each generation's excess demand depends on its own price. We define

$$[z^t_t(q_t), z^t_{t+1}(q_t)] = (x^t_t - e^t_t, x^t_{t+1} - e^t_{t+1})$$

for x^t satisfying (4), as the excess demand of generation t, when young and when old. We can accordingly rewrite equilibrium condition (1) as

$$z^{t-1}_t(q_{t-1}) + z^t_t(q_t) = 0 \text{ for all } t \in \mathbf{Z}. \tag{5}$$

Let us now investigate the equilibria of the above economy when preferences and endowments are perfectly stationary. To be concrete, let

$$a^t = a \text{ for all } t \in A$$

and let

$$e^t_t = e, \text{ and } e^t_{t+1} = 1 - e, \text{ for all } t \in A$$

where $e > a \ge 1/2$. Agents are born with a larger endowment when young than when old, but the aggregate endowment of the economy is constant at 1 in every time period. Furthermore, each agent regards consumption when young as at least as important as consumption when old $(a \ge 1/2)$, but on account of the skewed endowment the marginal utility of consumption at the endowment allocation when young is lower than when old:

$$\frac{a}{e} < \frac{1-a}{1-e}.$$

If we choose

$$q_t = \bar{q} = \frac{(1-a)e}{(1-e)a} > 1$$

for all $t \in$ **Z**, then we see clearly that at these prices each agent will just consume his endowment; $q = (\ldots, \bar{q}, \bar{q}, \ldots)$ is an equilibrium price vector, with $x^t = e^t$ for all $t \in A$. Note that if we had used the price normalization $p_0=1$, the equilibrium prices would be described by

$$(\ldots, p_0, p_1, p_2, \ldots) = (\ldots, 1, \bar{q}, \bar{q}^2, \ldots)$$

where $p_t \to \infty$ as $t \to \infty$. With $a=1/2$ and $e=3/4$, we get $\bar{q} = 3$ and $p_t = 3^t$.
But there are other equilibria as well. Take $q=(\ldots,1,1,1,\ldots)$, and

$$(x^t_t, x^t_{t+1}) = (a, 1-a) \text{ for all } t \in A$$

This 'golden rule' Pareto equilibrium dominates the autarkic equilibrium previously calculated. With $a=1/2$ and $e=3/4$, we see that (1/2,1/2) is much better for everyone than (3/4,1/4). This raises the most important puzzle of overlapping generations economies: why is it that equilibria can fail to be

Pareto optimal? We shall discuss this question at length in Section 4.
For now, let us observe one more curious fact. We can define the *core* of our economy in a manner exactly analogous to the finite commodity and consumer case. We say that a feasible allocation $x = (x^t; t \in A)$ is in the core of the economy E if there is no subset of traders $A' \subset A$, and an allocation $y = (y^t; t \in A')$ for A' such that

$$\sum_{t \in A'} y^t = \sum_{t \in A'} e^t,$$

and

$$u^t(y^t) > u^t(x^t) \text{ for all } t \in A'.$$

A simple argument can be given to show that the core of this economy is empty. For example, the golden rule equilibrium allocation is Pareto optimal, but not in the core. Since $a<e$, every agent is consuming less when young than his initial endowment. Thus for any $t_0 \in A$, the coalition $A' = \{t \in A | t \geq t_0\}$ consisting of all agents born at time t_0 or later can block the golden rule allocation.
Let us continue to investigate the set of equilibria of our simple, stationary economy. Gale (1973) showed that for any $\bar{q}_0$, with $1 < \bar{q}_0 < \bar{q}$, there is an equilibrium price sequence

$$q = (\ldots, q_{-1}, q_0, q_1, \ldots)$$

with $q_0 = \bar{q}_0$. In other words, there is a whole continuum of equilibria, containing a nontrivial interval of values. Incidentally, it can also be shown that for all such equilibria $q, q_t \to \bar{q}$ as $t\to\infty$, and $q_t \to 1$ as $t\to-\infty$. Moreover, these equilibria, together with the two steady state equilibria, constitute the entire equilibrium set.
This raises the second great puzzle of overlapping generations economies. There can be a non-degenerate continuum of equilibria, while in finite commodity and finite agent economies there is typically only a finite number. Thus if we considered the finite truncated economy $E_{-T,T}$ consisting of those agents born between $-T$ and T, and no others, then it can easily be seen that there is only a unique equilibrium $(q_{-T}, \ldots, q_T) = (\bar{q}, \ldots, \bar{q})$, no matter how large T is taken. On the other hand, in the overlapping generations economy, there is a continuum of equilibria. Moreover, the differences in these equilibria are not to be seen only at the tails. In the OLG economy, as $\bar{q}_0$ varies from 1 to $\bar{q}$, the consumption of the young agent at time zero varies from a to e, and his utility from $a \log e + (1-a) \log (1-e)$ (which for e near 1 is close to $-\infty$), all the way to $a \log a + (1-a) \log (1-a)$. By pushing the 'end of the world' further into the future, one does not approximate the world which does not end. We shall take up this theme again in Section 4.
It is very important to understand that the multiplicity of equilibria is not due to the stationarity of the economy. If we imagined a non-stationary economy with each a^t near a and each (e^t_t, e^t_{t+1}) near $(e, 1-e)$, we would find the same multiplicity. One might hold the opinion that in a steady-state economy one should only pay attention to steady-state equilibria, that is, only to the autarkic and golden rule equilibria. In non-steady-state economies, there is no steady-state equilibrium to stand out among the continuum. One must face up to the multiplicity.
Let us reconsider how one might demonstrate the multiplicity of equilibria, even in a non-stationary economy. This will lead to a first economic explanation of indeterminacy similar to the one originally proposed by Gale. Suppose that in our non-stationary example we find one equilibrium $\hat{q} = (\ldots, \hat{q}_{-1}, \hat{q}_0, \hat{q}_1, \ldots)$ satisfying:

$$Z^{t-1}_t(\hat{q}_{t-1}) + Z^t_t(\hat{q}_t) = 0 \text{ for all } t \in \mathbf{Z}. \tag{6}$$

Let us look for 'nearby' equilibria.

We shall say that generation t is expectations sensitive at $\hat{q}_t$ if both $[\partial Z^t_t(\hat{q}_t) / \partial q_t] \neq 0$ and $[\partial Z^t_{t+1}(\hat{q}_t) / \partial q_t] \neq 0$. If the first inequality holds, then the young's behaviour at time t can be influenced by what they expect to happen at time t+1. Similarly, if the second inequality holds, then the behaviour of the old agent at time t+1 depends on the price he faced when he was young, at time t. Recalling the logarithmic preferences of our example, it is easy to calculate that the derivatives of excess demands, for any q_t>0, satisfy

$$\frac{\partial Z^t_t(q_t)}{\partial q_t} = a^t e^t_{t+1} \neq 0$$

and

$$\frac{\partial Z_{t+1}^t(q_t)}{\partial q_t} = \frac{-(1-a^t)e_t^t}{q_t^2} \neq 0.$$

Hence, by applying the implicit function theorem to (1) we know that there is a non-trivial interval I_{t-1}^F containing $\hat{q}_{t-1}$ and a function F_t with domain I_{t-1}^F such that $F_t(\hat{q}_{t-1}) = \hat{q}_t$, and more generally,

$$Z_t^{t-1}(q_{t-1}) + Z_t^t[F_t(q_{t-1})] = 0 \text{ for all } q_{t-1} \in I_{t-1}^F.$$

Similarly there is a non-trivial interval I_t^B containing $\hat{q}_t$, and a function B_t with domain I_t^B such that $B_t(\hat{q}_t) = \hat{q}_{t-1}$, and more generally, $Z_t^{t-1}[B_t(q_t)] + Z_t^t(q_t) = 0$, for all $q_t \in I_t^B$. Of course, if $F_t(q_{t-1}) = q_t \in I_t^B$, then $B_t(q_t)=q_{t-1}$.

These forward and backward functions F_t and B_t, respectively, hold the key to one understanding of indeterminacy. Choose any relative price $q_0 \in I_0^F \cap I_0^B$ between periods 0 and 1. The behaviour of the generation born at 0 is determined, including its behaviour when old at period 1. If $q_0 \neq \hat{q}_0$, and generation 1 continues to expect relative prices $\hat{q}_1$ between 1 and 2, then the period 1 market will not clear. However, it will clear if relative prices q_1 adjust so that $q_1=F_1(q_0)$. Of course, changing relative prices between period 1 and 2 from $\hat{q}_1$ to q_1 will upset market clearing at time 2, if generation 2 continues to expect $\hat{q}_2$. But if expectations change to $q_2=F_2(q_1)$, then again the market at time 2 will clear. In general, once we have chosen $q_t \in I_t^F$, we can take $q_{t+1}=F_{t+1}(q_t)$ to clear the $(t+1)$ market. Similarly, we can work backwards. The change in q_0 will cause the period 0 market not to clear, unless the previous relative prices between period −1 and 0 were changed from $\hat{q}_{-1}$ to $q_{-1}=B_0(q_0)$. More generally, if we have already chosen $q_t \in I_t^B$, we can set $q_{t-1}=B_t(q_t)$ and still clear the period t market.

Thus we see that it is possible that an arbitrary choice of $q_0 \in I_0^F \cap I_0^B$ could lead to an equilibrium price sequence q. What happens at time 0 is undetermined because it depends on expectations concerning period 1, and also the past. But what can rationally be expected to happen at time 1 depends on what in turn is expected to happen at time 2, and so on.

There is one essential element missing in the above story. Even if $q_t \in I_t^F$, there is no guarantee that $q_{t+1}=F_{t+1}(q_t)$ is an element of I_{t+1}^F. Similarly, $q_t \in I_t^B$ does not necessarily imply that $q_{t-1} = B_t(q_t) \in I_{t-1}^B$. In our steady state example, this can easily be remedied. Since all generations are alike,

$$F_t = F_1,\ B_t = B_0,\ I_t^F = I_0^F \text{ and } I_t^B = I_0^B \text{ for all } t \in \mathbf{Z}.$$

One can show that the interval $(1, \bar{q}) \subset I_0^F \cap I_0^B$, and that if $q_0 \in (1, \bar{q})$, then $F_1(q_0) \in (1, \bar{q})$, and $B_0(q_0) \in (1, \bar{q})$. This establishes the indeterminacy we claimed.

In the general case, when there are several commodities and agents per period, and when the economy is non-stationary, a more elaborate argument is needed. Indeed, one wonders, given one equilibrium $\hat{q}$ for such an economy, whether after a small perturbation to the agents there is any equilibrium at all of the perturbed economy near $\hat{q}$. We shall take this up in Section 8.

It is worth noting that we can define two more complete markets OLG economies with present value prices. In the economy $E_{0,\infty}$ only agents born at time $t \geq 1$ participate. The definition of OLG equilibrium is the same as before, except that now the set of agents is restricted to the participants, and market clearing is only required for $t \geq 1$. In the q-normalized form, equilibrium is defined by $q=(q_1, q_2,\ldots)$ such that

$$Z_1^1(q_1) = 0 Z_t^{t-1}(q_{t-1}) + Z_t^t(q_t) = 0\ \forall t \geq 2.$$

It is immediately apparent (with one agent born per period and one good) that $E_{0,\infty}$ has a unique equilibrium, at which no agent trades and which is Pareto inefficient.

We could also define an economy $E_{0,\infty}^M$ in which only agents $t \geq 1$ participate, but where we require (in the normalized price version) that

$$Z_1^1(q_1) = -M Z_t^{t-1}(q_{t-1}) + Z_t^t(q_t) = 0\ \forall t \geq 2.$$

Equilibrium in $E_{0,\infty}^M$ is as if we gave an outside agent who had no endowment the purchasing power of M at time 1, and still managed to clear all markets $t \geq 1$. As long as $0 \leq M \leq Z_1^0(\bar{q})$, $E_{0,\infty}^M$ has an equilibrium. Take q_0 solving $M = Z_1^0(q_0)$, and $q_1=F_1(q_0)$ and $q_t=F_t(q_{t-1})$ for $t \geq 2$. We examine these two models more closely in Section 3.

2 Endogenous cycles

Let us consider another remarkable and suggestive property that one-commodity, stationary OLG economies can exhibit. We shall call the equilibrium $q = (\ldots, q_{-1}, q_0, q_1, \ldots)$ periodic of period *n* if $q_0, q_1, \ldots, q_{n-1}$ are all distinct, and if for all integers *i* and *j*, $q_i = q_{i+jn}$. The possibility that a perfectly stationary economy can exhibit cyclical ups and downs, even without any exogenous shocks or uncertainty, is reminiscent of 1930s–1950s business cycle theories. In fact, it is possible to construct a robust one-commodity per period economy which has equilibrium cycles of every order *n*. Let us see how.

As before, let each generation *t* consist of one agent, with endowment $e^t = (\ldots, 0, e, 1-e, 0, \ldots)$ positive only in period *t* and *t*+1, and utility $u^t(x) = u_1(x_t) + u_2(x_{t+1})$. Again, suppose that $\bar{q} = u_2'(1-e) / u_1'(e) > 1$. It is an immediate consequence of the separability of u^t, that for $q_t \le \bar{q}$,

$$Z_t^t(q_t) \le 0,\ Z_{t+1}^t(q_t) \ge 0,\ \frac{\partial Z_{t+1}^t(q_t)}{\partial q_t} < 0.$$

From monotonicity, we know that $Z_{t+1}^t(q_t) \to \infty$ as $q_t \to 0$. Hence it follows that for any $0 < q_0 < \bar{q}$, there is a unique $q_{-1} = B_0(q_0)$ with

$$Z_0^{-1}[B_0(q_0)] + Z_0^0(q_0) = 0.$$

From the fact that $Z_0^0(q_0) \ge -e$ for all q_0, it also follows that there is some $\underline{q} \le 1$ such that if $q_0 \in [\underline{q}, \bar{q}]$, then $B_0(q_0) \in [\underline{q}, \bar{q}]$.

Now consider the following theorem due to the Russian mathematician Sarkovsky and to the mathematicians Li and Yorke (1975).

Sarkovsky–Li–Yorke Theorem: . Let $B: [\underline{q}, \bar{q}] \to [\underline{q}, \bar{q}]$ *be a continuous function from a nontrivial closed interval into itself. Suppose that there exist a three-cycle for B, that is, distinct points* q_0, q_1, q_2, *in* $[\underline{q}, \bar{q}]$ *with* $q_1 = B(q_0)$, $q_2 = B(q_1)$, $q_0 = B(q_2)$. *Then there are cycles for B of every order n.*

Grandmont (1985), following related work of Benhabib and Day (1982) and Benhabib and Nishimura (1985), gave a robust example of a one-commodity, stationary economy (u_1, u_2, e) giving rise to a three-cycle for the function B_0. Of course a cycle for B_0 is also a cyclical equilibrium for the economy, hence there are robust examples of economies with cycles of all orders.

Theorem: (Benhabib–Day, 1982; Benhabib–Nishimura,1985; Grandmont, 1985). *There exist robust examples of stationary, one-commodity OLG economies with cyclical equilibria of every order n.*

This result is extremely suggestive of macroeconomic fluctuations arising for endogenous reasons, even in the absence of any fundamental fluctuations. Note first, however, that all of the cyclical equilibria, except the autarkic one-cycle $(\ldots, \bar{q}, \bar{q}, \bar{q}, \ldots)$, can be shown to be Pareto optimal (see Section 4), while the theory of macroeconomic business cycles is concerned with the welfare losses from cyclical fluctuations. (On the other hand, the fact that cyclical behaviour is not incompatible with optimality is perhaps an important observation for macroeconomics.) More significantly, it must be pointed out that Sarkovsky's theorem is a bit of a mathematical curiosity, depending crucially on one dimension. And of course non-stationary economies, even with one commodity, will typically not have any periodic cycles. By contrast, the multiplicity and suboptimality of non-periodic equilibria that we saw in Section 1 are robust properties that are maintained in OLG economies with multiple commodities and heterogeneity across time. The main contribution of the endogenous business cycle literature is that it establishes the extremely important, suggestive principle that very simple dynamic models can have very complicated ('chaotic') dynamic equilibrium behaviour.

In the next section we turn to another phenomenon that can generally occur in overlapping generations economies, but never in finite horizon models.

3 Money and the sequential economy

Money very often has value in an overlapping generations model, but it never does in a finite horizon Arrow–Debreu model. The reason for its absence in the latter model is familiar: money would enable some agents to spend more on goods than they received from sales of their goods. But that would mean in the aggregate that spending on goods would exceed revenue from the sale of goods, contradicting market clearing in goods.

This argument can be given another form. Without uncertainty, Arrow–Debreu equilibrium can be reinterpreted as a sequential equilibrium with contemporaneous prices. But if the number of periods is finite, then in the last period the marginal utility of money to every consumer is zero, hence so is its price. In the second-to-last period nobody will pay to end up holding any money, because in the last period it will be worthless. By induction it will have no value even in the first period.

Evidently both these arguments fail in an infinite horizon setting. There is no last period, so the backward induction argument has no place to begin. And with an infinite number of consumers, aggregate spending and revenue might both be infinite, preventing us from comparing their sizes. On the other hand, there are infinite horizon models where money cannot have value. The difference between the OLG model and these other infinite horizon models will be discussed in Section 7.

Strictly speaking, the overlapping generations model we have discussed so far has been modelled along the lines of Arrow–Debreu: each agent faced only one budget constraint and equilibrium was defined as if all markets met simultaneously at the beginning of time (−∞). In such a model money has no function. However, we can define another model, similar to the first considered by Samuelson, in which agents face a sequence of budget constraints and markets meet sequentially, and where money does have a store-of-value role. Surprisingly, this model turns out to have formally the same properties as the OLG model we have so far considered. To distinguish the two models we shall refer to this latter monetary model as the Samuelson model.

Suppose that we imagine a one-good per period economy in which the markets meet sequentially, according to their dates, and not simultaneously at the beginning of time. Suppose also that there are no assets or promises to trade. In such a setting it is easy to see that there could be no trade at all, since, as Samuelson put it, there is no double coincidence of wants. The old and the young at any date *t* both have the same kind of commodity, so they have no mutually advantageous deal to strike. But as Samuelson pointed out, introducing a durable good called money, which affects no agent's utility, might allow for much beneficial trade. The old at date *t* could sell their money to the young for commodities, who in turn could sell their money when old to the next period's young. In this manner new and more efficient equilibria might be created. The 'social contrivance of money' is thus connected to both the indeterminacy of equilibrium and the Pareto suboptimality of equilibrium, at least near autarkic equilibria. The puzzle, we have said, is how to explain the positive price of money when it has no marginal utility.

A closer examination of the equilibrium conditions of Samuelson's sequential monetary equilibrium reveals that, although it appears much more complicated, it reduces to the timeless OLG model we have defined above, but with one difference, namely, that the budget constraint of the generation endowed with money is increased by the value of money. The introduction of the asset money thus ‘completes the markets’ in the sense of Arrow (1953), by which we mean that the equilibrium of the sequential economy can be understood as if it were an economy in which money did not appear and all the markets cleared at the beginning of time (except, as we said, that the incomes of several agents are increased beyond the value of their endowments). The puzzle of how money can have positive value in the Samuelson model can thus be reinterpreted in the OLG model as follows. How is it possible that we can increase the purchasing power of one agent beyond the value of his endowment, without decreasing the purchasing power of any other agent below his, and yet continue to clear all the markets? Before giving a more formal treatment of the foregoing, let me re-emphasize an important point. It has often been said that the paradoxical properties of equilibrium in the sequential Samuelson consumption loan model can be explained on the basis of incomplete markets. Adding money to the model, however, completes the markets, in the precise sense of Arrow–Debreu, but the result is the OLG model in which the puzzles remain.

Let us now formally define the sequential one-commodity Samuelson model with money, $E^{M,S}_{0,\infty}$. Consider a truncated economy in which there is a new agent ‘born’ at each date $t \geq 0$, whose utility depends only on the two goods dated during his lifetime, and whose endowment is positive only in those same commodities. At each date $t \geq 1$ there will be two agents alive, a young one and an old one. Let us suppose that trade does not begin until period 1, so that the date 0 generation must consume its endowment when it is young. To this truncation of our earlier model we now add one extra commodity, which we call money. Money is a perfectly durable commodity that affects no agent's utility. Agents are endowed with money (M^t_t, M^t_{t+1}), in addition to their commodity endowments.
A (contemporaneous) price system is defined as a sequence

$$(\pi;\ p) = (\pi_1, \pi_2, \ldots;\ p_1, p_2, \ldots)$$

of contemporaneous money prices π_t and contemporaneous commodity prices p_t for each $t \geq 1$. The budget set for any agent $t \geq 1$ is defined by

$$\left\{(m_t, m_{t+1}, x_t, x_{t+1}) \geq 0 \mid \pi_t m_t + p_t x_t \leq \pi_t M^t_t + p_t e^t_t \text{ and } \pi_{t+1} m_{t+1} + p_{t+1} x_{t+1} \leq \pi_{t+1} M^t_{t+1} + p_{t+1} e^t_{t+1} + \pi_{t+1} m_t\right\}.$$

For agent 0 the budget constraint is

$$\left\{(m_0, m_1, x_0, x_1) \geq 0 \mid m_0 = M^0_0,\ x_0 = e^0_0, \text{ and } \pi_1 m_1 + p_1 x_1 \leq \pi_1 M^0_1 + p_1 e^0_1 + \pi_1 m_0\right\}.$$

The budget constraints express the principle that in the Samuelson model agents cannot borrow at all, and cannot save, that is, purchase more when old than the value of their old endowment, except by holding over money m_t from when they were young. Let $m^t_t(\pi, p)$ and $m^t_{t+1}(\pi, p)$ be the utility maximizing choices of money holdings by generation t when young and when old. As before, the excess commodity demand is defined by $z^t_t(\pi, p)$ and $z^t_{t+1}(\pi, p)$.
To keep things simple, we suppose that agent 0 is endowed with $M^0_1 = M$ units of money when he is old, but all other endowments M^t_s are zero. Since money is perfectly durable, total money supply in every period is equal to M. Equilibrium is defined by a price sequence (π, p) such that for all $t \geq 1$,

$$m^{t-1}_t(\pi, p) + m^t_t(\pi, p) = M \text{ and } z^{t-1}_t(\pi, p) + z^t_t(\pi, p) = 0.$$

At first glance this seems a much more complicated system than before.
But elementary arguments show that in equilibrium either π_t=0 for all t, and there is no intergenerational trade of commodities, or π_t>0 for all t, or π_t<0 for all t. In the case where π_t>0, no generation will choose to be left with unspent cash when it dies, hence $m^t_{t+1}(\pi, p) = 0$ for all t, hence money market clearing is reduced to

$$m^t_t(\pi, p) = M \text{ for all } t \geq 1.$$

By homogeneity of the budget sets, if π_t>0, we might as well assume π_t=1 for all t. But then the prices p_t become the same as the present value prices from Section 1. From period by period Walras’ Law, we deduce that, if the goods market clears at date t, so must the money market. So we never have to mention money market clearing.
Moreover, by taking $q_t = (\pi_t p_{t+1}) / (\pi_{t+1} p_t)$ we can write the commodity excess demands for agent $t \geq 1$ just as in Section 1, by

$$[z^t_t(q_t),\ z^t_{t+1}(q_t)]$$

and they are the same as

$$[Z_t^t(\pi, p), Z_{t+1}^t(\pi, p)].$$

The only agent who behaves differently is agent 0, whose budget set must now be written

$$B^0(\mu, M) = \left\{(x_0, x_1) | x_0 = e_0^0, x_1 \le e_1^0 + \mu M\right\},$$

where

$$\mu = \frac{\pi_1}{p_1}.$$

We can then write agent 0's excess demand for goods at time 1 as

$$Z_1^0(\mu, q, M) = Z_1^0(\mu M) = \mu M.$$

Thus any sequential Samuelson monetary equilibrium can be described by (μ, q), $\mu \ge 0$, satisfying

$$Z_1^0(\mu M) + Z_1^1(q_1) = 0,$$

and

$$Z_t^{t-1}(q_{t-1}) + Z_t^t(q_t) = 0 \text{ for all } t \ge 2.$$

But of course that is precisely the same as the definition of an OLG equilibrium for $E_{0,\infty}^{\mu M}$ given in Section 1.

4 Understanding OLG economies as lack of market clearing at infinity

In this section we point out that the suboptimality of competitive equilibria, the indeterminacy of non-stationary equilibria, the non-existence of the core, and the positive valuation of money can all occur robustly in possibly non-stationary OLG economies with multiple consumers and L>1 commodities per period. We also note the important principle that the potential dimension of indeterminacy is related to L. In the two-way infinity model, it is 2L−1. In the one-way infinite model without money it is L−1; in the one-way infinity model with money the potential dimension of indeterminacy is L.

None of these properties can occur (robustly) in a finite consumer, finite horizon, Arrow–Debreu model. In what follows we shall suggest that a proper understanding of these phenomena lies in the fact that the OLG model is isomorphic, in a precise sense, to a '*-finite' model in which not all the markets are required to clear.

One of the first explanations offered to account for the differences between the Arrow–Debreu model and the sequential Samuelson model with money centred on the finite lifetimes of the agents and the multiple budget constraints each faced. These impediments to intergenerational trade (for example, the fact that an agent who is 'old' at time t logically cannot trade with an agent who will not be 'born' until time t+s) were held responsible. But as we saw in the last section, without uncertainty the presence of a single asset like money is enough to connect all the markets. Formally, as we saw, the model is identical to what we called the OLG model in which we could imagine all trade taking place simultaneously at the beginning of time, with each agent facing a single budget constraint involving all the commodities. What prevents trade between the old and the unborn is not any defect in the market, but a lack of compatible desires and resources.

Another common explanation for the surprising properties of the OLG model centres on the 'paradoxes' of infinity, as suggested by Shell (1971). In finite models, one proves the generic local uniqueness of equilibrium by counting the number of unknown prices, less 1 for homogeneity, and the number of market clearing conditions, less 1 for Walras' Law, and notes that they are equal. In the OLG model there is an infinity of prices and markets, and who is to say that one infinity is greater than another? We already saw that the backward induction argument against money fails in an infinite horizon setting, where there is no last period. Surely it is right that infinity is at the heart of the problem. But this explanation does not go far enough. In the model considered by Bewley (1972) there is also an infinite number of time periods (but a finite number of consumers). In that model all equilibria are Pareto optimal, and money never has value, even though there is no last time period. The problem of infinity shows that there may be a difference between the Arrow–Debreu model and the OLG model. In itself, however, it does not predict the qualitative features (like the potential dimension of indeterminacy) that characterize OLG equilibria.

Consider now a general OLG model with many consumers and commodities per period. We index utilities $u^{t,h}$ by the time of birth t, and the household h H, a finite set. Household (t, h) owns initial resources $e_t^{t,h}$ when young, an L-dimensional vector, and resources $e_{t+1}^{t,h}$ when old, also an

L-dimensional vector, and nothing else. As before utility $u^{t,h}$ depends only on commodities dated either at time t or t+1. Given prices

$$q_t = (q_{ta}, q_{tb}) \in \Delta_{++}^{2L-1} = \left\{ q \in R_{++}^{2L} \middle| \sum_{\ell=1}^{L} (q_\ell + q_{L+\ell}) = 2 \right\}$$

consisting of all the 2L prices at date t and t+1, each household in generation t has enough information to calculate the relevant part of its budget set

$$B^{t,h}(q_t) = \left\{ (x_t, x_{t+1}) \in R_+^{2L} | q_{ta} \cdot x_t + q_{tb} \cdot x_{t+1} \le q_{ta} \cdot e_t^{t,h} + q_{tb} \cdot e_{t+1}^{t,h} \right\}.$$

Hence we can write household excess demand $[z_t^{t,h}(q_t), z_{t+1}^{t,h}(q_t)]$ and the aggregate excess demand of generation t as $[z_t^t(q_t), z_{t+1}^t(q_t)]$, where

$$z_{t+s}^t(q_t) = \sum_{h \in H} z_{t+s}^{t,h}(q_t),\ s = 0, 1.$$

Of course we need to put restrictions on the q_t to ensure their compatibility, since q_{tb} and $q_{t+1,a}$ refer to the same period t+1 prices. But this is easily done by supposing that

$$q_{tb} = \lambda_t q_{t+1a} \text{ for some } \lambda_t > 0,\ \forall t \in \mathbb{Z}.$$

Present value OLG prices p can always be recovered from the normalized prices q via the recursion

$$p_1 = q_{1a} p_t = q_{ta}(\lambda_1 \lambda_2 \ldots \lambda_{t-1}) \text{ for } t \ge 2\ p_t = q_{ta}(\lambda_0^{-1} \lambda_{-1}^{-1} \ldots \lambda_t^{-1}) \text{ for } t \le 0.$$

We shall now define three variations of the OLG model and equilibrium, depending on when time starts, and whether or not there is money. Suppose first that time goes from $-\infty$ to ∞. We can write the market clearing condition for equilibrium exactly as we did in the one-commodity, one-consumer case, as

$$z_t^{t-1}(q_{t-1}) + z_t^t(q_t) = 0,\ t \in \mathbb{Z}.$$
(A)

Similarly we can define the one-way infinity economy $E_{0,\infty}$, in which time begins in period 0, but trade begins in time 1. We simply retain the same market clearing conditions for $t \ge 2$,

$$z_t^{t-1}(q_{t-1}) + z_t^t(q_t) = 0,\ t \ge 2$$
(A′)

$$\sum_{h \in H} \tilde{z}_1^{0,h}(q_{1a}) + z_1^1(q_1) = 0,$$
(7)

it being understood that $z_1^{0,h}$ has been modified to $\tilde{z}_1^{0,h}(q_{1a})$ because every agent (0,h) is forced to consume his own endowment at time 0, so that he maximizes over his budget set

$$B^{0,h}(q_{1a}) = \left\{ (x_0, x_1) \in R_+^{2L} | x_0 = e_0^{0,h},\ q_{1a} \cdot x_1 \le q_{1a} \cdot e_1^{0,h} \right\}.$$

Finally, let us define equilibrium in a one-way infinity model with money, $E^M_{0,\infty}$, when agents (0,*h*) are endowed with money M^h, in addition to their commodities, by (μ, q), $\mu \geq 0$, satisfying

$$\sum_{h\in H} \hat{z}_1^{0,h}(q_{1a}, \mu M^h) + z_1^1(q_1) = 0,$$

(A″)

and

$$z_t^{t-1}(q_{t-1}) + z_t^t(q_t) = 0, \text{ for } t \geq 2.$$

Again it is understood that the agents (0, *h*) born in time 0 cannot trade in time 0, and they maximize over the budget set

$$B^{0,h}(q_{1a}, \mu M^h) = \left\{(x_0, x_1) \in R_+^{2L} | x_0 = e_0^{0,h}, q_{1a} \cdot x_1 \leq q_{1a} \cdot e_1^{0,h} + \mu M^h\right\}.$$

These are the natural generalizations of the one-good economies defined in Section 1. (There is one small difference. With many agents born per period we can no longer conclude that if one agent holds a positive amount of money when young, then so must every other agent – no matter when he is born. We shall ignore this complication and allow some agents to hold negative money.)

We must now try to understand very generally why there may be many dimensions of OLG equilibria, why they might not be Pareto efficient, and how it is possible that some agents can spend beyond their budgets without upsetting market clearing.

Our explanation amounts to 'lack of market clearing at infinity'. We illustrate this for the case $E_{0,\infty}$.

Consider the truncated economy $E_{0,T}$ consisting of all the agents born between periods 0 and *T*. Market clearing in $E_{0,T}$ is defined to be identical to that in $E_{0,\infty}$ for t=1 to t=*T*. But at t=*T*+1, we require $z^T_{T+1}(q_t) = 0$ in $E_{0,T}$. This is a perfectly conventional Arrow–Debreu economy, and so necessarily has some competitive equilibria, all of which are Pareto efficient; generically its equilibrium set is a 0-dimensional manifold.

We have already seen in Section 1 what a great deal of difference there is between the economies $E_{0,T}$ (no matter how large *T* is) and $E_{0,\infty}$. The interesting point is that, by appealing to non-standard analysis, which makes rigorous the mathematics of infinite and infinitesimal numbers, one can easily show that the economy $E_{0,T}$, for *T* an infinite number, inherits any property that holds for all finite $E_{0,T}$. Thus the paradoxical properties of the economy $E_{0,\infty}$ do not stem from infinity alone, since the infinite economy $E_{0,T}$ does not have them. We shall need to modify $E_{0,T}$ before it corresponds to $E_{0,\infty}$. Nevertheless, the economies $E_{0,T}$ do provide some information about $E_{0,\infty}$.

Theorem: (Balasko–Cass–Shell, 1980; Wilson, 1981). *Under mild conditions, at least one equilibrium for* $E_{0,\infty}$ *always exists*.

To see why this is so, note that $E_{0,T}$ is well-defined for any finite *T*. From non-standard analysis we know that the sequence $E_{0,T}$ for $T \in$ **N** has a unique extension to the infinite integers. Now fix *T* at an infinite integer. We know that $E_{0,T}$ has at least one equilibrium, since $E_{0,s}$ does for all finite *s*. But if *T* is infinite, $E_{0,T}$ includes all the finite markets t=1,2,…, so all those must clear at an equilibrium q^* of $E_{0,T}$. Taking the standard parts of the prices q^*_t for the finite *t* (and ignoring the infinite *t*) gives an equilibrium *q* for $E_{0,\infty}$.

To properly appreciate the force of this proof, we shall consider it again, when it might fail, in Section 7, where we deal with infinite lived consumers.

In terms of the existence of equilibrium, $E_{0,\infty}$ (and similarly $E^M_{0,\infty}$ and $E_{-\infty,\infty}$) behaves no differently from an Arrow–Debreu economy. But the indeterminacy is a different story.

Definition: A classical equilibrium for the economy $E_{0,T}$ is a price sequence $q^*=(q_1,\ldots, q_T)$ that clears the markets for $1 \leq t \leq T$, but at t=*T*+1, market clearing $z^T_{T+1}(q_T) = 0$ is replaced by

$$z^T_{T+1}(q_T) \leq \sum_{h\in H} e^{T+1,h}_{T+1}.$$

Thus in a classical equilibrium there is lack of market clearing at the last period. The aggregate excess demand in that period, however, must be less than the endowment the young of period *T*+1 would have had, were they part of the economy. Economies in which market clearing is not required in every market are well understood in economic theory. Note that in a classical equilibrium the agents born at time *T* are not rationed at *T*+1; their full Walrasian (notional) demands are met, out of the dispossessed endowment of the young. But we do not worry about how this gift from the *T*+1 young is obtained. The significance of our classical equilibrium for the OLG models can be summarized in the following theorem from Geanakoplos and Brown (1982):

Theorem: (Geanakoplos–Brown, 1982). *Fix T at an infinite integer. The equilibria q for* $E_{0,\infty}$ *correspond exactly to the standard parts of classical equilibria* q^* *of* $E_{0,T}$.

The Walrasian equilibria of the economy $E_{0,\infty}$, which apparently is built on the usual foundations of agent optimization and market clearing, correspond to the 'classical equilibria' of another finite-like economy $E_{0,T}$ in which the markets at T+1 ('at infinity') need not clear. The existence of a classical equilibrium in $E_{0,T}$, and thus an equilibrium in $E_{0,\infty}$, is not a problem, because market clearing is a special case of possible non-market clearing, and $E_{0,T}$, being finite-like, always has market clearing equilibria.

Thus even though the number of prices and the number of markets in $E_{0,\infty}$ are both infinite, by looking at $E_{0,T}$ it is possible to say which is bigger, and by how much. There are exactly L more prices than there are markets to clear. From Walras' Law we know that if all the markets but one clear, that must clear as well. Hence having L markets that need not clear provides for $L-1$ potential dimensions of indeterminacy.

Corollary: (Geanakoplos–Brown, 1982). *For a generic economy $E_{0,\infty}$, there are at most $L-1$ dimensions of indeterminacy in the equilibrium set.*

Though the classical equilibria of $E_{0,T}$ generically have $L-1$ dimensions of indeterminacy, it is by no means true that there must be $L-1$ dimensions of *visible* indeterminacy. If we consider any classical equilibrium q^* for a generic economy $E_{0,T}$, then we will be able to arbitrarily perturb some set of $L-1$ prices near their q^* values, and then choose the rest of the prices to clear all the markets up through time T. But which $L-1$ prices these are depends on which square submatrix N (of derivatives of excess demands with respect to prices) is invertible. For example, call the economy $E_{0,\infty}$ intertemporally separable if each generation t consists of a single agent whose utility for consumption at date t is separable from his utility for consumption at date t+1. Then the $L-1$ free parameters must all be chosen at date T+1 (as part of $q_{T,b}$), that is, way off at infinity.

Corollary: (Geanakoplos–Polemarchakis, 1984). *Intertemporally separable economies $E_{0,\infty}$ generically have locally unique equilibria (in the product topology).*

For example, a natural generalization of the example in Section 1 would be to generations consisting of a single Cobb–Douglas consumer of L>1 goods when young and when old. The corollary shows that this economy has no indeterminacy of equilibrium. Since Cobb–Douglas economies seem so central, one might guess that multi-good OLG economies $E_{0,\infty}$ do not generate indeterminacy. But that is incorrect. Separability with one agent drastically reduces the effect expectations about future prices can have on the present, because changes in future consumption do not change marginal utilities today. In the separable case, changing all L prices tomorrow only affects today through the one dimension of income.

Even when the $L-1$ degrees of freedom may be chosen at time t=1, there still may be no visible indeterminacy, if the matrix N has an inverse (in the non-standard sense) with infinite norm. But when the free $L-1$ parameters may be chosen at t=1 and also the matrix N has an inverse with finite norm, then all nearby economies must also display $L-1$ dimensions of indeterminacy.

Theorem: (Kehoe–Levine, 1984; Geanakoplos–Brown, 1982). *In the $E_{0,\infty}$ OLG model there are robust examples of economies with $L-1$ dimensions of indeterminacy. In the monetary economy, $E^M_{0,\infty}$, there are robust examples of economies with L dimensions of indeterminacy.*

Let us now turn our attention to the question of Pareto optimality.

Definition: An allocation $\bar{x} = (x^{t,h};\ 0 \le t \le T)$ is classically feasible for the economy $E_{0,T}$ if $\sum_{(t,h)\in A} x_s^{t,h} \le \sum_{(t,h)\in A} e_s^{t,h}$, for $0 \le s \le T+1$. The classically feasible allocation $\bar{x}$ for $E_{0,T}$ is a classic Pareto optimum if there is no other classically feasible allocation $\bar{y}$ for $E_{0,T}$ with $u^t(y^{t,h}) > u^t(x^{t,h})$ for all (t,h) A with $0 \le t \le T$, with at least one inequality $(0,h)$ representing a non-infinitesimal difference.

Theorem: (Geanakoplos–Brown, 1982). *The Pareto-optimal allocations $\bar{x}$ for the OLG economy $E_{0,\infty}$ are precisely the standard parts of classical Pareto-optimal allocations $\bar{x}^*$ for $E_{0,T}$, if T is fixed at an infinite integer.*

The upshot of this theorem is that the effective social endowment includes the commodities e^{T+1}_{T+1} of the generation born at time s=T+1, even though they are not part of the economy $E_{0,T}$. Since the socially available resources exceed the aggregate of private endowments, it is no longer a surprise that a Walrasian equilibrium, in which the value of aggregate spending every period must equal the value of aggregate private endowments, is not Pareto optimal.

On the other hand, this does not mean that all equilibria are Pareto suboptimal. If the (present value) equilibrium prices $p_t \to 0$, as $t\to\infty$ (or, more generally, if p_{T+1} is infinitesimal), then the value of the extra social endowment is infinitesimal, and there are no possible non-infinitesimal improvements. To see this, let $(p, \bar{x})$ be an equilibrium in present value prices for the OLG economy $E_{0,\infty}$. Consider the concave–convex programming problem of maximizing the utility of agent $(0, \bar{h})$, holding all other utilities of agents (t, h) with $0 \le t \le T$ at the levels $u^{t,h}(x^t)$ they get with $\bar{x}$, over all possible allocations in $E_{0,T}$ that do not use more resources, even at time T+1, than $\bar{x}$. Clearly $\bar{x}$ itself is a solution to this problem. But now let us imagine raising the constraints at time T+1 from

$$\sum_{h\in H} x^{T,h}_{T+1} \text{ to } \sum_{h\in H} (e^{T,h}_{T+1} + e^{T+1,h}_{T+1}).$$

What is the rate of change of the utility $u^{0,\bar{h}}$? From standard concave programming theorems, for the first infinitesimal additions to period T+1 resources, the rate of change of $u^{0,\bar{h}}$ is on the order of p_{T+1}, assuming p_1 is normalized to equal the marginal utility of consumption for agent $(0, \bar{h})$ at date 1. Additional resources bring decreasing benefits. This shows that if p_{T+1} is infinitesimal, then there are no possible non-infinitesimal improvements with a finite amount of extra resources.

An important example of $p_t \to 0$ occurs when the prices are summable, as they are when they decline geometrically to zero. Thus in a stationary equilibrium with a positive real interest rate, equilibrium must be Pareto efficient. Another proof of efficiency in the case of geometric present value prices is to observe that then the present value of the aggregate endowment must be finite, so the standard proof of Pareto efficiency in a finite horizon model goes through.

If p_t increases geometrically to infinity, then it is evident that equilibrium cannot be Pareto efficient. Thus, in a stationary equilibrium with a negative real interest rate, equilibrium must be Pareto inefficient.

When $p_t \nrightarrow 0$ but also does not increase exponentially to infinity, the calculation becomes much more delicate. An infinitesimal increase ε in resources at time T+1 can be used to increase utility of $(0, \bar{h})$ on the order of $p_{T+1}\varepsilon$, which is still infinitesimal if p_{T+1} is non-infinitesimal but finite. As the increases ε get larger, this rate of change could drop quickly, as higher derivatives come into play (assuming that agents have strictly concave utilities), leaving infinitesimal (and thus invisible) increases in utility even with a finite increase in resources. Second derivatives, and their uniformity, come into play. But this subtle case has been brilliantly dealt with:

Theorem: (Cass, 1972; Benveniste–Gale, 1975; Balasko–Shell, 1980; Okuno–Zilcha, 1980). *If agents have uniformly strictly concave utilities, and if the aggregate endowment is uniformly bounded away from* 0 *and* ∞, *then the equilibrium* $(p, \bar{x})$ *with present value prices p for an OLG economy* $E_{0,\infty}$ *is Pareto optimal if and only if* $\sum_{t=0}^{\infty} 1/\|p_t\| = \infty$.

Note that in this theorem it is the present value prices that play the crucial role. It follows immediately from this theorem that the golden rule equilibrium $q = (\ldots, 1, 1, 1, \ldots)$ for the simple one good, stationary economy of Section 1 is Pareto optimal, since the corresponding present value price sequence is also $(\ldots, 1, 1, 1, \ldots)$. In fact, a moment's reflection shows that any periodic, non-autarkic equilibrium must also be periodic in the present value prices *p*. Hence, as we have said before, but without a proof, the cyclical equilibria of Section 2 are all Pareto optimal.

Having explained the indeterminacy and Pareto suboptimality of equilibria for $E_{0,\infty}$ in terms of lack of market clearing at infinity, let us re-examine the monetary equilibria of OLG economies $E^M_{0,\infty}$, where $M = (M^h; h \in H)$ is the stock of money holdings by the agents (0, *h*) at time 0.

The next theorem shows that any monetary equilibrium allocation of $E^M_{0,\infty}$ corresponds to the standard part of a non-monetary economy $E_{0,T}(z)$ obtained from $E_{0,T}$ by augmenting the endowments of the first generation $(0,h)_{h \in H}$ by a vector of goods *z* at time *T*+1.

Definition: Let $z \in R^L$ be a vector of commodities for time *T*+1. Suppose that $-\sum_{h \in H} e^{T,h}_{T+1} \le \sum_{h \in H} M^h z \le \sum_{h \in H} e^{T+1,h}_{T+1}$. Let the augmented non-monetary economy $E_{0,T}(z, M)$ be identical to the non-monetary economy $E_{0,T}$, except that the endowment of each agent (0, *h*) is augmented by $M^h \cdot z$ units of commodities at time *T*+1.

Theorem: (Geanakoplos–Brown, 1982). *Fix an infinite integer T. The equilibria q of the monetary economy* $E^M_{0,\infty}$ *are precisely obtained by taking standard parts of full market clearing equilibria* q^* *of all the augmented non-monetary economies* $E_{0,T}(z,M)$.

The above theorem explains how it is possible to give agents (0, *h*) extra purchasing power without disturbing market clearing in the economy $E^M_{0,\infty}$. The answer is that the purchasing power comes from owning extra commodities at date *T*+1, and equilibrium in $E^M_{0,\infty}$ does not require market clearing in date *T*+1 commodities.

The above theorem gives another view of why there are potentially *L* dimensions of monetary equilibria: the augmenting endowment vector *z* can be chosen from a set of dimension *L*. It also explains how money can have positive value: it corresponds to the holding of extra physical commodities. The theorem also explains how the ‘social contrivance of money’ can lead to Pareto-improving equilibria, even in OLG economies where there is already perfect financial intermediation. The holding of money can effectively bring more commodities into the aggregate private endowment. The manifestation of the ‘real money balances’ is the physical commodity bundle *z* at date *T*+1. Money plays more than just an intermediation role.

Before concluding this section let us consider a simple generalization. Suppose that agents live for three periods. What plays the analogous role to $E_{0,T}$? The answer is that prices need to be specified through time *T*+2, but markets are only required to clear through time *T*. There are therefore 2*L*−1 potential dimensions of indeterminacy, even in the one-sided economy. In general, we must specify the price vector up until some time *s*, and then require market clearing only in those commodities whose excess demands are fully determined by those prices.

This reasoning has an important generalization to production. Suppose that capital invested at time *t* can combine with labour at time *t*+1 to produce output at time *t*+1, and suppose that all agents live two periods. Is there any difference between the case where labour is inelastically supplied, and the case where leisure enters the utility? In both cases the number of commodities is the same, but in the latter case the potential dimension of indeterminacy is one higher, since the supply of labour at any time might depend on further prices.

5 Land, the real rate of interest, and Pareto efficiency

Allais and Samuelson argued that the infinity of both time periods and agents radically changed the nature of equilibrium. Samuelson suggested that equilibrium might not be Pareto efficient, and that the real rate of interest might be negative, even if the economy did not shrink over time. In our one-good example from Section 1, the autarkic equilibrium has a negative real interest rate since each $q_t<1$, and the real interest rate is $1/q_t-1$.

They also thought that a second, new kind of equilibrium would emerge in which the real rate of interest is divorced from any of the considerations like impatience that Irving Fisher had stressed. They thought that in this new kind of equilibrium the real rate of interest would turn out to be equal to the rate of population growth, irrespective of the impatience of the consumers or the distribution of their endowments. Indeed, in the example from Section 1, the ‘golden rule’ equilibrium had real interest rate $1/q_t-1=0$ in every period, irrespective of the utilities or the endowments, but equal to the population growth rate.

Furthermore, as we saw in Section 3, Samuelson argued that it might not be necessary for an asset to be valued according to the present value of its dividends, contradicting yet another one of Fisher's central concepts. Samuelson suggested that a piece of green paper might be worth a lot, even though it pays no dividends, because the holder might think he could sell it to somebody later, who would buy it on the expectation that he could sell it to somebody else later, ad infinitum. Later authors called this a rational bubble.

It turns out that these views are incorrect if one includes in the model infinitely lived assets like land, that do pay dividends in every period. Imagine an OLG economy as before with

$$u^t(x_t, x_{t+1}) = \frac{1}{2}\log x_t + \frac{1}{2}\log x_{t+1}(e^t_t, e^t_{t+1}) = (3, 1).$$

But let us also suppose there is one acre of *land* in the economy that produces a dividend $D_t=1$ apple every period for ever. Suppose the economy begins in period 1, with an old agent who owns the land and has an endowment of one apple, and a newly born agent as above. We suppose that buying the land at time *t* gives ownership of all dividends from time *t*+1 up to and including the dividends in the period in which the asset is sold. The apple dividend from the land at time 1 is owned by the old agent at time 1 (who presumably acquired the land at time 0 and hence has the claim on the apple).

At every period *t* we need to find the contemporaneous price q_t of the commodity and the price Π_t of the land.

Every agent in the economy must decide how much to consume when young, and what assets to hold when young, and how much to consume when old. The decision in old age is trivial, since the agent cannot do better than selling every asset he has and using the proceeds to buy consumption goods.

Thus for every $t \ge 1$ we can describe the decision problem of generation *t* by

$$\max_{y,z,\theta} u^t(y, z) = \frac{1}{2}\log y + \frac{1}{2}\log z \text{such that} q_t y + \Pi_t \theta = q_t e_t^t = q_t 3 q_{t+1} z = q_{t+1} e_{t+1}^t + \theta D_{t+1} + \Pi_{t+1}\theta = q_{t+1} 1 + \theta 1 + \Pi_{t+1}\theta.$$

For the original old generation, he optimizes simply by setting

$$x_1^0 = e_1^0 + D_1 + \Pi_1 = 1 + 1 = 2 + \Pi_1.$$

Denote the optimal choice of agents $t \geq 1$ by $(x_t^t, x_{t+1}^t, \theta^t)$. Market clearing requires for each $t \geq 2$ that consumption of the old plus consumption of the young is equal to total output of goods, and also that demand equals the supply of land

$$x_t^{t-1} + x_t^t = e_t^{t-1} + e_t^t + D_t = 1 + 3 + 1 = 5 \theta^t = 1.$$

In period t=1 we must have

$$x_1^0 + x_1^1 = e_1^0 + e_1^1 + D_1 = 1 + 3 + 1 = 5, \theta^1 = 1.$$

Sequential equilibrium is thus a vector $(x_1^0, (q_t, \Pi_t, (x_t^t, x_{t+1}^t, \theta^t))_{t=1}^{\infty})$ satisfying the above conditions on agent maximization and market clearing. Fisher's recipe for computing equilibrium with assets is to put the asset dividends into the endowments of their owners, and then find the usual general equilibrium with present value prices ignoring the assets. In this example that means giving agent 0 an endowment e^0=(2,1,…) of two apples in period 1 and one apple every period thereafter, and ignoring the land. Equilibrium with present value prices is then described exactly as in Section 1.

To solve for the present value prices $(p_1, p_2, \ldots)$ we can guess that since the economy is stationary, there will be stationary equilibrium $(p_1, p_2, \ldots) = (1, p, p^2, \ldots)$. For each $t \geq 2$, we must solve

$$\frac{1}{2}\frac{[3 + p1]}{p} + \frac{1}{2}[3 + p1] = 1 + 3 + 1,$$

which gives a quadratic equation

$$p^2 - 6p + 3 = 0$$

which is solved by

$$p = \frac{6 \pm (36 - 12)^{.5}}{2} = .55, r = 1 / p - 1 = 81.7\%.$$

The other root is greater than one, and could not be right, because it would give a real interest rate less than zero, which would make the present value of land infinite. Hence consumption when young and old is

$$(y, z) = (1.775, 3.225).$$

Clearly these values clear the consumption market for all $t \geq 2$. We know by Walras's Law that, if all markets but one clears, then the last will as well, so we don't really have to check the period 1 market. But we will check it anyway. The present value of agent 0's endowment is

$$2 + p1 + p^2 1 + \ldots = 2 + p / (1 - p) = 3.225$$

and so indeed the period t=1 market clears.

We can now translate this general equilibrium back into a sequential equilibrium. Taking q_t=1 for every period and the real interest rate solving p=1/

$(1+r)$, the present value of land is

$$PVLand = p1 + p^2 1 + \ldots = p\,/\,(1 - p) = \frac{1}{1+r}1 + \frac{1}{(1+r)^2}1 + \ldots = 1.225.$$

In every period the old will consume their endowment of 1 plus the dividend of 1 plus the value of the land they will sell, which gives exactly 3.225. The sequential equilibrium is $(x_1^0, (q_t, \Pi_t, (x_t^t, x_{t+1}^t, \theta^t))_{t=1}^{\infty}) = (3.225, (1, 1.225, (1.775, 3.225, 1))_{t=1}^{\infty})$.

Despite what Allais and Samuelson said, the rate of interest at the unique steady state is positive, higher than the growth rate of population. Moreover, as noted in Geanakoplos (2005), the real interest rate does respond to shocks in exactly the way Fisher argued. Consider the same model as before, but make all the consumers more impatient

$$U(y, z) = \frac{2}{3}\log y + \frac{1}{3}\log z.$$

Then our master equation would become

$$\frac{1}{3}\frac{[3 + p1]}{p} + \frac{2}{3}[3 + p1] = 1 + 3 + 1$$

giving

$$p = .419, \quad r = 139\%, \quad PVLand = .721.$$

As Fisher would have predicted, the real rate of interest does indeed increase, and the price of land decreases.

5.1 Pareto efficiency and bubbles

Observe that in our example the dividends of land represent 20 per cent of all endowments every period. Since the price of land must be finite, that means in any equilibrium the present value of all endowments must be finite. We know that implies equilibrium must be Pareto efficient. Furthermore, if the value of aggregate endowments is finite, then money cannot have value and there can be no bubbles, because the old argument is correct that markets cannot clear if some agents are spending more than the value of their commodity endowments and nobody is spending less. Land makes the OLG economy look much more like an Arrow–Debreu economy.

5.2 Social security

The overlapping generations model is the workhorse model for examining social security. There is not space here to describe these studies. Observe simply that a pay-as-you-go system amounts to a simple transfer of endowments from each young person to each old person. We can immediately calculate the effects of such a transfer on our steady state interest rate and land value by recomputing the equilibrium for the OLG economy in which endowments are adjusted to (2,2) for every generation $t \geq 1$, and assuming the old generation 0 has an endowment of 2 apples at time 1 plus the land, which pays 1 apple every period. We get

$$p = .38, r = 161\%, PVLand = .62.$$

This also confirms Fisher's contention that decreasing early endowments and increasing later endowments should raise the rate of interest and lower land values.

Notice that the pay-go system gives each agent the same number of apples when he is old that he gave up when he is young, which is a below market return on his original contribution. Social security lowers the utility of every agent except the first generation. Samuelson had argued that social security could make every agent better off. But his conclusion is false in the model with land.

It is often said that if only every generation had more children, social security would give better returns, since the young would be able to share the burden of helping the old. The trouble with that reasoning is that it ignores the fact that higher population and output growth would mean higher real interest rates, which tend to make the social security rate of return as bad as before relative to market interest rates. There is no space to discuss this here.

6 Demography in OLG

In America since the early 20th century, the generations have alternated in size between big and small. Everybody knows about the baby boom and echo baby boom, but the same pattern happened before. Recently many authors have suggested that the retiring of the baby boom generation will force stock prices to fall. This has been criticized on the grounds that demography is easy to predict. If agents knew that stock prices would fall when the baby boomers retired, they would fall now. These two opposing views can be analysed in the OLG model by allowing generation sizes to fluctuate.

Suppose the small generation is exactly as before, but now we alternate that small generation with a large generation that is identical in every respect, except that it is twice as big

$$u^b(y, z) = \frac{1}{2}\log y + \frac{1}{2}\log z (e^b_y, e^b_z) = (6, 2).$$

As before, suppose that land produces 1 unit of output each period. Begin at time 1 with a small generation of young, and suppose the old owns the land.
We investigate whether the price of land and the real interest rate alternate between periods.
Let r_b be the interest rate that prevails when the big generation b is young, and r_a prevail when the small generation a is young. Equilibrium can be reduced to two equations. The first describes market clearing for goods in odd periods when the small generation is young and the big generation is old, and the second equation describes market clearing in even periods, when the big generation is young and the small generation is old. As before, we let p_a=1/(1+r_a) and p_b=1/(1+r_b). Then

$$\frac{1}{2}\frac{[6 + p_b 2]}{p_b} + \frac{1}{2}[3 + p_a 1] = 2 + 3 + 1 \frac{1}{2}\frac{[3 + p_a 1]}{p_a} + \frac{1}{2}[6 + p_b 2] = 1 + 6 + 1.$$

These can be simultaneously solved to get

$$p_a = .418, r_a = 139\%, PV_{Land a} = 1.29$$

$$p_b = .912, r_b = 9.6\%, PV_{Land b} = 2.09.$$

It is evident that the price of land is higher in the periods when b is young, since the interest rate is lower. Even though it is perfectly anticipated that when the big generation gets old, the price of land will fall, the price does not fall earlier because the interest rate is so low. (This point has been made by Geanakoplos, Magill and Quinzi, 2004.)

7 Impatience and uniform impatience

We have already suggested that it is useful in understanding the OLG model to consider variations, for example in which consumers live for ever. By doing so we shall also gain an important perspective on what view of consumers is needed to restore the usual properties of neoclassical equilibrium to an infinite horizon setting, a subject to which we return in Section 8.
Let us now allow for consumers t A who have endowments e^t that may be positive in all time periods, and also for arbitrary utilities u^t defined on uniformly bounded vectors $x \in L = \mathbf{R}^{\mathbf{N}}_{+}$. For ease of notation we assume one good per period. A minimal assumption we need about utilities u^t is continuity on finite segments, that is, fixing x_s for all $s>n$, $u^t(x)$ should be continuous in $(x_1,\ldots, x_n)$. We also need continuity on L, in some topology, but we will not go into these details. We also assume $\Sigma_{t \in A} e^t$ is uniformly bounded. In short, we suppose consumers may live for ever.
We shall find that in order to have Walrasian equilibria the consumers must be impatient. Suppose we try to form the truncated economy $E_{0},_{T}$ as before, say for T finite. Since utility potentially depends on every commodity, we could not define excess demands in $E_{0},_{T}$ unless we knew all the prices. To make it into a finite economy, let us call $E^{'}_{0,T}$ the version of $E_{0},_{T}$ in which every agent is obliged to consume his initial endowment during periods $t \leq 0$ and $t>T$. Clearly $E^{'}_{0,T}$ has an equilibrium. For this to give information about the original economy $E_{0,\infty}$, we need that consumers do not care very much about what happens to them after T, as T gets very far away. This requires a notion of impatience.
For any vector x, let ${}_n\hat{x}$ be the vector which is zero for $t>n$, and equal to x up until n. Thus ${}_n\hat{x}$ is the initial n-segment of x. To say that agent t A is impatient means that for any two uniformly bounded consumption streams x and y, if $u^t(x) > u^t(y)$, then for all big enough n, $u^t({}_n\hat{x}) > u^t(y)$. Let us suppose that all consumers are impatient. If these segments can be taken uniformly across agents, then we say the economy is uniformly impatient. Any finite economy with impatient consumers is uniformly impatient.
Note that the OLG agents are all impatient, since none of them cares about consumption after he dies, but the economy is not uniformly impatient.
Even with an economy consisting of all impatient consumers, the truncation argument, applied at an infinite $E^{'}_{0,T}$, does not guarantee the existence of an equilibrium. For, once we take standard parts, ignoring the infinitely dated commodities, it may turn out that the income from the sale of an agent's endowed commodities at infinite t, which he used to finance his purchase of commodities at finite t, is lost to the agent. It must also be guaranteed that the equilibria of $E^{'}_{0,T}$ give infinitesimal total value to the infinitely dated commodities. Wilson (1981) has given an example of an economy, composed entirely of impatient agents, that does not have an equilibrium precisely for this reason.
On the other hand, if there are only finitely many agents, even if they are infinitely lived, then we have:
Theorem: (Bewley, 1972). *Let the economy E be composed of finitely many, impatient consumers. Then there exists an equilibrium, and all equilibria are Pareto optimal.*
The Pareto efficiency of equilibria in these Bewley economies can be derived from the standard proof of efficiency: since there is a finite number of agents, the value of the aggregate endowments is a finite sum of finite numbers, and therefore finite itself.
In the special case with separable, commonly discounted utilities of the form $u^h(x) = \Sigma^{\infty}_{t=0}\delta^t v^h(x_t)$, with $\delta<1$, we have:

Theorem: (Kehoe–Levine, 1985). *In finite agent, separable commonly discounted utility economies, there is generically a finite number of equilibria.*
This theorem has been extended by Shannon (1999) and Shannon and Zame (2002).
Returning to the case of an infinite number of consumers, Pareto efficiency of equilibria, if they exist, can be guaranteed as long as a finite number of the agents collectively hold a non-negligible fraction of total endowment. But that also would guarantee the existence of equilibrium, since in the economy $E'_{0,T}$ we would then get the summability of the prices, meaning the endowments at infinity would have zero value, as Wilson (1981) pointed out.
It is extremely interesting to investigate the change in behaviour of an economy that evolves from individually impatient to uniformly impatient. Wilson (1981) considered an example with one infinitely lived agent, and infinitely many, overlapping, finite-lived agents, and showed that equilibria must exist, and all must be Pareto efficient. By the foregoing remarks, no matter what the proportion of sizes of the two kinds of consumers, equilibria must exist and be Pareto efficient. Muller and Woodford (1988) showed in a particular case that, when the single agent's proportion of the aggregate endowment is low enough, there is a continuum of equilibria, but if it is high enough there is no local indeterminacy.

8 Comparative statics for OLG economies

A celebrated theorem of Debreu assets that almost any Arrow–Debreu economy is regular, in the sense that it has a finite number of equilibria, each of which is locally unique. Small changes to the underlying structure of the economy (tastes, endowments, and so on) produce small, unique changes in each of the equilibria.
We have already seen that there are robust OLG economies with a continuum of equilibria. If attention is focused on one of them, how can one predict to which of the continuum of new equilibria the economy will move if there is a small change in the underlying structure of the economy, perhaps caused by deliberate government intervention? In what sense is any one of the new equilibria near the original one? In short, is comparative statics possible?
It is helpful at this point to recall that the OLG model is, in spirit, meant to represent a dynamic economy. Trade may occur as if all the markets cleared simultaneously at the beginning of time, but the economy is equally well described as if trade took place sequentially, under perfect foresight or rational expectations. Indeed, this is surely what Samuelson envisaged when he introduced money as an asset into his model. Accordingly, when a change occurs in the underlying structure of the economy, we can interpret it as if it came announced at the beginning of time, or as if it appeared at the date on which it actually affects the economy.
We distinguish two kinds of changes to the underlying structure of an economy $\bar{E}_{-\infty,\infty}$ starting from an equilibrium $\bar{q}$. Perfectly anticipated changes, after which we would look for a new equilibrium that cleared all the markets from the beginning of time, represent one polar case, directly analogous to the comparative statics experiments of the Arrow–Debreu economy. At the other extreme we consider perfectly *un*anticipated changes, say at date t=1. Beginning at the original economy and equilibrium $\bar{q} = (\ldots, \bar{q}_{-1}, \bar{q}_0, \bar{q}_1, \ldots)$, we would look, after the change from $\bar{E}_{-\infty,\infty}$ to $E_{-\infty,\infty}$, at time t=1 (say to the endowment or preferences of the generation born at time 1), for a price sequence $q = (\ldots, q_{-1}, q_0, q_1, \ldots)$ in which $q_t = \bar{q}_t$ for $t \le 0$, and $Z_t^{t-1}(q_{t-1}) + Z_t^t(q_t) = 0$ for $t \ge 2$. But at date t=1 we would require q_1 to satisfy $Z_1^0(q_{1a}|\bar{q}_0) + Z_1^1(q_1) = 0$, where $Z_1^0(q_{1a}|\bar{q}_0)$ represents the excess demand of the old at time 1, given that when they were young they purchased commodities on the strength of the conviction that they could surely anticipate prices $\bar{q}_{0b}$ when they got old, only to discover prices q_{1a} instead.
To study these two kinds of comparative statics, we must describe what we mean by saying that two price sequences are nearby. Our definition is based on the view that a change at time t=1 ought to have a progressively smaller impact the further away in time from t=1 we move. We say that q is near $\bar{q}$ if the difference $|q_t - \bar{q}_t|$ declines geometrically to zero, both as $t \to \infty$ and as $t \to -\infty$.
We have already noted in Section 1 that the multiplicity of OLG equilibria is due to the fact that at any time t the aggregate behaviour of the young generation is influenced by their expectations of future prices, which (under the rational expectations hypothesis) depends on the next generation's expectations, and so on. Accordingly we restrict our attention to generations whose aggregate behaviour Z^t satisfies the expectations sensitivity hypothesis:

$$\operatorname{rank} \frac{\partial Z_t^t(p_t, p_{t+1})}{\partial p_{t+1}} = \operatorname{rank} \frac{\partial Z_{t+1}^t(p_t, p_{t+1})}{\partial p_t} = L.$$

For economies composed of such generations we can apply the implicit function theorem, exactly as in Section 1, around any equilibrium q to deduce the existence of the forward and backward functions F_t and B_t. We write their derivatives at $\bar{q}$ as D_t and D_t^{-1}, respectively.
For finite Arrow–Debreu economies, Debreu gave a definition of regular equilibrium based on the derivative of excess demand at the equilibrium. He showed that comparative statics is sensible at a regular equilibrium, and then he showed that a 'generic' economy has regular equilibria. We follow the same program.
We say that the equilibrium $\bar{q}$ for the expectations sensitive OLG economy $\bar{E}$ is Lyapunov regular if the long-run geometric mean of the products $D_t^* D_t D_{t-1}^* D_{t-1} \cdots D_1^* D_1$ and $D_{-t}^{-1*} D_{-t}^{-1} \cdots D_{-1}^{-1*} D_{-1}^{-1}$ converge and if to these products we can associate $2L-1$ eigenvalues, called Lyapunov exponents. The equilibrium is also non-degenerate if in addition none of these Lyapunov exponents is equal to 1.
Theorem: (Geanakoplos–Brown, 1985). *Let $\bar{E} = \bar{E}_{-\infty,\infty}$ be an expectations-sensitive economy with a regular non-degenerate equilibrium $\bar{q}$. Then for all sufficiently small perfectly anticipated perturbations E of $\bar{E}$ (including $\bar{E}$ itself) E has a unique equilibrium q near $\bar{q}$.*
Thus the comparative statics of perfectly anticipated changes in the structure of $\bar{E}$, around a regular, non-degenerate equilibrium, is directly analogous to the Arrow–Debreu model. The explanation for the theorem is that a perfectly anticipated change at time 0 gives rise to price changes that have a forward stable manifold (on which prices converge exponentially back to where they started) and a backward stable manifold, and that there is only one price at time 0 that is on both the forward and the backward stable manifolds. Note, incidentally, that one implication of the above theorem is that neutral policy changes, like jawboning or changing animal spirits, that is, those for which $\bar{q}$ itself remains an equilibrium, cannot have any effect if they are perfectly anticipated and move the economy to nearby equilibria.
Theorem: (Geanakoplos–Brown, 1985). *Let $\bar{E}$ be an expectations-sensitive economy with a regular equilibrium $\bar{q}$. Then, for all sufficiently small perfectly unanticipated perturbations E of $\bar{E}$ (including $\bar{E}$ itself), the set of unanticipated equilibria q of E near $\bar{q}$ is either empty, or a manifold of dimension r, $0 \le r \le L$ (L−1 if there is no money in the economy), where r is independent of the perturbation.*
The above theorem allows for the possibility that an unanticipated change may force the economy onto a path that diverges from the original

equilibrium; the disturbance could be propagated and magnified through time. And if there are nearby equilibria, then there may be many of them. (Indeed, that is basically what was shown in Section 4.) In particular, an unanticipated neutral policy change could be compatible with a continuum of different equilibrium continuations. The content of the theorem is that, if there is a multiplicity of equilibrium continuations, it is parameterizable. In other words, the same r variables can be held fixed, and for any sufficiently small perturbation, there is exactly one nearby equilibrium which also leaves these r variables fixed. We shall discuss the significance of this in the next section.

This last theorem was proved first, in the special case of steady-state economies, by Kehoe–Levine, in the same excellent paper to which we have referred already several times. The theorem quoted here, together with the previous theorem on the comparative statics of perfectly anticipated policy changes, refers to economies in which the generations may be heterogeneous across time.

Let us suppose that A is a compact collection of generational characteristics, all of which obey the expectations-sensitive hypothesis. Let us suppose that each generation's characteristics are drawn at random from A, according to some Borel probability measure. If the choices are made independently across time, then the product measure describes the selection of economies. Almost any such collection will have a complex demographic structure, changing over time. The equilibrium set is then endogenously determined, and will be correspondingly complicated. It can be shown, however, that

Theorem: (Geanakoplos–Brown, 1985). *If the economy E is randomly selected, as described above, then with probability* 1, *E has at least one Lyapunov regular equilibrium*.

Note that the regularity theory for infinite economies stops short of Arrow–Debreu regularity. In the finite economies, with probability one all the equilibria are regular.

9 Keynesian macroeconomics

Keynesian macroeconomics is based in part on the fundamental idea that changes in expectations, or animal spirits, can affect equilibrium economic activity, including the level of output and employment. It asserts, moreover, that publicly announced government policy also has predictable and significant consequences for economic activity, and that therefore the government should intervene actively in the marketplace if investor optimism is not sufficient to maintain full employment.

The Keynesian view of the indeterminacy of equilibrium and the efficacy of public policy has met a long and steady resistance, culminating, in the sharpest attack of all, from the so-called new classicals, who have argued that the time-honoured microeconomic methodological premises of agent optimization and market clearing, considered together with rational expectations, are logically inconsistent with animal spirits and the non-neutrality of public monetary and bond-financed fiscal policy.

The foundation of the new classical paradigm is the Walrasian equilibrium model of Arrow–Debreu, in which it is typically possible to prove that all equilibria are Pareto optimal and that the equilibrium set is finite; at least locally, the hypothesis of market clearing fixes the expectations of rational investors. In that model, however, economic activity has a definite beginning and end. Our point of view is that for some purposes economic activity is better described as a process without end. In a world without a definite end, there is the possibility that what happens today is underdetermined, because it depends on what people expect to happen tomorrow, which in turn depends on what people tomorrow expect to happen the day after tomorrow, and so on.

Consider the simple one-good per period overlapping generations economy with money $E_0^{M,S}$, which we discussed in Section 3. Generation 0 is endowed with money when old, and equilibrium can be described with the contemporaneous commodity prices $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots)$ where we take the price of money to be fixed at 1. (In this case, as we saw in Section 3, contemporaneous prices are also present value prices.) It is helpful to reinterpret the model as a simple production economy. Imagine that the endowment e_t^t in the first period of life is actually labour, which can be transformed into output, y_t, according to the production function, $y_t=e_t$. We would then think of any purchases of goods by the old generation as demand for real output to be produced by the young. The young in turn now derive utility from leisure in their youth and consumption in their old age. Equilibrium in which consumption of the old is higher can be interpreted as an equilibrium with less leisure and higher output.

The indeterminacy of rational expectations equilibrium has the direct interpretation that optimistic expectations *by themselves* can cause the economy's *output* to expand or contract. In short the economy has an inherent volatility. The Keynesian story of animal spirits causing economic growth or decline can be told without invoking irrationality or non-market clearing.

In fact, the indeterminacy of equilibrium expectations is especially striking when seen as a response to public (but unanticipated) policy changes. Suppose the economy is in a long-term rational expectations equilibrium $\bar{p}$, when at time 1 the government undertakes some expenditures, financed, say, by printing money. How should rational agents respond? The environment has been changed, and there is no reason for them to anticipate that $(\bar{p}_2, \bar{p}_3, \ldots)$ will still occur in the future. Indeed, in models with more than one commodity (such as we will shortly consider) there may be no equilibrium $(p_1, p_2, p_3, \ldots)$ in the new environment with $p_2 = \bar{p}_2$, $p_3 = \bar{p}_3$, and so on. There is an ambiguity in what can be rationally anticipated.

We argue that it is possible to explain the differences between Keynesian and monetarist policy predictions by the assumptions each makes about expectational responses to policy, and not by the one's supposed adherence to optimization, market clearing, and rational expectations, and the other's supposed denial of all three.

Consider now the government policy of printing a small amount of money, ΔM, to be spent on its own consumption of real output – or equivalently to be given to generation $t=0$ (when old) to spend on its consumption. Imagine first that agents are convinced that this policy is not inflationary, that is, that $\bar{p}_1$ will remain the equilibrium price level during the initial period of the new equilibrium. This will give generation $t=0$ consumption level $(M + \Delta M) / \bar{p}_1$. As long as ΔM is sufficiently small and the initial equilibrium was one of the Pareto-suboptimal equilibria described in Section 1, there is indeed a new equilibrium price path p beginning with $p_1 = \bar{p}_1$. Output at time 1 rises by $\Delta M / \bar{p}_1$, and in fact this policy is Pareto improving.

On the other hand, imagine instead that agents are convinced that the path of real interest rates $p_t/p_{t+1}-1$ will remain unchanged. In this economy, price expectations are a function of p_1. Recalling the initial period market-clearing equation, it is clear that p_1 and all future prices rise proportionally to the growth in the money stock. The result is that output is unchanged and the old at $t=1$ must pay for the government's consumption. If the government's consumption gives no agent utility, the policy is Pareto worsening.

This model is only a crude approximation of the differences between Keynesian and monetarist assumptions about expectations and policy. It is quite possible to argue, for example, that holding $p_2 / p_1 = \bar{p}_2 / \bar{p}_1$ (the future inflation rate) fixed is the natural *Keynesian* assumption to make. This ambiguity is unavoidable when there is only one asset into which the young can place their savings. We are thereby prevented from distinguishing between the inflation rate and the interest rate. Our model must be enriched before we can perform satisfactory policy analysis. Nevertheless, the model conveys the general principle that expected price paths are not locally unique. There is consequently no natural assumption to make about how expectations are affected by policy. A sensible analysis is therefore impossible without externally given hypotheses about expectations. These can be Keynesian, monetarist, or perhaps some combination of the two.

Geanakoplos and Polemarchakis (1985) build just such a richer model of macroeconomic equilibrium by adding commodities, including a capital good, and a neoclassical production function. With elastically supplied labour, there are two dimensions of indeterminacy. It is therefore possible to fix both the nominal wage, and the firm's expectations (‘animal spirits’), and still solve for equilibrium as a function of policy perturbations to the

economy. These institutional rigidities are more convincingly Keynesian, and they lead to Keynesian policy predictions. Moreover, taking advantage of the simplicity of the two-period lived agents, the analysis can be conducted entirely through the standard Keynesian (Hicksian) IS–LM diagram.
Keynesians themselves often postulate that the labour market does not clear. For Keynesians, lack of labour market clearing has at least a threefold significance, which it is perhaps important to sort out. First, since labour is usually taken to be inelastically supplied, it makes it possible to conceive of (Keynesian) equilibria with different levels of output and employment. Second, it makes the system of demand and supply underdetermined, so that endogenous variables like animal spirits (that is, expectations) which are normally fixed by the equilibrium conditions can be volatile. Third, it creates unemployment that is involuntary. By replacing lack of labour market clearing at time 1 with elastic labour supply and lack of market clearing 'at infinity' one can drop what seems to many an ad hoc postulate, yet retain at least the first two desiderata of Keynesian analysis.

10 Neoclassical equilibrium vs. classical equilibrium

The Arrow–Debreu model of general equilibrium, based on agent optimization, rational expectations, and market clearing, is universally regarded as the central paradigm of the neoclassical approach to economic theory. In the Arrow–Debreu model, consumers and producers, acting on the basis of individual self-interest, combine, through the aggregate market forces of demand and supply, to determine (at least locally) the equilibrium distribution of income, relative prices, and the rate of growth of capital stocks (when there are durable goods). The resulting allocations are always Pareto optimal.
Classical economists at one time or another have rejected all of the methodological principles of the Arrow–Debreu model. They replace individual interest with class interest, ignore (marginal) utility, especially for waiting, doubt the existence of marginal product, and question whether the labour market clears. But by far the most important difference between the two schools of thought is the classical emphasis on the long-run reproduction of the means of production, in a never-ending cycle.
Thus the celebrated classical economist Sraffa writes in Appendix D to his book:

> It is of course in Quesnay's *Tableau Economique* that is found the original picture of the system of production and consumption as a circular process, and it stands in striking contrast to the view presented by modern theory, of a one-way avenue that leads from 'Factors of Production' to 'Consumption Goods.'

The title of his book, *Production of Commodities by Means of Commodities*, itself suggests a world that has no definite beginning, and what is circular can have no end.
In the Arrow–Debreu model time has a definite end. As we have seen, that has strong implications. With universal agreement about when the world will end, there can be no reproduction of the capital stock. In equilibrium it will be run down to zero. Money, for example, can never have positive value. Rational expectations will fix, at each moment, and for each kind of investment, the expected rate of profit.
In the classical system, by contrast, the market does not determine the distribution of income. Sraffa (1960, p. 33) writes:

> The rate of profits, as a ratio, has a significance which is independent of any prices, and can well be 'given' before the prices are fixed. It is accordingly susceptible of being determined from outside the system of production, in particular by the money rates of interest. In the following sections the rate of profits will therefore be treated as the independent variable.

Other classical writers concentrate instead on the real wage as determined outside the market forces of supply and demand, for example by the level of subsistence or the struggle between capital and labour. Indeterminacy of equilibrium seems at least as central to classical economists as it is to Keynesians.
Like Keynesians, classicals often achieve indeterminacy in their formal models by allowing certain markets not to clear in the Walrasian sense. (Again like Keynesians, the labour market is usually among them.) Thus we have called the equilibrium in Section 4 in which some of the markets were allowed not to clear a 'classical equilibrium'.
What the OLG model shows is that, by incorporating the classical view of the world without definite beginning or end, it is possible to maintain all the neoclassical methodological premises and yet still leave room for the indeterminacy which is the hallmark of both classical and Keynesian economics. In particular this can be achieved while maintaining labour market clearing. The explanation for this surprising conclusion is that the OLG model is isomorphic to a finite-like model in which indeed not all the markets need to clear. But far from being the labour markets, under pressure to move towards equilibrium from the unemployed clamouring for jobs, these markets are off 'at infinity', under no pressure towards equilibrating.
We have speculated that, once one has agreed to the postulate that the resources of the economy are potentially as great at any future date as they are today, then uniform impatience of consumers is the decisive factor, according to Walrasian principles, which may influence whether the market forces of supply and demand determine a locally unique, Pareto-optimal equilibrium, or leave room for extra-market forces to choose among the continuum of inefficient equilibria. In these terms, the Arrow–Debreu model supposes a short-run impatient economy, and OLG a long-run patient economy.

11 Sunspots

So far we have not allowed uncertainty into the OLG model. As a result we found no difference in interpreting trade sequentially, with each agent facing two budget constraints, or 'as if' the markets all cleared simultaneously at the beginning of time, with each agent facing one budget constraint. Once uncertainty is introduced these inpts become radically different. In either case, however, there is a vast increase in the number of commodities, and hence in the potential for indeterminacy.
If we do not permit agents to make trades conditional on moves of nature that occur before they are born, then agents will have different access to asset markets. Even in finite horizon economies, differing access to asset markets has been shown by Cass and Shell (1983) to lead to 'sunspot effects'.
A 'sunspot' is a visible move of nature which has no real effect on consumers, on account of preferences, or endowments, or through production. In the Arrow–Debreu model it also could have no effect on equilibrium trade; this is no longer true when access to asset markets differs.
The sunspot effect is intensified when combined with the indeterminacy that can already arise in an OLG economy. Consider the simple one good, steady state OLG economy of Section 2. Suppose that there is an equilibrium two cycle in present value prices $p = (\ldots, p_{-1}, p_0, p_1, \ldots)$ with $p_{2t}=p^S$ and $p_{2t+1}=p^R$, for all $t \in T$. Now suppose that the sun is known to shine on even periods, and hide behind rain on odd periods. The above equilibrium is perfectly correlated with the sun, even though no agent's preferences or endowments are. As usual, the same prices for $t \geq 0$ support an equilibrium, given the right amount of money, in $E^{M,S}_{0,\infty}$.

More generally, suppose that the probability of rain or shine, given the previous period's weather, is given by the Markov matrix $\pi = (\pi_{SS}, \pi_{SR}, \pi_{RS}, \pi_{RR})$. A steady state equilibrium for $E^{M,S}_{0,\infty}$, given π, is an assignment of a money price for the commodity, depending only on that period's weather, such that, if all agents maximize their expected utility with respect to π, then in each period the commodity market and money market clears. Azariadis (1981) essentially showed that, if there is a two-cycle of the certainty economy, then there is a continuum of steady state sunspot equilibria.
The sunspot equilibria, unlike the cyclical equilibria of Section 2, are Pareto suboptimal whenever the matrix π is non-degenerate.
The combination of the dynamic effects of the infinite horizon OLG model with the burgeoning theory of incomplete markets under real uncertainty, is already on the agenda for the next generation's research.

See Also

- Arrow–Debreu model of general equilibrium

Bibliography

Allais, M. 1947. *Economie et intérêt*. Paris: Imprimerie Nationale.

Arrow, K.J. 1953. The role of securities in the optimal allocation of risk-bearing. Repr. in K.J. Arrow, *Essays in the theory of Risk Bearing*. Chicago: Markham, 1971.

Azariadis, C. 1981. Self-fulfilling prophecies. *Journal of Economic Theory* 25, 380–96.

Balasko, Y., Cass, D. and Shell, K. 1980. Existence of competitive equilibrium in a general overlapping-generations model. *Journal of Economic Theory* 23, 307–22.

Balasko, Y. and Shell, K. 1980. The overlapping-generations model, I: the case of pure exchange without money. *Journal of Economic Theory* 23, 281–306.

Benhabib, J. and Day, R.H. 1982. A characterization of erratic dynamics in the overlapping generations model. *Journal of Economic Dynamics and Control* 4, 37–44.

Benhabib, J. and Nishimura, N. 1985. Competitive equilibrium cycles. *Journal of Economic Theory* 35, 284–306.

Benveniste, L. and Gale, D. 1975. An extension of Cass' characterization of infinite efficient production programs. *Journal of Economic Theory* 10, 229–38.

Bewley, T. 1972. Existence of equilibria in economies with infinitely many commodities. *Journal of Economic Theory* 4, 514–40.

Cass, D. 1972. On capital overaccumulation in the aggregative, nonclassical model of economic growth. *Journal of Economic Theory* 4, 200–23.

Cass, D. and Shell, K. 1983. Do sunspots matter? *Journal of Political Economy* 91, 183–227.

Cass, D. and Yaari, M.E. 1966. A re-examination of the pure consumption loan model. *Journal of Political Economy* 74, 200–23.

Debreu, G. 1970. Economies with a finite set of equilibria. *Econometrica* 38, 387–92.

Diamond, P.A. 1965. National debt in a neo-classical growth model. *American Economic Review* 55, 1126–50.

Fisher, I. 1907. *The Rate of Interest*. New York: Macmillan.

Fisher, I. 1930. *The Theory of Interest*. New York: Macmillan.

Gale, D. 1973. Pure exchange equilibrium of dynamic economic models. *Journal of Economic Theory* 6, 12–36.

Geanakoplos, J. 1978. Sraffa's *Production of Commodities by Means of Commodities*: indeterminacy and suboptimality in neoclassical economics. Working Paper, RIAS. Also Chapter 3 in 'Four Essays on the Model of Arrow–Debreu', Ph.D. thesis, Harvard University, 1980.

Geanakoplos, J. 2005. The ideal inflation indexed bond and Irving Fisher's Theory of Interest with Overlapping Generations. In *Celebrating Irving Fisher: The Legacy of a Great Economist*, ed. R. Dimand and J. Geanakoplos. *Journal of Economics and Sociology* 64, 257–305. Oxford: Blackwell.

Geanakoplos, J. and Brown, D. 1982. Understanding overlapping generations economies as lack of market clearing at infinity. Mimeo, Yale University, revised 1985, 1986.

Geanakoplos, J. and Brown, D. 1985. Comparative statics and local indeterminacy in OLG economies: an application of the multiplicative ergodic theorem. Discussion Paper No. 773, Cowles Foundation.

Geanakoplos, J., Magill, M. and Quinzi, M. 2004. Demography and the long-run predictability of the stock market. *Brookings Papers on Economic Activity* 2004(1), 241–325.

Geanakoplos, J. and Polemarchakis, H.M. 1984. Intertemporally separable overlapping generations economies. *Journal of Economic Theory* 34, 207–

15.

Geanakoplos, J. and Polemarchakis, H.M. 1985. Walrasian indeterminacy and Keynesian macroeconomics. *Review of Economic Studies* 53, 755–99.

Grandmont, J.M. 1985. Endogenous, competitive business cycles. *Econometrica* 53, 995–1046.

Kehoe, T.J. and Levine, D.K. 1984. Regularity in overlapping generations exchange economies. *Journal of Mathematical Economics* 13, 69–93.

Kehoe, T.J. and Levine, D.K. 1985. Comparative statics and perfect foresight in infinite horizon models. *Econometrica* 53, 433–53.

Kehoe, T.J., Levine, D.K. and Romer, P.M. 1990. Determinacy of equilibrium in dynamic models with finitely many consumers. *Journal of Economic Theory* 50, 1–21.

Keynes, J.M. 1936. *The General Theory of Employment, Interest and Money*. London: Macmillan.

Li, T.Y. and Yorke, J.A. 1975. Period three implies chaos. *American Mathematical Monthly* 8, 985–92.

Lucas, R.E. 1972. Expectations and the neutrality of money. *Journal of Economic Theory* 4, 102–21.

Muller, W.J. and Woodford, M. 1988. Determinacy of equilibrium in stationary economies with both finite and infinite lived consumers. *Journal of Economic Theory* 46, 255–90.

Okuno, M. and Zilcha, I. 1980. On the efficiency of a competitive equilibrium in infinite horizon monetary economies. *Review of Economic Studies* 47, 797–807.

Samuelson, P.A. 1958. An exact consumption loan model of interest, with or without the social contrivance of money. *Journal of Political Economy* 66, 467–82.

Shannon, C. 1999. Determinacy of competitive equilibria in economies with many commodities. *Economic Theory* 14, 29–87.

Shannon, C. and Zame, B. 2002. Quadratic concavity and determinacy of equilibrium. *Econometrica* 70, 631–62.

Shell, K. 1971. Notes on the economics of infinity. *Journal of Political Economy* 79, 1002–11.

Sraffa, P. 1960. *Production of Commodities by Means of Commodities*. Cambridge: Cambridge University Press.

Wilson, C. 1981. Equilibrium in dynamic models with an infinity of agents. *Journal of Economic Theory* 24, 95–111.

How to cite this article

Geanakoplos, John. "overlapping generations model of general equilibrium." The New Palgrave Dictionary of Economics. Second Edition. Eds. Steven N. Durlauf and Lawrence E. Blume. Palgrave Macmillan, 2008. The New Palgrave Dictionary of Economics Online. Palgrave Macmillan. 10 August 2009 <http://www.dictionaryofeconomics.com/article?id=pde2008_O000067> doi:10.1057/9780230226203.1236