BAYES SOLUTIONS OF THE STATISTICAL INVENTORY PROBLEM[1]

BY HERBERT SCARF

Stanford University

1. Introduction. The inventory problem as discussed in the paper of Arrow, Harris, and Marschak [1] is a sequential decision problem. At the beginning of each time period a decision is made to stock a quantity of a specific item in anticipation of demand during that period. If the demand exceeds the available supply, the next period is begun with an initial stock which is either zero or negative. If the demand, on the other hand, is less than the available supply, the subsequent period begins with a stock level which is equal to the excess of supply over demand. In both cases the stock level may be augmented by additional purchasing.

A number of costs are assumed to be operative in this situation, among them, a cost of purchasing stock, a cost of holding or maintaining the stock in inventory, and a cost which arises whenever current inventory is insufficient to meet demand. The main problem is the determination of a sequence of purchasing or ordering decisions which minimizes some criterion built up from these costs. The criterion adopted in this paper is to discount costs incurred n periods in the future by an amount α^n, and to select that sequence of stockage decisions which minimizes the sum of all discounted costs. An elaborate discussion of the costs and the general structure of inventory models may be found in [2]. In the paper of Arrow, Harris, and Marschak and in a number of other papers in this area, the assumption has been made that the quantity demanded during any time period is a random variable whose distribution is known and unchanging from period to period. However, in the second of two papers by Dvoretsky, Kiefer and Wolfowitz [3] a more general situation in which the demand distribution is not known precisely is examined from the point of view of statistical decision theory. In the present paper our concern will be with this latter problem. The treatment, in distinction to that given in the paper by Kiefer, et al will be very specific, in the sense that we shall restrict our attention to very simple types of cost functions in order to obtain some detailed results about the optimal stockage policies.

The costs will be as follows:

(a) The ordering cost $c(z)$, as a function of the amount ordered, will be assumed to be linear, i.e., $c(z) = cz$.

(b) If the inventory at the close of the period is positive a holding cost $h(x)$ will be incurred, which in this paper is assumed to be a linear function of the quantity of inventory on hand at the end of the period, i.e., $h(x) = hx$.

(c) If the quantity of stock at the end of any period is negative a linear pen-

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alty cost will be incurred. The marginal penalty cost is represented by the constant p. It will be assumed that $p > c(1 - \alpha)$.

It should be noted that under these assumptions, if the demand distribution is known precisely the optimal stockage policy is exceedingly simple to determine. In fact there exists a single critical stock level $\bar{x}$ so that if the stock on hand at the beginning of any period is greater than $\bar{x}$ we do not order, and if the stock level x at the beginning of the period is less than $\bar{x}$, we order an amount $\bar{x} - x$. The number $\bar{x}$ is obtained as the solution of the equation

$$\frac{p - c(1 - \alpha)}{h + p} = \int_0^{\bar{x}} \varphi(\xi)\, d\xi, \tag{1}$$

where $\varphi(\xi)$ is the density of the distribution of demand.

In this paper, rather than making the assumption that the density of demand is known precisely, we assume that it may be described by a density $\varphi(\xi, \omega)$ with an unknown statistical parameter ω. It will be convenient to restrict our attention to densities $\varphi(\xi, \omega)$ which are members of the exponential class, i.e.,

$$\varphi(\xi, \omega) = \beta(\omega) e^{-\xi\omega} r(\xi), \tag{2}$$

with $r(\xi) = 0$ for $\xi < 0$. The reason for this restriction is that at the nth stage all of the relevant information may be summarized in the single statistic $s = (\xi_1 + \cdots + \xi_{n-1})/(n - 1)$. We shall assume that $r(\xi)$ is bounded, and that $r(\xi) > 0$ for $\xi > 0$. We shall also assume that an a priori Bayes distribution is chosen for the unknown parameter ω. At the beginning of the nth period the available information will consist of two parts, one a knowledge of the current stock level x and the second part a knowledge of the statistic s. The optimal policy at the nth period will be that function of the two variables x and s which indicates the proper quantity of the items to be purchased. Our first result, which is crucially dependent upon the assumptions made about the cost functions, is that the optimal policies are again defined by single critical numbers. In fact there is a sequence of functions $\bar{x}_n(s)$ so that the quantity to order is equal to

$$\operatorname{Max}\{\bar{x}_n(s) - x, 0\} \qquad \text{(see Theorem 1)}. \tag{3}$$

The critical functions $\bar{x}_n(s)$ are themselves quite difficult to obtain analytically. Some properties of the functions may be obtained, for example, it will be shown that $\bar{x}_n(s)$ is monotone increasing, a fact which is essentially a consequence of the monotone likelihood ratio property possessed by all members of the exponential class [7]. This result is demonstrated in Theorem 2.

Theorem 3 presents an additional result. If at the beginning of the nth period, the stock level is less than $\bar{x}_n(s)$, we bring the stock level up to this number. If the demand during the nth period is ξ, the new stock level is $\bar{x}_n(s) - \xi$, and the new average demand is $((n - 1)s + \xi)/n$. In Theorem 3 it is demonstrated that no ordering is done at the beginning of the $(n + 1)$st period if ξ is sufficiently small. Mathematically this is equivalent to

$$\bar{x}_{n+1}([(n-1)s+\xi]/n)$$

for ξ sufficiently small.

Our primary interest however will be in obtaining an asymptotic expansion for the critical functions $\bar{x}_n(s)$, for large n. It will be shown, under the assumption of certain regularity conditions, that

$$\bar{x}_n(s) \sim \bar{x}(s) + \frac{a(s)}{n}. \tag{4}$$

The function $\bar{x}(s)$ is the critical level for the inventory problem in which the demand density is known precisely, in fact known to be $\varphi(\xi, \omega)$ where ω is selected so that the mean demand is actually s. This is, of course, the value of ω which satisfies the equation

$$\frac{d \log \beta}{d\omega} = s, \qquad \text{which we shall write as } \omega(s). \tag{5}$$

Therefore, as (1) indicates, the function $\bar{x}(s)$ is given by the solution of the equation

$$\frac{p - c(1-\alpha)}{h+p} = \int_0^{\bar{x}(s)} \varphi(\xi, \omega(s))\, d\xi. \tag{6}$$

The term $a(s)$ may also be determined explicitly. Let $\Phi(x, \omega)$ be the cumulative demand distribution, and let $\sigma^2(\omega)$ be the variance of demand if the true parameter value is $\omega(\sigma^2 = -(d^2 \log \beta/d\omega^2))$. Then

$$a(s) = -\frac{\dfrac{\partial}{\partial\omega}\left(\dfrac{f^2}{\sigma^2}\dfrac{\partial\Phi}{\partial\omega}\right)}{2f^2\varphi} \tag{7}$$

with x replaced by $\bar{x}(s)$ and ω replaced by $\omega(s)$ after the differentiation is carried out. The expansion is valid if f has two continuous derivatives near the point $\omega(s)$, $f(\omega(s)) > 0$, and $0 < s < \mathrm{Max}_\omega\, (\beta'/\beta)(\omega)$.

The sign of $a(s)$ may be either positive or negative. Since $\omega(s)$ is the maximum likelihood estimator of the true parameter value, $\bar{x}(s)$ will be the maximum likelihood estimate of the critical stockage level, and equation (5) indicates that the optimal Bayes stockage policy approaches the maximum likelihood policy occasionally from above and occasionally from below, depending on the costs and the original Bayes estimate.

It is important to realize that in this discussion neither s nor $\bar{x}_n(s)$ are random variables. $\bar{x}_n(s)$ is for each n a specific function which minimizes total cost under the assumption that a specific Bayes distribution is operative. These functions are difficult to compute directly and our purpose in obtaining an asymptotic expansion is solely to approximate these functions in a simple way for large values of n.

The crucial part of the proof of (5) is the determination of the first several terms in the asymptotic expansion of the a posteriori Bayes density. and there are many points in common with some recent work done by Johns and Guthrie [5], in a problem of acceptance sampling.

2. A review of the problem when the demand distribution is known. In this section we shall assemble those facts about the non-statistical version of our problem, which will be needed in the subsequent sections of this paper. We begin by defining $C(x)$ to be the expected cost incurred by the use of an optimal policy if the initial stock level is x. Costs which occur n periods in the future are discounted by a factor α^n and incorporated in the function $C(x)$. In the model that we are considering x may take on negative values, representing unfilled orders.

If the policy used is to order an amount $y - x$ when the initial stock is x, then the cost incurred during the first period is $c(y - x) + L(y)$, where

$$L(y) = \begin{cases} h \int_0^y (y - \xi)\varphi(\xi)\, d\xi + p \int_y^\infty (\xi - y)\varphi(\xi)\, d\xi; & y > 0 \\ p \int_0^\infty (\xi - y)\varphi(\xi)\, d\xi; & y < 0. \end{cases} \tag{8}$$

At the end of the first period the stock on hand is $y - \xi$. Since the situation faced at the beginning of the second period is precisely that faced at the beginning of the first period with the exception that the stock level is now $y - \xi$, we see that the expectation of all future costs may be summarized by $\alpha \int_0^\infty C(y - \xi)\varphi(\xi)\, d\xi$, and therefore $C(x) \leqq \{c(y - x) + L(y) + \alpha \int_0^\infty C(y - \xi)\varphi(\xi)\, d\xi\}$. The choice of y which minimizes the right hand side of this equation is clearly optimal and we obtain

$$C(x) = \inf_{y \geqq x} \left\{ c(y - x) + L(y) + \alpha \int_0^\infty C(y - \xi)\varphi(\xi)\, d\xi \right\}. \tag{9}$$

It is sometimes convenient to consider an inventory problem with precisely the same costs as the above problem, but which is engaged in for a total of N periods rather than continuing indefinitely in the future. No costs, either holding or penalty costs, are incurred after N periods have elapsed and no salvage value is to be attributed to any excess stock at the end of N periods. If the minimum expected costs for such a problem are denoted by $C^N(x)$, then $C^N(x)$ tends to $C(x)$ at every point [6] and we have the following equation:

$$C^N(x) = \inf_{y \geqq x} \left\{ cy - cx + L(y) + \alpha \int_0^\infty C^{N-1}(y - \xi)\varphi(\xi)\, d\xi \right\}, \tag{10}$$

where $C^{-1}(x) \equiv 0$.

Equation (10) may be used to show that $C^N(x)$ is convex, and therefore $C(x)$ is convex. A simple argument, based on (9) and the convexity of $C(x)$ shows that the optimal stockage policy is indeed defined by the single critical number $\bar{x}$ which minimizes

$$cy + L(y) + \alpha \int_0^\infty C(y - \xi)\varphi(\xi)\, d\xi. \tag{11}$$

(This argument is given in detail in the next section for the statistical problem.) Equation (9) shows us that for $x < \bar{x}$, $C'(x) = -c$, and it therefore follows from (11) that $\bar{x}$ satisfies the equation

$$0 = c(1 - \alpha) + L'(y).$$

This is merely a restatement of equation (1).

This method permits us to make a direct computation of the optimal stockage policy, without making an explicit determination of the cost $C(x)$. We shall find it necessary in the treatment of the statistical problem, to utilize the fact that in some sense the minimum cost function depends continuously upon the demand distribution. In order to obtain this result we shall derive an expression for $C(x)$.

If $x > \bar{x}$, equation (9) yields

$$\begin{aligned} C(x) &= L(x) + \alpha \int_0^\infty C(x - \xi)\varphi(\xi)\, d\xi \\ &= L(x) + \alpha \int_0^{x-\bar{x}} C(x - \xi)\varphi(\xi)\, d\xi + \alpha \int_{x-\bar{x}}^\infty C(x - \xi)\varphi(\xi)\, d\xi. \end{aligned} \tag{12}$$

In the latter integral $x - \xi < \bar{x}$ so that $C(x - \xi) = c(\bar{x} - x + \xi) + C(\bar{x})$ and we obtain for $x > \bar{x}$

$$C(x) = L(x) + \alpha(c\bar{x} - cx + C(\bar{x})) \int_{x-\bar{x}}^\infty \varphi(\xi)\, d\xi + \alpha c \int_{x-\bar{x}}^\infty \xi\varphi(\xi)\, d\xi + \alpha \int_0^{x-\bar{x}} C(x - \xi)\varphi(\xi)\, d\xi.$$

If we put $x = \bar{x}$, we obtain

$$C(\bar{x}) = \frac{L(\bar{x}) + \alpha c \int_0^\infty \xi\varphi(\xi)\, d\xi}{1 - \alpha}, \tag{13}$$

and if we define $U(x) = C(x + \bar{x})$, we obtain

$$U(x) = A(x) + \alpha \int_0^x U(x - \xi)\varphi(\xi)\, d\xi. \tag{14}$$

where

$$A(x) = L(x + \bar{x}) + \alpha(-cx + C(\bar{x})) \int_x^\infty \varphi(\xi)\, d\xi + \alpha c \int_x^\infty \xi\varphi(\xi)\, d\xi.$$

Equation (14) is a Volterra equation which may be solved for $U(x)$ as follows: We define $M(x, \alpha) = \sum_1^\infty \alpha^n \Phi^{(n)}(x)$, where $\Phi^{(n)}$ is the n-fold convolution of the cumulative demand distribution. Then

$$U(x) = C(x + \bar{x}) = A(x) + \int_0^x A(x - \xi)\, dM_\xi(\xi, \alpha). \tag{15}$$

This discussion may be used to prove the following lemma.

LEMMA 1. *Let $\varphi_k(x)$ be a sequence of positive density functions which converges uniformly in every finite interval to the positive density function $\varphi(x)$. Furthermore, assume that the functions have finite means which converge to the finite mean of φ. If we define $C^k(x)$ to be the cost function for the inventory problem with demand density $\varphi_k(x)$, then $C^k(x)$ converges to $C(x)$ at every point.*

If we define $\bar{x}^k$ to be the critical stockage level for the problem with demand distribution $\varphi_k(x)$, then it is clear from equation (1), that $\bar{x}^k \to \bar{x}$. By uniform convergence and (8) we see that $L^k(\bar{x}^k) \to L(\bar{x})$ and therefore by (13) $C^k(\bar{x}^k) \to C(\bar{x})$. If $x < \bar{x}$ then for large k, $x < \bar{x}^k$ and $C^k(x) = c(\bar{x}^k - x) + C^k(\bar{x}^k)$ which tends to $C(x)$. On the other hand if $x > \bar{x}$ and consequently $> \bar{x}^k$ for large k, then we may use (15) to show that $C^k(x)$ tends to $C(x)$. The only point that needs mention is that if we define $M^k(x, \alpha) = \sum_1^\infty \alpha^n \Phi_k^{(n)}(x)$, then $M^k(x, \alpha)$ converges uniformly in any finite x-interval to $M(x, \alpha)$.

3. Properties of Bayes solutions to the statistical problem. In this section we shall set up a functional equation analogous to (9) and discuss some simple properties of Bayes solutions to the problem. The asymptotic expansion of the Bayes solutions will be discussed in the subsequent section.

We assume that the demand density is of the form

$$\varphi(\xi, \omega) = \beta(\omega) e^{-\xi\omega} r(\xi). \tag{16}$$

We shall also assume that an a priori distribution has been selected for the unknown parameter ω. We denote the density of this distribution by $f(\omega)$.

At the beginning of the nth period, the information available to the decision maker is a knowledge of the present stock level x, and a record of all the previous demands $\xi_1, \xi_2, \cdots, \xi_{n-1}$. All of the demand information may be summarized in the sufficient statistic

$$s = \frac{\sum_{i=1}^{n=1} \xi_i}{n - 1}. \tag{17}$$

We define $C_n(x, s)$ to be the discounted sum of costs incurred if an optimal policy is followed. We first of all remark that the demand density with which the decision maker is faced during the nth period is prob (demand $= \xi \mid$ average demand in last $n - 1$ periods $= s$), and this is

$$\varphi_n(\xi \mid s) = r(\xi) \frac{\int_0^\infty \beta^n(\omega) e^{-\omega\xi} e^{-(n-1)\omega s} f(\omega)\, d\omega}{\int_0^\infty \beta^{n-1}(\omega) e^{-(n-1)\omega s} f(\omega)\, d\omega}. \tag{18}$$

This distribution summarizes the decision makers expectations as far as demand during the nth period is concerned.

If the policy used is to order an amount $y - x$, then the expected cost during the nth period is $c(y - x) + L_n(y \mid s)$, where

$$(19) \quad L_n(y \mid s) = \begin{cases} h \int_0^y (y - \xi)\varphi_n(\xi \mid s)\, d\xi + p \int_y^\infty (\xi - y)\varphi_n(\xi \mid s)\, d\xi; & y > 0 \\ p \int_0^\infty (\xi - y)\varphi_n(\xi \mid s)\, d\xi; & y < 0. \end{cases}$$

If demand in the nth period is ξ, then the stock level at the beginning of the $(n + 1)$st period is $y - \xi$, and the statistic s which represents average demand in the preceding periods becomes $((n - 1)s + \xi)/n = s + ((\xi - s)/n)$. Therefore the discounted expected cost from the $(n + 1)$st period onward, if an optimal policy is followed in those periods is $C_{n+1}(y - \xi, s + ((\xi - s)/n))$. We therefore obtain the following functional equation

$$(20) \quad \begin{aligned} C_n(x, s) = \inf_{y \geq x} \Big\{ & c(y - x) + L_n(y \mid s) \\ & + \alpha \int_0^\infty C_{n+1}\left(y - \xi, s + \frac{(\xi - s)}{n}\right) \varphi_n(\xi \mid s)\, d\xi \Big\}. \end{aligned}$$

It is worth remarking that this heuristic derivation may be made quite rigorous by considering the abstract decision spaces [6].

In order to analyze equation (20) we first define $C_n^N(x, s)$ to be optimal discounted expected costs from time period n onwards if the inventory problem is engaged in for a total of N periods. It may be shown that

$$(21) \quad \begin{aligned} C_n^N(x, s) = \inf_{y \geq x} \Big\{ & c(y - x) + L_n(y \mid s) \\ & + \alpha \int_0^\infty C_{n+1}^N\left(y - \xi, s + \frac{(\xi - s)}{n}\right) \varphi_n(\xi \mid s)\, d\xi \Big\}, \end{aligned}$$

and that $\lim_{N\to\infty} C_n^N(x, s) = C_n(x, s)$. If $n > N$, then $C_n^N(x, s) = 0$.

LEMMA 2. *$C_n^N(x, s)$ has a continuous derivative with respect to x, and is convex with respect to x. The optimal policies are defined by single critical numbers $\bar{x}_n^N(s) \geqq 0$. $C_n^N(x, s)$ has a continuous second derivative with respect to x at all points except perhaps $x = \bar{x}_n^N(s)$ at which point both the right and left hand second derivatives exist.*

The lemma is clearly true when $n = N + 1$. Let us assume that it is true for $n + 1$ and show that it remains true for n. The function

$$(22) \quad cy + L_n(y \mid s) + \alpha \int_0^\infty C_{n+1}^N\left(y - \xi, s + \frac{(\xi - s)}{n}\right) \phi_n(\xi \mid s)\, d\xi$$

is a differentiable convex function. For $y \to +\infty$ this function becomes positively infinite. To see this we notice that this function is larger than

$$cy + L_n(y \mid s)$$

and for $y > 0$ this latter function has a derivative equal to $c - p + (h + p) \int_0^y \varphi_n(\xi \mid s)\, d\xi$, which is certainly positive for $y \to +\infty$. For $y < 0$, the stock level $y - \xi < \bar{x}_{n+1}^N(s + ((\xi - s)/n))$ (by the assumption that all of the critical numbers are positive), and therefore

$$C_{n+1}^N\left(y - \xi, s + \frac{(\xi - s)}{n}\right) = c\left[\bar{x}_{n+1}^N\left(s + \frac{(\xi - s)}{n}\right) - y + \xi\right] + C_{n+1}^N\left(\bar{x}_{n+1}^N\left(s + \frac{(\xi - s)}{n}\right), s + \frac{(\xi - s)}{n}\right).$$

It follows that $(\partial/\partial y)C_{n+1}^N(y - \xi,\ s + ((\xi - s)/n)) = -c$ for $y < 0$, and therefore the derivative of (22) with respect to y is $c(1 - \alpha) - p < 0$. It follows that the minimum of (22) occurs at a point $\bar{x}_n^N(s) > 0$. From the convexity of (22) and equation (21) it follows that $\bar{x}_n^N(s)$ is indeed the critical number. In order to show the convexity of $C_n^N(x, s)$ we notice first of all that for $x < \bar{x}_n^N(s)$, the function $C_n^N(x, s)$ is linear in x. For $y > \bar{x}_n^N(s)$, $C_n^N(y, s) = (22)$ with the term cy replaced by zero, and is therefore convex.

The only problem in showing the differentiability of $C_n^N(x, s)$ with respect to x, is in verifying that the right and left hand derivatives are the same at the point $\bar{x}_n^N(s)$, and this follows from the fact that the derivative of (22) equals zero when $y = \bar{x}_n^N(s)$.

It follows from this lemma that $C_n(x, s)$ is a convex function with respect to x, and also that its derivative with respect to x is equal to $-c$, when $x < 0$. A repetition of the above arguments yields the following result.

THEOREM 1. *There is a sequence of non-negative functions $\bar{x}_n(s)$ so that the optimal ordering rule is to purchase* Min $(\bar{x}_n(s) - x, 0)$. *In fact,* $\bar{x}_n(s) = \lim_{N\to\infty} \bar{x}_n^N(s)$.

In the course of proving this theorem, the critical number $\bar{x}_n(s)$ will be found to be the *unique* minimum of

$$(23) \qquad cy + L_n(y \mid s) + \alpha \int_0^\infty C_{n+1}\left(y - \xi, s + \frac{(\xi - s)}{n}\right) \phi_n(\xi \mid s)\, d\xi.$$

The uniqueness of the minimum is a consequence of the fact that (23) has strictly increasing first differences, which is itself implied by $L_n''(y \mid s) = (h + p)\, \varphi_n(y \mid s) > 0$.

Unlike the case in which the demand distribution is known precisely there is no simple equation along the lines of (1) which determines the critical numbers $\bar{x}_n(s)$. Certain properties of the critical functions may, however, be derived from a direct examination of equation (20). In the following argument we shall show that $\bar{x}_n(s)$ is increasing in s, a fact which is interesting in its own right and necessary for the considerations of Section 4.

LEMMA 3. *$\partial C_n^N(x, s)/\partial x$ is decreasing in s.*

The proof of this lemma is by backwards induction on the subscript n. The proof uses, in a crucial way, a property of densities with a monotone likelihood ratio [7]. A probability density $p(x \mid \omega)$ is said to have a monotone likelihood ratio if

$$\det \begin{vmatrix} p(x_1 \mid \omega_1) & p(x_1 \mid \omega_2) \\ p(x_2 \mid \omega_1) & p(x_2 \mid \omega_2) \end{vmatrix} \geqq 0, \quad \text{for } x_1 > x_2 \text{ and } \omega_1 > \omega_2 . \tag{24}$$

We shall use the fact, demonstrated in [7], that if $p(x \mid \omega)$ has a monotone likelihood ratio, then $h(\omega) = \int p(x \mid \omega) g(x)\, dx$ is a monotone decreasing function of ω if $g(x)$ is a monotone decreasing function of x.

The particular densities that we shall be interested in are the densities

$$\varphi_n(\xi \mid s) = r(\xi) \frac{\int_0^\infty \beta^n(\omega)\, e^{-\omega\xi} e^{-(n-1)\omega s} f(\omega)\, d\omega}{\int_0^\infty \beta^{n-1}(\omega)\, e^{-(n-1)\omega s} f(\omega)\, d\omega}$$

defined in equation (18).

If the determinant (24) is formed, using this density, it will be seen to have the same sign as the determinant

$$\begin{vmatrix} \int_0^\infty \beta^n e^{-\omega\xi_1} e^{-(n-1)\omega s_1} f(\omega)\, d\omega & \int_0^\infty \beta^n e^{-\omega\xi_1} e^{-(n-1)\omega s_2} f(\omega)\, d\omega \\ \int_0^\infty \beta^n e^{-\omega\xi_2} e^{-(n-1)\omega s_1} f(\omega)\, d\omega & \int_0^\infty \beta^n e^{-\omega\xi_2} e^{-(n-1)\omega s_2} f(\omega)\, d\omega \end{vmatrix}$$

and this may be seen to be positive, by virtue of the fact that

$$\log \int_0^\infty e^{-\omega x} g(\omega)\, d\omega$$

is a convex function whenever g is positive (in this case we take

$$g(\omega) = \beta^n(\omega) f(\omega) e^{-\omega\xi_2} e^{-(n-1)\omega s_2}).$$

We now proceed with the induction proof of Lemma 3. Let us assume that $\partial C_{n+1}^N(x, s)/\partial s$ is decreasing in s. From Lemma 2, we see that

$$\frac{\partial C_n^N(x, s)}{\partial x} = \begin{cases} \dfrac{\partial L_n(x \mid s)}{\partial x} + \alpha \displaystyle\int_0^\infty \frac{\partial C_{n+1}^N}{\partial x}\left(x - \xi, s + \frac{(\xi - s)}{n}\right) \varphi_n(\xi \mid s)\, d\xi; & x \geqq \bar{x}_n^N(s), \\ -c; & x \leqq \bar{x}_n^N(s). \end{cases} \tag{25}$$

Let us first of all show that

$$G(x, s) = \frac{\partial L_n(x \mid s)}{\partial x} + \alpha \int_0^\infty \frac{\partial C_{n+1}^N}{\partial x}\left(x - \xi, s + \frac{(\xi - s)}{n}\right) \varphi_n(\xi \mid s)\, d\xi \tag{26}$$

is decreasing in s. Since $\partial L_n(x, s)/\partial x$ is equal to $-p + (h + p) \int_0^x \varphi_n(\xi \mid s)\, d\xi$, we may rewrite the expression for $G(x, s)$ as

$$G(x, s) = \tag{27}$$
$$-p + \int_0^\infty \left\{ \alpha \frac{\partial C_{n+1}^N\left(x - \xi, s + \frac{(\xi - s)}{n}\right)}{\partial x} + (h + p)\chi_x(\xi) \right\} \varphi_n(\xi \mid s)\, d\xi,$$

where $\chi_x(\xi)$ is the characteristic function of $(0, x)$. If $s_1 < s_2$, then by the induction assumption, we have

$$G(x, s_1) \geqq \tag{28}$$
$$-p + \int_0^\infty \left\{ \alpha \frac{C_{n+1}^N\left(x - \xi, s_2 + \frac{(\xi - s_2)}{n}\right)}{\partial x} + (h + p)\chi_x(\xi) \right\} \varphi_n(\xi \mid s_1)\, d\xi.$$

The integrand in this expression, is a monotone decreasing function of ξ, by virtue of the convexity of C_{n+1}^N with respect to its first argument, and because of the induction assumption. Using the property of distributions with a monotone likelihood ratio we conclude that the integral is larger than the corresponding integral with s_1 replaced by s_2. This demonstrates that $G(x, s_1) \geqq G(x, s_2)$.

The critical number $\bar{x}_n^N(s)$ is the solution of the equation $-c = G(x, s)$. This permits us to conclude that $\bar{x}_n^N(s_1) \leqq \bar{x}_n^N(s_2)$. To see this we write

$$G_n^N(\bar{x}_n^N(s_2), s_2) = -c = G_n^N(\bar{x}_n^N(s_1), s_1) \geqq G_n^N(\bar{x}_n^N(s_1), s_2),$$

and recall that G is an increasing function of its first argument (see the proof of Lemma 2).

In order to complete the proof of this step of the induction, we consider three cases.

(a) $x \leqq \bar{x}_n^N(s_1)$. In this case

$$\frac{\partial C_n^N(x, s_1)}{\partial x} - \frac{\partial C_n^N(x, s_2)}{\partial x} = 0.$$

(b) $\bar{x}_n^N(s_1) \leqq x \leqq \bar{x}_n^N(s_2)$. In this case

$$\frac{\partial C_n^N(x, s_1)}{\partial x} - \frac{\partial C_n^N(x, s_2)}{\partial x} = G(x, s_1) + c \geqq G(\bar{x}_n^N(s_1), s_1) + c = 0.$$

(c) $\bar{x}_n^N(s_2) \leqq x$. In this case

$$\frac{\partial C_n^N(x, s_1)}{\partial x} - \frac{\partial C_n^N(x, s_2)}{\partial x} = G(x, s_1) - G(x, s_2) \geqq 0.$$

This completes the proof of Lemma 3.

As a consequence of Lemma 3, we see that for $b > a$, $C_n^N(b, s) - C_n^N(a, s)$ is decreasing in s, and passing to the limit we obtain $C_n(b, s) - C_n(a, s)$ is decreasing in s. By a slight modification of the arguments of Lemma 3, we obtain

THEOREM 2. *The critical stockage numbers $\bar{x}_n(s)$ are increasing functions of the statistic s. The critical numbers $\bar{x}_n^N(s)$ are also increasing functions of s.*

The next question that we shall examine is the relationship between the critical numbers in successive time periods. Suppose that at the beginning of the nth period the stock level x is less than the critical level $\bar{x}_n(s)$, where s represents the average demand in the preceding $n - 1$ periods. The optimal rule is to bring the stock level up to $\bar{x}_n(s)$. If demand during the nth period is ξ, then the stock level at the beginning of the $(n + 1)$st period is $\bar{x}_n(s) - \xi$, and the average demand is $s + ((\xi - s)/n)$. We shall address ourselves to the question of whether the optimal rule calls for positive ordering in the $(n + 1)$st period if the demand ξ is small. To say that there will be no positive ordering is equivalent to saying that $\bar{x}_n(s) - \xi \geqq \bar{x}_{n+1}(s + ((\xi - s)/n))$ for all small ξ, and in view of the conclusion of Theorem 2, this is equivalent to saying that

$$\bar{x}_n(s) > \bar{x}_{n+1}(s(n - 1)/n).$$

This result is actually true.

THEOREM 3. *The critical levels have the property that* $\bar{x}_n(s) > \bar{x}_{n+1}(s(n - 1)/n)$, *or in other words, we do not order in any period if the demand has been sufficiently small during the previous period.*

Let us assume to the contrary that there exist values of n and s, so that

$$\bar{x}_n(s) \leqq \bar{x}_{n+1}\left(\frac{n-1}{n}\, s\right). \tag{29}$$

We know that (see (23)) $\bar{x}_n(s)$ is the unique minimum point of

$$cy + L_n(y \mid s) + \alpha \int_0^\infty C_{n+1}\left(y - \xi, s + \frac{(\xi - s)}{n}\right) \varphi_n(\xi \mid s)\, d\xi . \tag{30}$$

For $y < \bar{x}_n(s)$ we will have, by assumption, $y - \xi < \bar{x}_{n+1}(s + ((\xi - s)/n))$ and therefore $C_{n+1}(y - \xi, s + ((\xi - s)/n))$ will be a linear function in y whose derivative is equal to $-c$. Therefore the derivative of (30) for any value of $y < \bar{x}_n(s)$, is equal to $c(1 - \alpha) + \partial L_n(y \mid s)/\partial y$. It may also be shown that both the right and left hand derivatives of (30) exist and are equal at the point $\bar{x}_n(s)$, and we conclude that

$$c(1 - \alpha) + \frac{\partial L_n(y \mid s)}{\partial y} = 0, \qquad \text{when } y = \bar{x}_n(s) . \tag{31}$$

Now let us examine the equation which gives $\bar{x}_{n+1}(s(n - 1)/n)$. Similarly as with (30) we know that $\bar{x}_{n+1}(s(n - 1)/n)$ minimizes

$$\begin{aligned} cy + L_{n+1}\left(y \,\middle|\, \frac{n-1}{n}\, s\right) + \alpha \int_0^\infty C_{n+2}\left(y - \xi, \frac{(n-1)s}{n+1} + \frac{\xi}{n+1}\right) \\ \varphi_{n+1}\left(\xi \,\middle|\, \frac{n-1}{n}\, s\right) d\xi . \end{aligned} \tag{32}$$

If we evaluate this function at $y = \bar{x}_{n+1}(s(n - 1)/n)$ subtract the value for $y = \bar{x}_{n+1}(s(n - 1)/n) - \delta$, divide by δ, and use the following consequence of the convexity of $C_{n+2} : C_{n+2}(y - \xi, .) - C_{n+2}(y - \delta - \xi, .) \geqq -c\delta$, then we obtain the conclusion

$$0 \geqq c(1-\alpha) + \frac{\partial L_{n+1}\left(\bar{x}_{n-1}\left(\frac{n-1}{n}s\right), \frac{n-1}{n}s\right)}{\partial x}. \tag{33}$$

Since the right hand side of this equation is an increasing function of its first argument, it follows that $\bar{x}_{n+1}(s(n-1)/n)$ is less than the solution of the equation

$$0 = c(1-\alpha) + \frac{\partial L_{n+1}\left(y, \frac{n-1}{n}s\right)}{\partial y}, \tag{34}$$

and therefore by (29) it follows that $\bar{x}_n(s)$ is less than the solution of (34).

We shall show that this last statement is false, thereby proving the theorem. In other words, we shall show that the solution of (31) is always strictly greater than the corresponding solution of (34). In view of the specific nature of the functions involved this will be correct if we can show that

$$\frac{\partial L_{n+1}\left(y \,\middle|\, \frac{n-1}{n}s\right)}{\partial y} > \frac{\partial L_n(y \mid s)}{\partial y} \qquad \text{or, see (19)},$$

$$\int_0^y \varphi_{n+1}\left(\xi \,\middle|\, \frac{n-1}{n}s\right) d\xi > \int_0^y \varphi_n(\xi \mid s)\, d\xi, \qquad \text{for } y > 0. \tag{35}$$

The proof of (35) is quite simple and again makes use of the monotonicity preserving property of densities with a monotone likelihood ratio. Since the characteristic function of the interval $(0, y)$ is a monotone decreasing function of ξ, it follows that

$$\int_0^y \varphi_{n+1}\left(\xi \,\middle|\, \frac{n-1}{n}s\right) d\xi = \frac{\int_0^\infty \beta^{n+1} e^{-(n-1)\omega s}\left[\int_0^y e^{-\xi\omega} r(\xi)\, d\xi\right] f(\omega)\, d\omega}{\int_0^\infty \beta^n e^{-(n-1)\omega s} f(\omega)\, d\omega}$$

is a monotone decreasing function of s, and therefore

$$\int_0^y \varphi_{n+1}\left(\xi \,\middle|\, \frac{n-1}{n}s\right) d\xi \geqq \int_0^y \varphi_{n+1}\left(\xi \,\middle|\, \frac{n-1}{n}s + \frac{\tau}{n-1}\right) d\xi$$

$$= \frac{\int_0^\infty \beta^{n+1} e^{-(n-1)\omega s} e^{-\tau\omega}\left[\int_0^y e^{-\xi\omega} r(\xi)\, d\xi\right] f(\omega)\, d\omega}{\int_0^\infty \beta^n e^{-(n-1)\omega s} e^{-\omega\tau} f(\omega)\, d\omega}$$

or

$$\begin{aligned} &\int_0^y \varphi_{n+1}\left(\xi \,\middle|\, \frac{n-1}{n}s\right) d\xi \cdot \int_0^\infty \beta^n e^{-(n-1)\omega s} e^{-\omega\tau} f(\omega)\, d\omega \\ &\qquad \geqq \int_0^\infty \beta^{n+1} e^{-(n-1)\omega s} e^{-\tau\omega}\left[\int_0^y e^{-\xi\omega} r(\xi)\, d\xi\right] f(\omega)\, d\omega, \end{aligned} \tag{36}$$

with strict inequality holding for some values of τ. If we multiply both sides of (36) by $r(\tau)$, integrate with respect to τ, from zero to infinity, and remember that $\int_0^\infty e^{-\omega\tau} r(\tau)\, d\tau = 1/\beta(\omega)$, we obtain

$$\int_0^y \varphi_{n+1}\left(\xi \,\middle|\, \frac{n-1}{n} s\right) d\xi \cdot \int_0^\infty \beta^{n-1} e^{-(n-1)\omega s} f(\omega)\, d\omega \tag{37}$$
$$> \int_0^\infty \beta^n e^{-(n-1)\omega s} \left[\int_0^y e^{-\xi\omega} r(\xi)\, d\xi \right] f(\omega)\, d\omega ,$$

which is the same as (35). This completes the proof of Theorem 3.

4. Asymptotic properties of the critical numbers. The sequence of past demands may be used to estimate the true demand distribution with increasing accuracy as the number of time periods becomes large. It seems reasonable to expect, therefore, that the critical number $\bar{x}_n(s)$ will, for large n, be quite close to the critical number $\bar{x}(s)$ obtained by assuming that the mean demand is s. We shall, in this section, demonstrate that $\bar{x}_n(s)$ does indeed tend to $\bar{x}(s)$, and then establish an asymptotic expansion of the form $\bar{x}_n(s) \sim \bar{x}(s) + (a(s)/n)$, with $a(s)$ given by (7). The proof of this fact will depend on obtaining certain asymptotic expansions for the a posteriori density of ω, given that the demand in the first $n - 1$ periods has a mean of s.

We shall summarize the necessary asymptotic results in the following two lemmas, without proof. In each case, the proof is a relatively straight forward application of Laplace's method for the evaluation of exponential integrals [5], [8]. The a posteriori density of w, given that $\xi_1 + \cdots + \xi_{n-1} = (n-1)s$, is

$$\frac{\beta^{n-1}(\omega) e^{-(n-1)\omega s} f(\omega)}{\int_0^\infty \beta^{n-1}(\omega) e^{-(n-1)\omega s} f(\omega)\, d\omega} = f_n(\omega \mid s) , \tag{38}$$

and the maximum likelihood estimate of ω is given by the solution of the equation

$$\frac{d \log \beta(\omega)}{d\omega} = s , \tag{39}$$

which we shall also write as $\omega(s)$. We shall only consider those values of s for which $0 < s < \lim_{\omega \to 0} (\beta'(\omega)/\beta(\omega))$.

LEMMA 4. *$\varphi_n(\xi \mid s) \to \varphi(\xi \mid \omega(s))$ uniformly in every finite ξ interval, if*

$$f(\omega(s)) > 0.$$

The proof of this fact may be obtained by minor modifications of the argument presented on p. 283 of [8].

LEMMA 5. *Let $f(\omega)$ and $h(\omega)$ have continuous 2nd derivatives in some neighborhood of the point $\omega(s)$, and let $f(\omega) > 0$. Then*

$$\int_0^\infty h(\omega) f_n(\omega \mid s)\, d\omega \sim h(\omega) + \frac{1}{n} \frac{\frac{d}{d\omega}\left(\frac{f^2}{\sigma^2} h'\right)}{2f^2} ,$$

where $\sigma^2 = -d^2 \log\beta/d\omega^2$, *and the coefficient of* $1/n$ *is evaluated at the point* $\omega(s)$. *If only the first term is required, then continuity of* h *and* f *are sufficient.*

This lemma may be demonstrated by an application of Laplace's method [5].

Let us apply these results to verify that $\bar{x}_n(s) \to \bar{x}(s)$. The proof depends on the following simple observations. Let us consider an inventory problem in which the true value of ω is revealed after $n - 1$ demands are experienced. In this situation the expectation of the discounted costs from the nth period onwards is

$$\int_0^\infty C(x, \omega) f_n(\omega \mid s)\, d\omega, \tag{40}$$

where $C(x, \omega)$ is the cost for the sequential inventory problem, when the true value of the parameter is known to be ω. It is possible, however, to use in this situation the strategy defined by the critical numbers $\bar{x}_n(s)$ which are optimal in the statistical problem. If this strategy is used it disregards the information which is made available as to the true value of ω, and consequently the expected cost incurred by the use of such a strategy will be greater than (40). This cost is, of course, $C_n(x, s)$, and therefore $\int_0^\infty C(x, \omega) f_n(\omega \mid s)\, d\omega \leqq C_n(x, s)$.

On the other hand the following situation may be considered. Let us assume that after $(n - 1)$ demands have occurred, no additional demand information will be made available to the decision maker. In this case the minimum expected cost from the nth period onward will be the same as the non-statistical problem with a demand distribution given by $\varphi_n(\xi \mid s)$. If we denote this cost by $C^n(x \mid s)$, then clearly

$$\int_0^\infty C(x, \omega) f_n(\omega \mid s)\, d\omega \leqq C_n\,(x, s) \leqq C^n(x, s)\,. \tag{41}$$

By Lemma 5, we see that $\int_0^\infty C(x, \omega) f_n(\omega \mid s)\, d\omega$ tends to $C(x, \omega(s))$. In order to appy Lemma 1, we must verify that $\varphi_n(\xi \mid s) \to \varphi(\xi \mid \omega(s))$ uniformly in every interval, and that the means converge. The first part of this statement is implied by Lemma 4, and the second part by an application of Lemma 5 to the function $h(\omega) = \beta(\omega) \int_0^\infty \xi e^{-\omega\xi} r(\xi)\, d\xi$. We have therefore demonstrated

LEMMA 6. *If* $f(\omega(s)) > 0$, *then* $C_n(x, s) \to C(x, \omega(s))$.

It is a simple consequence of this lemma that $\bar{x}_n(s) \to \bar{x}(s)$.

We have therefore established

THEOREM 4. *Under the assumption that* $f(\omega(s)) > 0$, *we have* $\bar{x}_n(s) \to \bar{x}(s)$, $\bar{x}(s)$ *being given by* (6).

We now address ourselves to the problem of showing that the asymptotic expansion for $\bar{x}_n(s)$ given by (7) is indeed valid. Our first result is embodied in the following lemma.

LEMMA 7. *For any fixed value of* x,

$$0 \geqq L_n'(\bar{x}_n(s) \mid s) + c(1 - \alpha) = o\left(\bar{x}_n(s) - \bar{x}_{n+1}\left(\frac{n-1}{n}\, s\right)\right)\,, \text{as } n \to \infty\,.$$

In order to demonstrate this lemma, we consider the cost function $C_n^N(x, s)$. According to Lemma 2, and its proof, we have

$$(42)\quad \begin{aligned} -c = L_n'(\bar{x}_n^N(s) \mid s) - \alpha c + \\ \alpha \int_0^\infty \left[\frac{\partial}{\partial x} C_{n+1}^N\left(\bar{x}_n^N(s) - \xi, s + \frac{\xi - s}{n}\right) + c\right] \varphi_n(\xi \mid s)\, d\xi. \end{aligned}$$

The integrand on the right hand side vanishes for all ξ with

$$\bar{x}_n^N(s) - \xi < \bar{x}_{n+1}^N(s + ((\xi - s)/n)),$$

and is positive for other values of ξ. For $N \to \infty$, we know that $\bar{x}_n^N(s) \to \bar{x}_n(s)$, and if $\xi > \bar{x}_n(s) - \bar{x}_{n+1}(s(n-1)/n)$, then $\bar{x}_n(s) - \xi < \bar{x}_{n+1}(s + ((\xi - s)/n))$. (This follows from Lemma 3.) Therefore for N large, the integrand vanishes for $\xi > \bar{x}_n^N(s) - \bar{x}_{n+1}^N(s(n-1)/n) + \delta_N$, where $\delta_N \to 0$ as N tends to infinity. Therefore, since the integrand is a decreasing function of ξ (Lemmas 2 and 3), we may write

$$(43)\quad \begin{aligned} 0 &\leqq \int_0^\infty \left[\frac{\partial}{\partial x} C_{n+1}^N\left(\bar{x}_n^N(s) - \xi, s + \frac{\xi - s}{n}\right) + c\right] \varphi_n(\xi \mid s)\, d\xi \\ &\leqq \Phi_n\left(\bar{x}_n^N(s) - \bar{x}_{n+1}^N\left(\frac{n-1}{n} s\right) + \delta_N \mid s\right) \\ &\qquad \cdot \left[\frac{\partial}{\partial x} C_{n+1}^N\left(\bar{x}_n^N(s), \frac{n-1}{n} s\right) + c\right]. \end{aligned}$$

If ϵ is any positive number then

$$(44)\quad \begin{aligned} &\frac{\partial}{\partial x} C_{n+1}^N\left(\bar{x}_n^N(s), \frac{n-1}{n} s\right) + c \\ &\leqq \frac{1}{\epsilon}\left\{C_{n+1}^N\left(\bar{x}_n^N(s) + \epsilon, \frac{n-1}{n} s\right) - C_{n+1}^N\left(\bar{x}_n^N(s), \frac{n-1}{n} s\right)\right\} + c. \end{aligned}$$

If we substitute this in (42) and let $N \to \infty$, we obtain $0 \geqq L_n'(\bar{x}_n(s) \mid s) + c(1 - \alpha) \geqq -\Phi_n(\bar{x}_n(s) - \bar{x}_{n+1}(s(n-1)/n) \mid s)$ multiplied by

$$\frac{1}{\epsilon}\left\{C_{n+1}\left(\bar{x}_n(s) + \epsilon, \frac{n-1}{n} s\right) - C_{n+1}\left(\bar{x}_n(s), \frac{n-1}{n} s\right)\right\} + c.$$

For n large $\Phi_n(\mu \mid s) \leqq K\mu$ for s away from 0 and Max $(\beta'/\beta)(\omega)$. Also $(1/\epsilon)$ $\{\cdots\} + c$ approaches $(1/\epsilon)\{C(\bar{x}(s) + \epsilon, \omega(s)) - C(\bar{x}(s), \omega(s))\} + c$, which may be made arbitrarily small. This proves the lemma.

Lemma 7 is not sufficient to demonstrate the main result of this section, except under certain cases. Let us suppose that there exists a subsequence of $\{\bar{x}_{n+1}(s(n-1)/n)\}$ which is greater than or equal to $\bar{x}(s)$. For this sequence $\bar{x}_n(s) - \bar{x}_{n+1}(s(n-1)/n) \leqq \bar{x}_n(s) - \bar{x}(s)$, and Lemma 7 may be rephrased as $L_n'(\bar{x}_n(s) \mid s) + c(1 - \alpha) = o\{\bar{x}_n(s) - \bar{x}(s)\}$. Then

$$L_n'(\bar{x}(s) \mid s) + c(1 - \alpha) + [\bar{x}_n(s) - \bar{x}(s)]L_n''(\bar{x}(s) + \theta_n \mid s) = o\{\bar{x}_n(s) - \bar{x}(s)\},$$

where $\theta_n \to 0$. But

$$L'_n(\bar{x}(s) \mid s) = -p + (h + p) \int_0^{\bar{x}(s)} \varphi_n(\xi \mid s) \, d\xi$$

$$\sim -p + (h + p)\Phi(\bar{x}(s) \mid \omega(s)) + \frac{(h + p)}{n} \frac{\frac{\partial}{\partial \omega}\left(\frac{f^2}{\sigma^2} \frac{\partial \Phi}{\partial \omega}\right)}{2f^2}$$

by Lemma 5, and $L''_n(\bar{x}(s) + \theta_n \mid s) \to (h + p)\varphi(\bar{x}(s) \mid \omega(s))$. Since

$$c(1 - \alpha) - p + (h + p)\Phi(\bar{x}(s) \mid \omega(s)) = 0$$

(equation 6) this demonstrates the asymptotic expansion for $\bar{x}_n(s)$, given in (7).

In order to demonstrate the asymptotic expansion for $\bar{x}_n(s)$, in general, we shall find it necessary to show that these functions satisfy a Lipschitz condition of order 1. This is dependent upon the following lemma.

Lemma 8. *Let s be any point between zero and* $\text{Max}_\omega (\beta'/\beta)(\omega)$. *Then for n sufficiently large $\partial^2 C_n^N(x, s)/\partial x \partial s$ exists except perhaps at the point $x = \bar{x}_n^N(s)$ and is bounded from below, independently of N.*

The existence of $\partial^2 C_n^N(x, s)/\partial x \partial s$ follows directly from its definition. From equation (25) we see that

$$(45) \quad \frac{\partial^2 C_n^N}{\partial x \partial s} = \begin{cases} 0; & x \leqq \bar{x}_n^N(s) \\ \dfrac{\partial}{\partial s}\left[\dfrac{\partial L_n}{\partial x} + \alpha \displaystyle\int_0^\infty \dfrac{\partial C_{n+1}^N}{\partial x}\left(x - \xi, s + \dfrac{\xi - s}{n}\right)\varphi_n(\xi \mid s) \, d\xi\right]; & \\ & x > \bar{x}_n^N(s). \end{cases}$$

The function

$$\frac{\partial L_n}{\partial x} + \alpha \int_0^\infty \frac{\partial C_{n+1}^N}{\partial x}\left(x - \xi, s + \frac{\xi - s}{n}\right)\varphi_n(\xi \mid s) \, d\xi$$

is decreasing in s, so that its derivative with respect to s is negative, for all values of x. We may therefore write

$$(46) \quad \frac{\partial^2 C_n^N}{\partial x \partial s} \geqq \frac{\partial}{\partial s}\left[\frac{\partial L_n}{\partial x} + \alpha \int_0^\infty \frac{\partial C_{n+1}^N}{\partial x}\left(x - \xi, s + \frac{\xi - s}{n}\right)\varphi_n(\xi \mid s) \, d\xi\right],$$

for all x. Now let us assume that we can find a sequence of functions $F_n(x \mid s)$ which satisfy the equations

$$(47) \quad \frac{\partial F_n}{\partial s} = \frac{\partial}{\partial s}\left[\frac{\partial L_n}{\partial x} + \alpha \int_0^\infty F_{n+1}\left(x - \xi, s + \frac{\xi - s}{n}\right)\varphi_n(\xi \mid s) \, d\xi\right],$$

and are increasing in x and decreasing in s. Then it is a simple matter to show that $\partial^2 C_n^N/\partial x \partial s \geqq \partial F_n/\partial s$, and this will furnish us with the required bounds. It will be sufficient to obtain a solution to the equation

$$(48) \quad F_n(x, s) = (h + p)\Phi_n(x \mid s) + \alpha \int_0^\infty F_{n+1}\left(x - \xi, s + \frac{\xi - s}{n}\right)\varphi_n(\xi \mid s) \, d\xi,$$

$$\text{for} \quad x > 0.$$

This equation may be shown to have the following solution. Define $\Phi^{(k)}(x \mid \omega)$ to be the the k-fold convolution of the demand distribution, and define

$$M(x, \omega) = (h + p) \sum_{k=0}^{\infty} \alpha^k \Phi^{(k+1)}(x \mid \omega). \tag{49}$$

Then

$$F_n(x \mid s) = \frac{\int_0^\infty (\beta e^{-\omega s})^{n-1} M(x \mid \omega) f(\omega)\, d\omega}{\int_0^\infty (\beta e^{-\omega s})^{n-1} f(\omega)\, d\omega}, \qquad \text{for} \quad x \geqq 0, \tag{50}$$

and equal to zero for x negative. This function may be substituted directly in (48) to demonstrate that it is indeed a solution. It may also be verified directly that $F_n(x \mid s)$ is increasing in x and decreasing in s.

In order to demonstrate Lemma 8 it is sufficient to show that $\partial F_n(x \mid s)/\partial s$ is bounded from below as n becomes infinite. Now

$$\lim_{n\to\infty} \frac{\partial F_n}{\partial s} = \lim_{n\to\infty} \left\{ \frac{(n-1) \int_0^\infty \beta^{n-1} e^{-\omega s(n-1)} \omega M(x \mid \omega) f(\omega)\, d\omega}{\int_0^\infty (\beta e^{-\omega s})^{n-1} f(\omega)\, d\omega} \right.$$
$$\left. - \frac{(n-1) \int_0^\infty (\beta e^{-\omega s})^{n-1} M(x \mid \omega) f(\omega)\, d\omega}{\int_0^\infty (\beta e^{-\omega s})^{n-1} f(\omega)\, d\omega} \frac{\int_0^\infty (\beta e^{-\omega s})^{n-1} \omega f(\omega)\, d\omega}{\int_0^\infty (\beta e^{-\omega s})^{n-1} f(\omega)\, d\omega} \right\}.$$

In this form we may apply Lemma 5 directly, and conclude that

$$\frac{\partial F_n}{\partial s} \sim (n-1)\left[\omega(s) M(x \mid \omega(s)) + O\left(\frac{1}{n}\right)\right]$$
$$- (n-1)\left[M(x \mid \omega(s) + O\left(\frac{1}{n}\right)\right]\left[\omega(s) + O\left(\frac{1}{n}\right)\right]$$
$$\sim O(1).$$

The application of Lemma 5 merely requires that $M(x \mid \omega)$ have a continuous second partial derivative with respect to ω. But by direct calculations

$$\frac{\partial^2 \Phi(x \mid \omega)}{\partial \omega^2} = \beta(\omega)\left\{-\sigma^2 \int_0^x e^{-\omega\xi} r(\xi)\, d\xi + \int_0^x (\xi - m)^2 e^{-\omega\xi} r(\xi)\, d\xi\right\},$$

$$\left|\frac{\partial^2 \Phi(x \mid \omega)}{\partial \omega^2}\right| \leqq 2\sigma^2(\omega), \qquad \text{and similarly}$$

$$\left|\frac{\partial^2 \Phi^{k+1}(x \mid \omega)}{\partial \omega^2}\right| \leqq 2(k+1)\sigma^2(\omega).$$

This proves Lemma 8.

LEMMA 9. *For any s and δ sufficiently small there exists a constant k such that* $\bar{x}_n(s + \delta) - \bar{x}_n(s) \leqq k\delta$.

Again we consider the cost functions C_n^N. We have

$$-c = \frac{\partial C_n^N}{\partial x} (\bar{x}_n^N(s), s)$$

$$= \frac{\partial C_n^N}{\partial x} (\bar{x}_n^N(s + \delta), s) + [\bar{x}_n^N(s) - \bar{x}_n^N(s + \delta)] \frac{\partial^2}{\partial x^2} C_n^N(y', s),$$

where y' is some intermediary point. Again this is equal to

$$\frac{\partial C_n^N}{\partial x} (\bar{x}_n^N(s + \delta), s + \delta) - \delta \frac{\partial^2 C_n^N}{\partial x \partial s} (\bar{x}_n^N(s + \delta), s') + [\bar{x}_n^N(s) - \bar{x}_n^N(s + \delta)] \frac{\partial^2}{\partial x^2} C_n^N(y', s).$$

In other words

$$[\bar{x}_n^N(s + \delta) - \bar{x}_n^N(s)] \frac{\partial^2 C_n^N}{\partial x^2} (y', s) = -\delta \frac{\partial^2 C_n^N}{\partial x \partial s} (\bar{x}_n^N(s + \delta), s').$$

But $\partial^2/\partial x^2 C_n^N(y', s)$ is positive, and is in fact larger than $L_n''(y' \mid s)$ (see (22)). Also $-\partial^2 C_n^N/\partial x \partial s$ is positive, and by Lemma 7 is bounded, first as $N \to \infty$ and then as $n \to \infty$. This proves the lemma.

Now let us give a proof of the general asymptotic expansion of the critical numbers. The method of proof is quite similar to that given above when

$$\bar{x}_{n+1}(s(n - 1)/n) \geqq \bar{x}(s),$$

and depends on showing, by means of Lemma 9, that $\bar{x}_n(s) - \bar{x}(s) = O(1/n)$.

Consider the sequence $\bar{x}_n(s)$ and assume first of all that there exists an infinite sequence of subscripts n_r so that

$$\bar{x}_{n_r}(s) \leqq \bar{x}_{n_r+1}(s), \tag{51}$$

with the reverse inequality holding for the other terms in the sequence. Then

$$0 \leqq \bar{x}_{n_r}(s) - \bar{x}_{n_r+1}\left(\frac{n_r - 1}{n_r} s\right)$$
$$= \bar{x}_{n_r}(s) - \bar{x}_{n_r+1}(s) + \left\{\bar{x}_{n_r+1}(s) - \bar{x}_{n_r+1}\left(\frac{n_r - 1}{n_r} s\right)\right\}$$
$$\leqq \frac{ks}{n_r}.$$

We use the techniques given above, based on Lemma 7, to show that $\bar{x}(s) - \bar{x}_{n_r}(s) = O(1/n_r)$. Now let us consider any subsequence $\bar{x}_n(s)$ all of whose elements are $\leqq \bar{x}(s)$. Then if $n_r \leqq n' \leqq n_{r+1}$ we have $\bar{x}_{n'} > \bar{x}_{n_{r+1}}$ and $\bar{x}(s) - \bar{x}_{n'}(s) < K/n_{r+1} < K/n'$. Therefore for any such subsequence $\bar{x}(s) - \bar{x}_n(s) = O(1/n)$. For any subsequence all of whose terms are $\geqq \bar{x}(s)$, it is a trivial matter to show, using Lemma 7, that $\bar{x}_n(s) - \bar{x}(s) = O(1/n)$.

On the other hand if we cannot find an infinite subsequence of the type described in (51), then we must have $\bar{x}_n(s)$ monotone decreasing, after awhile. Then $\bar{x}_n(s) > \bar{x}(s)$ for n sufficiently large. This is enough to show directly from Lemma 7, that $\bar{x}_n(s) - \bar{x}(s) = O(1/n)$.

We have therefore shown that $\bar{x}_n(s) - \bar{x}(s) = O(1/n)$, regardless of any assumptions as to the form of the sequence $\bar{x}_n(s)$. Then

$$\begin{aligned}\bar{x}_n(s) - \bar{x}_{n+1}\left(\frac{n-1}{n}s\right) &= [\bar{x}_n(s) - \bar{x}(s)] \\ &\quad + [\bar{x}(s) - \bar{x}_{n+1}(s)] + \left[\bar{x}_{n+1}(s) - \bar{x}_{n+1}\left(\frac{n-1}{n}s\right)\right] \\ &= O\left(\frac{1}{n}\right).\end{aligned}$$

If we apply Lemma 7, the general asymptotic expansion is obtained.

REFERENCES

[1] KENNETH J. ARROW, THEODORE E. HARRIS, AND JACOB MARSCHAK, "Optimal inventory policy," *Econometrica*, Vol. 19 (July, 1951), pp. 250–272.

[2] KENNETH J. ARROW, SAMUEL KARLIN, AND HERBERT SCARF, *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press, Stanford, 1958.

[3] ARYEH DVORETSKY, J. KIEFER, AND JACOB WOLFOWITZ, "The inventory problem," *Econometrica*, Vol. 20 (April, July, 1952), pp. 187–222, 450–466.

[4] A. ERDELYI, *Asymptotic Expansions*, Dover, 1956.

[5] DONALD GUTHRIE, JR. AND M. V. JOHNS, JR., "Bayes acceptance sampling procedures for large lots," to be published.

[6] SAMUEL KARLIN, "The structure of dynamic programming problems," *Naval Research Logistics Quarterly*, Vol. 2 (December, 1955), pp. 285–294.

[7] SAMUEL KARLIN, AND HERMAN RUBIN, "The theory of decision procedures for distributions with monotone likelihood ratio," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 272–299.

[8] DAVID VERNON WIDDER, *The Laplace Transform*, Princeton University Press, Princeton, 1941.