AN ANALYSIS OF MARKETS WITH A LARGE NUMBER OF PARTICIPANTS[1]

Herbert Scarf
Stanford University
Stanford, California

1. Introduction

Edgeworth, in 1881 [7], introduced the contract curve as a description of the possible exchanges of two commodities that might arise in a market with a relatively small number of participants. A somewhat loose geometrical argument is presented in Mathematical Psychics for the statement that, as the number of participants becomes infinite, the contract curve becomes smaller and approaches the competitive trade in the limit. The situation discussed is of a rather special nature, involving only two commodities and two types of participants in the market. At the start of Edgeworth's analysis, one participant of each type is considered and a version of the familiar contract curve is obtained. The number of participants of each type is then increased, with additional parts of the contract curve removed from consideration at each step. Roughly speaking, exchanges are ruled out if a group of participants can do better by "recontracting."

The basic concepts of the analysis described above can be thought of as applying to an entire economy, rather than to an isolated market. This would require the simultaneous consideration of many goods and many types of participants. In addition, some generalization would have to be given of the concept of recontracting, and of the contract curve.

Full credit must go to Martin Shubik for the insight that what we are studying here is a problem in n-person game theory. In a very interesting paper [11], Shubik analyzes the Edgeworth problem from the point of view of n-person game theory, employing the von Neumann-Morgenstern concept of solution, and a more appropriate concept -- the core of an n-person game.[2] As we shall see when we examine the core in detail, the trades which are not eliminated by recontracting are precisely those trades in the core.

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[1] I was very fortunate to have participated in a conversation with Lloyd Shapley and Martin Shubik, in which I first became aware of the problems discussed in this paper. Since that time we have had many conversations on this topic and it has become increasingly difficult for me to separate my own ideas from those contributed by Shapley and Shubik. I have also benefitted greatly from talks with Kenneth J. Arrow, Gerard Debreu, Abba Lerner, and Marc Nerlove. Professor Debreu has recently communicated to me an exceptionally elegant and simple proof of the main theorem of this paper, which will be used in place of the current proof when these results are eventually published in a journal.

[2] Reference should also be made to several papers of Shapley [8,9], in which finite market games are analyzed.

One distinction must be made in identifying the concept of "contracting out," with the current formulation of the core in n-person game theory. Most work in n-person game theory has been formulated in terms of transferable utility, which is unfortunately a concept quite foreign to current economic thought. Some recent work, however, has been done on a version of n-person game theory which involves no side payments in utilities [3,4,10], and which includes a definition of the non-transferable core. It is this latter concept which will form the basis of our subsequent discussion.

Let us begin by introducing some formal notation and definitions. We consider a market composed of N individuals $(1, 2, \ldots, N)$, each with a specific set of preferences for commodity bundles consisting of m commodities. We shall denote the typical commodity bundle by the vector $x = (x_1, \ldots, x_m)$ with the x_i being non-negative numbers; the preference orderings of the i^{th} consumer will be denoted by the customary symbol $\underset{i}{\geq}$. The interpretation of $x \underset{i}{\geq} y$ is, of course, that the i^{th} consumer either prefers x to y or is indifferent to the choice. If $x \underset{i}{\geq} y$ and $y \underset{i}{\geq} x$, then the commodity bundles x and y are indifferent. A number of assumptions that are quite familiar will be placed on the various preference orderings. (For a more complete discussion the reader may wish to consult [6].)

1. The preference ordering for each consumer is reflexive, transitive, and complete; i.e., $x \underset{i}{\geq} x$ and if $x \underset{i}{\geq} y$ and $y \underset{i}{\geq} z$, then $x \underset{i}{\geq} z$, and for any two commodity bundles x and y, either $x \underset{i}{\geq} y$ or $y \underset{i}{\geq} x$.

2. The ordering is continuous, i.e., for any y the set of commodity bundles preferred or indifferent to y is a closed set, and similarly for the set of commodity bundles which are indifferent to y or <u>not</u> preferred to y.

3. The preferences are convex, in the sense that for any fixed y, the commodity bundles preferred or indifferent to y, form a convex set.

4. The preferences are monotone. If all of the components of the commodity bundle y are greater than or equal to the corresponding components of x, then $y \underset{i}{\geq} x$. We shall also assume that if <u>all</u> of the components of x are positive, then $x \underset{i}{>} (0, \ldots, 0)$. Also let y be any commodity bundle which is strictly preferred to $(0, \ldots, 0)$. We shall assume that for any $a > 1$, $ay \underset{i}{>} y$.

In our analysis we shall focus our attention on the exchange aspects of the economy, that is to say no production will be considered. The analogue of our main result is correct when production is also included. The economic meaning, however, of coalition formation in the case in which production is included seems to me to be considerably more subtle than in the case of pure trade, and I would like to defer the discussion of this case for a subsequent paper.

In the present discussion consumers will be equipped only with specific initial holdings which they are interested in exchanging for commodity bundles of higher utility. The initial holdings of individual i will be denoted by the vector $I^i = (I_1^i, \ldots, I_m^i)$. It will be convenient to assume that every consumer holds a positive quantity of each item. Occasionally we shall find it useful to refer to the total initial holdings of all of the consumers in a particular set of consumers, and for this we shall use the notation $I(S)$ to indicate the vector obtained by summing the vectors I^i over all members of the set of consumers S . The entire set of consumers will be denoted by X , so that $I(X)$ refers to the total supply available in the market.

We are now in a position to describe what is meant by the core or equivalently, the contract surface. In an informal way, the core may be described as the collection of all allocations of the total market supply which cannot be improved upon by any subgroup of the consumers on the basis of their own initial holdings. Let us be somewhat more formal about this definition. We consider allocations of the total market supply to the various consumers:

$$x^1 + x^2 + \ldots + x^N = I(X) ,$$

with the commodity bundle x^i designated for the i^{th} consumer. Let S be any subset of the total collection of consumers (on the one extreme S may consist of a single consumer, and on the other extreme S may be taken as the entire set of consumers.) We shall say that the allocation $x^1, \ldots, x^N$ is blocked by the set S if there is some way of allocating the total holdings $I(S)$ into commodity bundles y^j , with

$$I(S) = \sum_{j \in S} y^j , \qquad \text{and} \qquad y^j \underset{j}{>} x^j ,$$
$$\text{for all } j \text{ in } S .$$

The set of those allocations which are not blocked by any subset S will be defined to be the core of the market, or the contract surface.

One immediate consequence of the definition is that every allocation in the core is a Pareto optimum allocation. (This is a slightly weaker definition of Pareto optimality than the one customarily given in which strict preference is required for only one individual.) We see this by taking the blocking set S to be the entire set of consumers. On the other hand there will be many allocations which are Pareto optimum and not in the core. An allocation may very well be Pareto optimum and yet assign to an individual consumer a commodity bundle which is worth less to him than his initial holdings. In this event the coalition consisting of this consumer himself would be sufficient to block the allocation. Even more generally, allocations which are Pareto optimum and assign to each consumer a commodity bundle preferred to his initial holding, may still be blocked by a coalition of several consumers.

If the number of participants in the market is large, there will be many coalitions which may possibly be available to block a given allocation. In some sense, the number of allocations in the core should therefore be relatively small. As we shall see in section 3, however, the dependence of the size of the core on the number of participants in the market is by no means simple to analyze.

There are always some allocations in the core -- the competitive allocations. Suppose that at prices $\pi_1, \ldots, \pi_m$, x^j is a commodity bundle which maximizes the preferences of the j^{th} consumer subject to the budget constraint $(\pi, y) \leq (\pi, I^j)$. Suppose in addition that $\Sigma x^j = I(X)$, the sum taken over all consumers in the market. (It may be seen, by adding the budget constraints, that in this case $(\pi, x^j) = (\pi, I^j)$.) Such a competitive allocation can never be blocked by a coalition S , as the following argument, communicated to me by Shapley, will demonstrate. Suppose that it were possible to find y^j such that

$$\sum_{j \in S} y^j = I(S) \quad \text{and} \quad y^j \underset{j}{>} x^j \, , \quad \text{for all } j \text{ in } S \, .$$

But then we must have $(\pi, y^j) > (\pi, I^j)$, for otherwise x^j would not maximize preference subject to the budget constraint. Therefore

$$(\pi, I(S)) = \sum_{j \in S} (\pi, y^j) > \sum_{j \in S} (\pi, x^j) = (\pi, I(S)) \, ,$$

which contradicts the assumption that the competitive allocation is blocked.

We know that under exceptionally general conditions [6] at least one competitive allocation will always exist. Since we have reason to suspect that for a large number of participants the core will be fairly small, it seems at least reasonable that the result of Edgeworth will be correct in the general situation discussed at present. As we shall demonstrate, in the remainder of this paper, the result is indeed correct. As the number of participants in the market tends to infinity (the precise meaning of this rather elusive notion will be clarified in section 3), the core will, in the limit, consist only of competitive allocations. A specific statement of this result is given in Theorem 4.

One of the ways of interpreting this result is that it describes a type of stability for the competitive equilibrium. If there are sufficiently many participants in the market, then any deviation from the competitive equilibrium will result in some group of consumers refusing to trade. This approach to the stability problem is, of course, quite different from that described by Arrow and Hurwicz [2] and subsequent writers.

2. A Characterization of the Core.

In this section we shall consider the general market with a finite number of participants. As we have mentioned before, any competitive allocation will be in the core, so that in general the core will not be empty. The core will almost always contain some allocations which are not competitive. The problem, to which we shall turn our attention, is to characterize those allocations which are actually in the core.

Let $x^1, \ldots, x^N$ be an allocation of the total market supply

$$\sum_{j=1}^{N} x^j = I(X) ,$$

with x^j the commodity bundle assigned to the j^{th} consumer and strictly preferred by the j^{th} consumer to the zero commodity bundle. In order to decide whether this allocation is in the core we shall construct a function $g(S,z)$ in terms of which the answer will be given. This function will be based on the specific allocation in question; for a different allocation the corresponding function would be defined in a similar fashion, but would assume different values. A more precise but more cumbersome notation would perhaps indicate the specific dependence of the function $g(S,z)$ upon the allocation $x^1, \ldots, x^N$.

The arguments of g are as follows: S will range over all possible subsets of consumers, including the set of all consumers in the market. The second argument z will range over all commodity bundles, that is all m-vectors with non-negative components. For a specific commodity bundle z and a subset of consumers S, we consider all possible allocations of z among the consumers in S, i.e., $z = \sum_{j \in S} y^j$. For any such allocation we consider the possible non-negative values of λ such that $\frac{y^j}{\lambda} \underset{j}{\geq} x^j$, for all $j \in S$. The largest possible value of λ, considering all allocations of z, is defined to be $g(S,z)$. More formally

$$g(S,z) = \max_{\lambda > 0} \left\{ \lambda \;\middle|\; \frac{y^j}{\lambda} \underset{j}{\geq} x^j \text{ with } \sum_{j \in S} y^j = z \right\} ,$$

or zero if no $\lambda > 0$ will do. The fact that we have a maximum rather than a supremum, follows from the continuity assumption.

The relevance of this function to the question of whether $x^1, \ldots, x^N$ is in the core should be clear from the following theorem.

<u>Theorem 1.</u> $(x^1, \ldots, x^N)$ with $\sum_{1}^{N} x^j = I(X)$ is in the core, if and only if $g(S,I(S)) \leq 1$ for all sets of consumers.

If $x^1, \ldots, x^N$ does constitute an allocation in the core, then clearly $g(S,I(S)) \leq 1$

for all S . If this were not correct, then there would exist a $\lambda > 1$ and an allocation of $I(S)$; say y^j with $\sum_{j \in S} y^j = I(S)$, such that $\frac{y^j}{\lambda} \underset{j}{\geq} x^j$ for all j in S . Now $y^j \underset{j}{>} (0, 0, \ldots, 0)$ for otherwise x^j would be indifferent to $(0, 0, \ldots, 0)$ and would be blocked by I^j . But then y^j is strictly preferred to $\frac{y^j}{\lambda}$ when $\lambda > 1$, and the coalition S will block the allocation.

On the other hand if $g(S,I(S)) \leq 1$ for all S , then $x^1, \ldots, x^N$ must be in the core. If not, there will exist an allocation $I(S) = \sum_{j \in S} y^j$ with $y^j \underset{j}{>} x^j$ for all j in S . Using our assumption as to the continuity of preferences we see that $\frac{y^j}{\lambda} \underset{j}{>} x^j$ for some λ strictly larger than 1 , and this demonstrates the theorem.

For any set S , $g(S,z)$ can be viewed as a social utility function of the set S for commodity bundles z , in which bundles are valued by the coalition according to the value of the bundle in blocking the underlying trade. Of course, the utilities are specifically dependent upon the underlying allocation $x^1, \ldots, x^N$; if the allocation were different, the valuation of commodity bundles would be different. The interpretation of z as a utility function will be somewhat more convincing after the following result is demonstrated.

<u>Lemma 1.</u> For each S , $g(S,z)$ is homogeneous of degree one, and concave.

The fact that $g(S,tz) = tg(S,z)$ is immediate from the definition of g . In order to demonstrate that g is concave we shall first show that

$$g(S,z + \bar{z}) \geq g(S,z) + g(S,\bar{z}) ,$$

where z and $\bar{z}$ are any commodity bundles. The theorem is obviously correct if either $g(S,z)$ or $g(S,\bar{z})$ is zero. Let us therefore assume that both $\lambda = g(S,z)$ and $\bar{\lambda} = g(S,\bar{z})$ are positive. Then we can find $\sum_{j \in S} y^j = z$ and $\sum_{j \in S} \bar{y}^j = \bar{z}$, so that

$$\frac{y^j}{\lambda} \underset{j}{\geq} x^j \quad \text{and} \quad \frac{\bar{y}^j}{\bar{\lambda}} \underset{j}{\geq} x^j \qquad \text{for } j \text{ in } S .$$

But then

$$z + \bar{z} = \sum_{j \in S} (y^j + \bar{y}^j)$$

and

$$\frac{y^j + \bar{y}^j}{\lambda + \bar{\lambda}} = \frac{\lambda}{\lambda + \bar{\lambda}}\left(\frac{y^j}{\lambda}\right) + \frac{\bar{\lambda}}{\lambda + \bar{\lambda}}\left(\frac{\bar{y}^j}{\bar{\lambda}}\right) \underset{j}{\geq} x^j .$$

Therefore $g(S,z + \bar{z}) \geq \lambda + \bar{\lambda}$, as we wanted to show. The fact that $g(S,z)$ is concave follows by combining this result with the homogeneity of z in the obvious way.

In the case in which S is the entire set of consumers, and in the more general situation in which production is included, the function $g(S,z)$ was introduced by Debreu [5], with, however, a slightly different emphasis than that given here. Debreu defines the "coefficient of resource allocation" to be

$$\rho = \frac{1}{g(X,I(X))} .$$

If the allocation $(x^1,\ldots,x^N)$ with $\sum_1^N x^j = I(X)$ is Pareto optimum then $\rho = 1$. On the other hand if the allocation is not Pareto optimum then ρ will be less than unity. More specifically if the resources $\rho I(X)$, rather than $I(X)$ are used, an allocation can be found which provides the j^{th} consumer with a commodity bundle of equal utility value as x^j. In this sense the quantity $1-\rho$ is a measure of the economic loss associated with the allocation $(x^1,\ldots,x^N)$.

In the analysis of the core, our attention will also be focused on $g(S,z)$ as a function of the set S. Let S be an arbitrary set of consumers. The next result describes the relationship between $g(S,z)$ and $g(S_1,z),\ldots,g(S_n,z)$ with $S_1,\ldots,S_n$ a partition of the set S.

Lemma 2. Let S be the union of disjoint sets $S_1,\ldots,S_n$. Then

$$g(S,z) = \max_{z^1+\ldots+z^n=z} \operatorname{Min}(g(S_1,z^1),\ldots,g(S_n,z^n)) .$$

For any z^i there will be commodity bundles y^{ij} such that

$$\sum_{j\in S_i} y^{ij} = z^i \quad \text{and} \quad \frac{y^{ij}}{g(S_i,z^i)} \underset{j}{\geq} x^j \qquad \text{for } j \in S_i .$$

Therefore if $z^1 + \ldots + z^n = z$, then $z = \sum_{i=1}^n \sum_{j\in S_i} y^{ij}$ with

$$\frac{y^{ij}}{\operatorname{Min}(g(S_1,z^1),\ldots,g(S_n,z^n))} \underset{j}{\geq} x^j .$$

It follows that

$$g(S,z) \geq \operatorname{Min}(g(S_1,z^1),\ldots,g(S_n,z^n)) .$$

Since this is correct for any $z^1 + \ldots + z^n = z$, we see that

$$g(S,z) \geq \max_{z^1+\ldots+z^n=z} \operatorname{Min}(g(S_1,z^1),\ldots,g(S_n,z^n)) .$$

The corresponding inequality, running in the other direction, is demonstrated as follows. By the definition of $g(S,z)$ we may write

$$z = \sum_{i=1}^n \sum_{j\in S_i} y^{ij} \quad \text{with} \quad \frac{y^{ij}}{g(S,z)} \underset{j}{\geq} x^j \qquad \text{if } j \in S_i .$$

Define $z^i = \sum_{j\varepsilon S_i} y^{ij}$, so that $z^1 + \dots + z^n = z$. It follows that $g(S_i,z^i) \geq g(S,z)$, and therefore

$$g(S,z) \leq \mathrm{Min}(g(S_1,z^1),\dots,g(S_n,z^n)) ,$$

for this particular decomposition of z into $z^1,\dots,z^n$. This demonstrates the lemma.

Before proceeding to an example let us derive two additional results which depend on specific assumptions as to the structure of the market.

Let us consider two subsets of consumers, say A and B . The number of consumers in each of these sets is assumed to be the same. In addition we shall assume that for each consumer in A there is a counterpart in B , and vice versa, with precisely the same tastes, and with precisely the same commodity bundle in the allocation used to define g . If these properties are fulfilled we shall call the sets A and B isomorphic. Clearly if A and B are isomorphic then $g(A,z) \equiv g(B,z)$.

<u>Lemma 3.</u> Let S be a subset of the consumers, which is the union of a disjoint collection of <u>isomorphic</u> subsets $S_1,\dots,S_n$. Then $g(S,z) = \frac{g(S_i,z)}{n}$.

If the functions $g(S_i,z)$ are denoted by $g(z)$, then from the result given above we have

$$g(S,z) = \underset{z^1 + \dots + z^n = z}{\mathrm{Max}} \mathrm{Min}(g(z^1),\dots,g(z^n)) .$$

Since $g(z)$ is concave, we have

$$g(\tfrac{z}{n}) = g\left(\frac{z^1 + \dots + z^n}{n}\right) \geq \sum_{j=1}^{n} \frac{g(z^j)}{n} \geq \underset{j}{\mathrm{Min}}(g(z^j)) ,$$

and therefore

$$g(S,z) \leq g(\tfrac{z}{n}) .$$

The reverse inequality is trivially obtained by taking all of the z^j the same.

<u>Lemma 4.</u> Let $S = \{1,2,\dots,n\}$ be a subset of consumers, all of whose members have identical preferences. Assume that the commodity bundles underlying the definition of g , may be ranked $x^1 \leq x^2 \leq x^3 \leq ,\dots,\leq x^n$ according to the common preference ranking. Then if $S_k = \{1,2,\dots,k\}$, with $k \leq n$, we have

$$g(S,z) \leq g(S_k, \tfrac{k}{n} z) .$$

In order to demonstrate this result, we introduce an artificial market consisting of nk consumers, all of whose preferences are identical. The consumers in this market will be labeled (i,j) with i running from 1 to n and with j from 1 to k. Let $\tilde{g}(A,z)$ be defined for this market on the basis of the allocation which gives the commodity bundle x^i to the consumer labeled (i,j). The entire set of consumers T in this new market, consists of a k-fold repetition of the original consumers in the set S. For this reason the previous lemma may be applied to deduce that

$$\tilde{g}(T,z) = \frac{1}{k} g(S,z) .$$

On the other hand the set T may be decomposed into the disjoint union of n sets $T_1,\ldots,T_n$ as follows. Let

$$T_1 = ((1,1),\ (2,1),\ldots,(k,1))$$

$$T_2 = ((1,2),\ (2,2),\ldots,(k,2))$$

$$\vdots$$

$$T_k = ((1,k),\ (2,k),\ldots,(k,k))$$

and

$$T_{k+1} = ((k+1,1),\ (k+1,2),\ldots,(k+1,k))$$

$$\vdots$$

$$T_n \ = ((n,1),\ (n,2),\ldots,(n,k)) \ .$$

The first k of these sets are obviously isomorphic and the common value of $\tilde{g}(T_j,z)$ for $j = 1,2,\ldots,k$ will be identical with $g(S_k,z)$. Consider the set T_{k+1}, all of whose members have been allocated the same commodity bundle x^{k+1}. Since $x^{k+1} \geq x^k \geq \ldots \geq x^1$, it follows directly from the definition that $\tilde{g}(T_{k+1},z) \leq \tilde{g}(T_j,z)$ for $j = 1,2,\ldots,k$, so that $\tilde{g}(T_{k+1},z) \leq g(S_k,z)$. A similar argument may be applied to any of the sets T_j for $j = k+1,\ldots,n$, and we therefore come to the conclusion that

$$\tilde{g}(T_j,z) \leq g(S_k,z) \qquad \text{for } j = 1,2,\ldots,n :$$

This remark permits us to make the appropriate evaluation of $\tilde{g}(T,z)$. Applying lemma 2, we have

$$\frac{1}{k}\, g(S,z) = \tilde{g}(T,z)$$

$$= \underset{z^1 + \ldots + z^n = z}{\text{Max}} \text{Min}(\tilde{g}(T_1,z^1),\ldots,\tilde{g}(T_n,z^n))$$

$$\leq \underset{z^1 + \ldots + z^n = z}{\text{Max}} \text{Min}(g(S_k,z^1),\ldots,g(S_k,z^n))$$

$$\leq \underset{z^1 + \ldots + z^n = z}{\text{Max}} \frac{g(S_k,z^1) + \ldots + g(S_k,z^n)}{n}$$

$$\leq g(S_k,\tfrac{z}{n}) \quad ,$$

because of the concavity of the g functions. Using the fact that g is homogeneous of degree one, the lemma is demonstrated.

This series of lemmas describes the formal properties of the g functions which will be used in the discussion of the core for large markets. These properties are somewhat abstract and it may be useful to illustrate them by means of an example.

Let us consider a market in which all of the consumers have identical preferences given by a utility function $U(x_1,\ldots,x_m)$ with the following properties

1) $U(x)$ is homogeneous of degree one.
2) U is concave, positive and increasing in each coordinate.
3) $U((I(X)) = 1$, where $I(X)$ is the vector of total market holdings.

Properties 2 and 3 are, of course, quite mild. On the other hand assumption one is quite strong. It should be remarked that we are <u>not</u> assuming that the initial holdings are also identical for the various consumers. If that were correct the core would consist of the competitive trade alone, that is no trade at all.

Our purpose is to construct the functions $g(S,z)$ based on a particular allocation $x^1,\ldots,x^N$ of the total supply, and then to determine the core by means of Theorem 1.

Let us begin with the case in which S refers to a single consumer, who receives the commodity bundle x^j as his share in the allocation of the total market supply I. $G(\{j\},z)$ is, by definition, the largest value of λ such that $U(\frac{z}{\lambda}) \geq U(x^j)$, or (using the homogeneity of the utility functions) $\frac{U(z)}{\lambda} \geq U(x^j)$. It follows that

$$g(\{j\},z) = \frac{U(z)}{U(x^j)} \quad .$$

Now let us consider an arbitrary set of consumers S. Applying lemma 2 we see that

$$g(S,z) = \underset{\sum_{j\in S} z^j = z}{\text{Max}} \quad \underset{j\in S}{\text{Min}} \left(\frac{U(z^j)}{U(x^j)}\right) .$$

It is easy to see that

$$g(S,z) = \frac{U(z)}{\sum_{j\in S} U(x^j)} .$$

For if we take

$$z^j = \frac{z\ U(x^j)}{\sum_{j\in S} U(x^j)}$$

we obtain

$$g(S,z) \geq \underset{j\in S}{\text{Min}} \left(\frac{U(z)}{\sum_{j\in S} U(x^j)} \right) = \frac{U(z)}{\sum_{j\in S} U(x^j)} ,$$

and on the other hand

$$\underset{j\in S}{\text{Min}} \frac{U(z^j)}{U(x^j)} = \underset{j\in S}{\text{Min}}\ U\left(\frac{z^j}{U(x^j)}\right) \leq \sum_{j\in S} \frac{U(x^j)\ U\left(\frac{z^j}{U(x^j)}\right)}{\sum_{j\in S} U(x^j)} \leq U\left(\frac{z}{\sum_{j\in S} U(x^j)}\right) ,$$

using convexity and homogeniety. Therefore

$$g(S,z) \leq \frac{U(z)}{\sum_{j\in S} U(x^j)}$$

and our statement is verified.

The simple form for the g functions in this example permits us to describe the allocations in the core as being those allocations for which

$$\sum_{j\in S} U(x^j) \geq U(I(S)) ,$$

for all subsets of consumers S. If we assume a version of strict concavity for the utility functions, appropriate to homogeneous functions, then the description of the core takes an even simpler form. Specifically let us assume that

$$U(\alpha x + (1-\alpha)y) > \alpha U(x) + (1-\alpha)\ U(y) ,$$

when $0 < \alpha < 1$, and if y and x are/not proportional. This, of course, implies that

$$U\left(\sum_1^n \rho_i x^i\right) > \sum_1^n \rho_i \, U(x^i) ,$$

if all $\rho_i > 0$ and if the x^i are not all proportional. In our case, if the allocation $x^1, \ldots, x^N$ is in the core then we must have

$$\sum_1^N U(x^j) \geq U\left(\sum x^j\right) ,$$

and therefore

$$\frac{\sum_1^N U(x^j)}{N} \geq U\left(\frac{\sum x^j}{N}\right) .$$

Under the assumption of strict concavity this implies that all of the allocations x^j are proportional, and therefore proportional to $I(X)$, the vector of total market supply.

The only allocations in the core are therefore those allocations $x^j = \alpha^j I(X)$, where α^j are non-negative numbers which sum to unity. α^j may be thought of as the share of the j^{th} consumer in the market. The conditions given above for an allocation to be in the core may be translated quite simply into conditions on the shares to each consumer. We are to have

$$\sum_{j \in S} U(x^j) \geq U(I(S)) .$$

Since $x^j = \alpha^j I(X)$, and $U(I(X)) = 1$ this is equivalent to

$$\sum_{j \in S} \alpha^j \geq U(I(S)) .$$

In other words, assuming that the utility of the total market supply is taken to be unity, then the share of the market allocated to any set of consumers must not be less than the utility of their combined initial holdings. The similarity of this result, for the special market considered here, to the definition of the core in the case of transferable utility should be clear. For this example, the shares of the market are measures of utility and are transferable.

3. The Core for Large Markets

We are now prepared to turn our attention to the question of whether the core tends to the collection of competitive allocations, as the size of the market increases. As was mentioned in the introduction this result, or versions of it, have been cited frequently in the economic literature. If the statement however is examined with some degree of care, it will be seen that its meaning is not at all clear. There are several possible interpretations that can be associated with this statement. First of all, it is possible that one intends to measure in some absolute sense (area, volume, etc.) the size of the set of allocations constituting the core or the contract surface, and to show that this measure of size is small if the number of consumers in the market is large, eventually tending to zero. This would probably be the most appropriate mathematical restatement of the intuitive idea, if the measure of size were plausible from an economic point of view. It seems, however, to be quite difficult to produce such a measure; Edgeworth does not work with this technique, nor do any of the subsequent writers on the subject.

As another possible mathematical statement of the problem we can attempt to demonstrate that as additional participants are added to the market, the core for the larger number of participants will actually be a subset (that is, directly contained in) the core for the smaller number of participants. It takes very little thought, however, to realize that this is a meaningless statement. If, as we are assuming, there are m commodities and n consumers, the core will consist of a subset of points in mn dimensional space, and there is no immediate way to compare such subsets on the basis of inclusion of sets for different values of n.

There is a modification of this approach which consists of focusing our attention on a particular collection of consumers, with their respective vectors of initial holdings $I^1,\dots,I^n$. We might then consider all markets obtained by the addition of consumers with additional initial holdings to this particular collection of consumers, and focus our attention on those commodity bundles $x^1,\dots,x^n$ which will be part of the larger core for <u>all</u> of these augmented markets. In this way we are comparing the cores for various markets by means of the possible allocations earmarked for a specific set of consumers. As we shall see in a subsequent section as part of a more general result, such an allocation $x^1,\dots,x^n$ will indeed turn out to be a competitive allocation of the resources of this group of consumers. I find this type of approach somewhat troubling, however, in as much as it forces the members of this particular group of consumers to become fully dependent upon their own resources.

This type of approach could be used without considering <u>all</u> markets containing the specific set of consumers. We might, for example, arrange the consumers in order as consumer number one, number two, three, etc., and then designate by S_n the collection of the first n

consumers. We would then compute the core for this market based on the total supply held by the first n consumers and then examine the point set in mN dimensions which describes those allocations in the core specifically earmarked for some definite set of N consumers. The relevant questions to be asked would be first of all whether this part of the core that we are studying becomes smaller as S_n increases, and a question as to the limiting set of points. The answer to the first question, however, is definitely no. The parts of the cores that we are studying, even though they are sets of points in a space of equal dimension, bear very little relationship to each other for different values of n. For an individual consumer, considered as a market by himself, the core will consist only of his initial holdings. If, however, this consumer forms part of a larger market, it is most probable that none of the allocations in the core for the larger market will assign to this consumer a commodity bundle identical to his initial holdings.

All of the difficulties in interpretation suggested above are forced upon us by the need to compare markets with increasingly larger numbers of consumers, that is, to compare fundamentally incommensurable objects. As I see it, there are basically two ways out of these difficulties. One is to permit the sequence of markets to increase in a sufficiently regular way so that comparisons are possible, and the other is to consider the ideal market with an infinite number of consumers. In the remainder of this section we shall examine the first approach which is very similar to that taken by Edgeworth.

Let us consider then a collection of a finite number of types of consumers, type 1, type 2,..., type N. All consumers of type j, if we refer to several of them, will be assumed to have identical preferences and identical vectors of initial holdings I^j. We shall then consider markets consisting of n consumers of type one, n of type two, and generally n consumers of every type. A typical consumer in this market will be described by a pair of indices (i,j) where j indicates the type of the consumer $(j = 1,2,\ldots,N)$ and $i = 1,2,\ldots,n$.

In such a market the allocations in the core may very well assign different commodity bundles to consumers of the same type. For the remainder of this section, however, we shall restrict our attention to those allocations in the core which assign the <u>same</u> commodity bundle to all consumers of the same type. There always will be such allocations in the core since the competitive allocation has this property.

In order to discuss such an allocation we need only indicate the commodity bundle x^j assigned to a typical consumer of type j . Since the commodity bundles in the core are meant to be an allocation of the total market supply it follows that

$$\sum_1^N x^j = \sum_1^N I^j \quad ,$$

regardless of the value of n . All allocations in the core, which are of the restricted type considered here, may therefore be described by a set of N commodity vectors $x^1,\ldots,x^N$ satisfying

$$\sum_1^N x^j = \sum_1^N I^j \ .$$

The sets of such allocations in the core, which we shall denote by C_n , does of course vary with n , the number of repetitions of each basic type of consumer, as the following theorem demonstrates.

Theorem 2. Under the assumptions given above

$$C_{n+1} \subseteq C_n \quad .$$

The allocations that we are considering in the market with n repetitions may be described as

repetition \ type	1	2	...	N
1	x^1	x^2	...	x^N
2	x^1	x^2	...	x^N
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	x^1	x^2	...	x^N

,

and that in the market with $n+1$ repetitions by an additional line at the bottom. The quantities $g(S,I(S))$, based on this allocation, will be identical for the two markets if S is a set whose members are among the first nN consumers. The theorem therefore follows immediately from Theorem 1.

That part of the core consisting of those allocations which assign identical commodities to identical consumers, will therefore become smaller as the number of repetitions increases.

The limiting set as n tends to infinity will consist of those allocations in the core for all finite n. As we have seen, any competitive allocation will be in the core for all n. The main result of this section will be that the only allocations of the type described above, which are in the core for all n, will be competitive allocations.

Let $x^1,\ldots,x^N$ with $\sum_1^N x^j = \sum_1^N I^j$, suitably repeated, be in the core for all finite n. If the functions g are based upon this allocation, then we must have

$$g(S,I(S)) \leq 1 \quad ,$$

where S is any set consisting of k_1 consumers of type one, k_2 consumers of type two, and generally k_j consumers of type j with $j = 1,2,\ldots,N$. Let S be partitioned into subsets $S_1,\ldots,S_N$, with S_j consisting of the k_j consumers of type j. Then since

$$g(S,I(S)) = \underset{\sum_1^N z^j = I(S)}{\mathrm{Max}} \quad \underset{j}{\mathrm{Min}} \; (g(S_j,z^j)) \; ,$$

we have

$$\underset{\sum_1^N z^j = I(S)}{\mathrm{Max}} \quad \underset{j}{\mathrm{Min}} \; (g(S_j,z^j)) \leq 1 \quad .$$

But S_j consists of k_j isomorphic consumers (see the discussion immediately preceding lemma 3), so that

$$g(S_j,z^j) = g(\{j\}, \frac{z^j}{k_j}) \quad ,$$

with $\{j\}$ referring to a typical consumer of type j. We also have

$$I(S) = \sum_{j=1}^N k_j I^j \quad ,$$

so that

$$\underset{\sum_1^N z^j = \sum_1^N k_j I^j}{\mathrm{Max}} \quad \underset{j}{\mathrm{Min}} \; \left(g_j(\frac{z^j}{k_j}) \right) \leq 1 .$$

(In this equation and in the remainder of this section we shall emply the more compact notation g_j, for the g function based on a typical consumer of type j.) If we replace z^j by $k_j t^j$, we obtain

$$\underset{\sum_1^N k_j(t^j - I^j) = 0}{\mathrm{Max}} \quad \underset{j}{\mathrm{Min}} \; (g_j(t^j)) \leq 1 \quad .$$

This inequality is to hold for any set of integers $(k_1, k_2, \dots, k_N)$. The constraining equality $\sum_1^N k_j(t^j - I^j) = 0$, may of course be written as

$$\sum_1^N \frac{k_j}{k} (t^j - I^j) = 0 ,$$

where $k = \sum_1^N k_j$, and this suggests that the ratios $\frac{k_j}{k}$ may be replaced by any set of positive numbers ρ_j ; in other words that

$$\underset{\sum_1^N \rho_j(t^j - I^j) = 0}{\text{Max}} \quad \text{Min}\ (g_j(t^j))$$

shall be less than or equal to one, whenever $\rho_j > 0$. This latter inequality, which shall be very important for us, is indeed correct.

In order to see this let $\rho_1, \dots, \rho_N$ be positive numbers, and let M be an integer eventually tending to infinity. We define $k_j = 1 + [M\rho_j]$, where $[M\rho_j]$ is the symbol for the greatest integer less than or equal to $M\rho_j$. Then $\frac{k_j}{M} \geq \rho_j$, and as $M \to \infty$, $\frac{k_j}{M} \to \rho_j$. Now let $t^1, \dots, t^N$ be <u>any</u> collection of N commodity bundles satisfying $\sum_1^N \rho_j(t^j - I^j) = 0$. We wish to show that $\underset{j}{\text{Min}}\ g_j(t^j) \leq 1$. First of all let us notice that

$$\tilde{t}^j = \frac{\rho_j M}{k_j} t^j + (1 - \frac{\rho_j M}{k_j}) I^j ,$$

satisfies the equation $\sum_1^N k_j(\tilde{t}^j - I^j) = 0$. It therefore follows from our previous result that

$$\underset{j}{\text{Min}} \left(g_j \left(\frac{\rho_j M}{k_j} t^j + (1 - \frac{\rho_j M}{k_j}) I^j \right) \right) \leq 1 .$$

However since each g_j is concave this implies that

$$\underset{j}{\text{Min}} \left(\frac{\rho_j M}{k_j} g_j(t^j) + (1 - \frac{\rho_j M}{k_j}) g_j(I^j) \right) \leq 1 .$$

Now let M tend to ∞ , and we obtain

$$\underset{j}{\text{Min}}\ (g_j(t^j)) \leq 1 ,$$

for any commodity bundles satisfying $\sum_1^N \rho_j(t^j - I^j) = 0$.

One more point is in order before we summarize this part of the discussion. If all of the numbers $\rho_1, \dots, \rho_N$ are the same, our inequality becomes

$$\underset{\sum_1^N t^j = \sum_1^N I^j}{\text{Max}} \quad \text{Min}\,(g_j(t^j)) \leq 1\ .$$

It is a simple matter to see, however, that the left hand side must actually be equal to one. For if we let S represent a set of consumers consisting of precisely one of each type, then the left hand side is the same as

$$g(S, \sum_1^N I^j)\ .$$

Since $\sum_1^N x^j = \sum_1^N I^j$ and of course $x^j \underset{j}{\geq} x^j$, we may conclude directly from the definition of the g function that

$$g(S, \sum_1^N I^j) \geq 1\ ,$$

and combining this with the previous inequality we obtain

$$\underset{\sum_1^N \rho_j(t^j - I^j)=0}{\text{Max}} \quad \underset{j}{\text{Min}}\,(g_j(t^j)) = 1\ ,$$

when all of the ρ_j are the same.

Before continuing with the argument, let us summarize these results in the following theorem.

Theorem 3. If $x^1,\ldots,x^N$ with $\sum_1^N x^j = \sum_1^N I^j$ is in the core for every finite n, then

$$\underset{\sum_1^N \rho_j(t^j - I^j)=0}{\text{Max}} \quad \underset{j}{\text{Min}}(g_j(t^j)) \leq 1\ ,$$

and is actually equal to 1, when all of the ρ_j are the same.

Let us now turn to the argument which will show us that $x^1,\ldots,x^N$ is actually a competitive allocation.

Our first step is to determine the set of prices $\pi_1,\ldots,\pi_m$ which will eventually turn out to be the competitive prices for the allocation $x^1,\ldots,x^N$. We consider a convex set T, defined to be the set of all commodity bundles $t = \sum_1^N t^j$ with $\underset{j}{\text{Min}}\, g_j(t^j) > 1$. Theorem 3 tells us that $\sum_1^N I^j$ is not in T, and we may therefore use the separating hyperplane theorem to obtain constants $\pi_1,\ldots,\pi_m$ (not all zero) such that $(\pi,t) \geq (\pi, \sum_1^N I^j)$, for all t in T. Since the coordinates of t may be selected as being arbitrarily large, we see that all

$\pi_j \geq 0$, and since the π_j are not all zero, we may normalize the prices so that $\sum_1^m \pi_j = 1$. There may, of course, be more than one such separating hyperplane, and we shall denote by Π the convex set of all hyperplanes which separate T from $\sum_1^N I^j$, normalized by the condition $\sum_1^m \pi_i = 1$.

If t is any commodity bundle such that $t = \sum_1^N t^j$ with $\operatorname*{Min}_j g_j(t^j) \geq 1$, then $(1+\epsilon)t$ is in T , and we may therefore conclude that $(\pi,t) \geq (\pi, \sum_1^N I^j)$.

It is a simple matter to see that x^j will actually maximize the preferences of the j^{th} consumer at the price's π . For if t^j is a commodity bundle with $t^j \underset{j}{>} x^j$ and $(\pi,t^j) \leq (\pi,x^j)$, then by the hypothesis of continuity of the preferences, there will be a commodity bundle $\tilde{t}^j$ with $\tilde{t}^j \underset{j}{\geq} x^j$ and $(\pi,\tilde{t}^j) < (\pi,x^j)$, unless $(\pi,x^j) = 0$. But then $t = \sum_{k \neq j} x^k + \tilde{t}^j$, will satisfy

$$(\pi,t) \geq \sum_1^N (\pi,I^j) ,$$

or

$$\sum_{k \neq j} (\pi,x^k) + (\pi,\tilde{t}^j) \geq \sum_{k=1}^N (\pi,x^k) ,$$

which is a contradiction, unless $(\pi,x^j) = 0$. As we shall see later, this exceptional case cannot occur.

In order to finish the proof that the x^j represent a competitive allocation, we need only show that

$$(\pi,x^j) \leq (\pi,I^j) ,$$

in other words that each consumer spends no more than the value of his initial holdings at the stated set of prices. This collection of N inequalities will not be valid for all prices $\pi_1,\ldots,\pi_m$ in the convex set Π ; it will however be valid for at least one such set of prices, and this is sufficient for our purpose.

The proof of this result will proceed in a somewhat indirect fashion. We shall, first of all, demonstrate that for every collection of non-negative numbers $\delta_1,\delta_2,\ldots,\delta_N$, there will be at least one set of prices π in Π , such that $(\pi, \sum_1^N \delta_j (I^j - x^j)) \geq 0$. To see this, let us appeal to Theorem 3, with $\rho_j = 1 - \frac{\delta_j}{M}$. M will be a positive number, eventually tending to infinity. For every value of M , however, we have

$$\operatorname*{Min}_j g_j(t^j) \leq 1 ,$$

for all $t^1,\dots,t^N$ with $\sum_1^N \rho_j t^j = \sum_1^N \rho_j I^j$. Let us define the convex set T_ρ to be the collection of all t , such that

$$t = \sum_1^N \rho_j t^j$$

with

$$\operatorname*{Min}_j g_j(t^j) > \operatorname*{Max}_{\Sigma \rho_j(z^j - I^j)=0} \; \operatorname*{Min}_j g_j(z^j) \; .$$

Obviously $\sum_1^N \rho_j I^j$ is not in this convex set, and we may therefore find prices $\pi_1(\rho),\dots,\pi_m(\rho)$ such that

$$(\pi(\rho),t) \geq \sum_1^N \rho_j \; (\pi(\rho),I^j) \; , \qquad \text{for all } t \text{ in } T_\rho \; .$$

Again these prices will be non-negative and may be normalized so that their sum is one. Moreover, we may also conclude that

$$(\pi(\rho),t) \geq \sum_1^N \rho_j \; (\pi(\rho),I^j) \; ,$$

for any $t = \sum_1^N \rho_j t^j$ with

$$\operatorname*{Min}_j g_j(t^j) \geq \operatorname*{Max}_{\Sigma \rho_j(z^j - I^j)=0} \; \operatorname*{Min} g_j(z^j) \quad .$$

On the other hand, let t be any element of the set T previously defined, so that $t = \sum_1^N t^j$ with $\operatorname*{Min}_j g_j(t^j) \geq 1$. It follows that $t = \sum_1^N \rho_j \left(\frac{t^j}{\rho_j}\right)$ with

$$\operatorname*{Min}_j g_j \left(\frac{t^j}{\rho_j} \right) = \operatorname{Min} \frac{g_j(t^j)}{\rho_j} \geq 1 \quad ,$$

since the numbers ρ_j are less than or equal to one. In other words if t is in T , it will certainly be in T_ρ and we will have

$$(\pi(\rho),t) \geq \sum_1^N \rho_j \; (\pi(\rho),I^j) \quad .$$

Now let $M \to \infty$, so that the numbers ρ tend to one. Since the $\pi(\rho)$ are all in some compact set it follows that there will be a limit point for the $\pi(\rho)$'s , say π , and that

$$(\pi,t) \geq \sum_1^N (\pi,I^j) \; , \qquad \text{for any } t \text{ in } T \; .$$

In other words any limit point for the $\pi(\rho)$'s will be a set of prices in Π , the set of separating hyperplanes for T .

We wish to show that

$$\sum_1^N \delta^j \; (\pi, I^j - x^j) \geq 0 \; ,$$

for any such π that appears as a limit of the $\pi(\rho)$. Consider $\sum_1^N \rho_j x^j$. Since

$$\operatorname*{Min}_j \; g_j(x^j) = 1 \geq \operatorname*{Max}_{\sum_1^N \rho_j(t^j - I^j)=0} \; \operatorname*{Min}_j \; g_j(t^j) \; ,$$

we see immediately that

$$\sum_1^N \rho_j(\pi(\rho),\; x^j) \geq \sum_1^N \rho_j(\pi(\rho),\; I^j)$$

or

$$\sum_1^N (1 - \frac{\delta_j}{M}) \; (\pi(\rho),\; I^j - x^j) \leq 0 \; .$$

But since $\sum_1^N (I^j - x^j) = 0$, we have $\sum_1^N \delta_j(\pi(\rho),\; I^j - x^j) \geq 0$. If we let ρ tend to one, thereby obtaining a set of prices in Π , we see that

$$\sum_1^N \delta_j(\pi,\; I^j - x^j) \geq 0 \; .$$

With this result in mind, it is quite simple to show that there will be at least one set of prices π in Π , such that $(\pi,\; I^j - x^j) \geq 0$ for all j , and this will demonstrate that x^j is indeed a competitive allocation. We shall use the following simple lemma.

Lemma 5. Let A be a closed convex cone in m-dimensional space and Π a closed, bounded convex set, also in m-dimensional space. Assume that for every point a in A , there exists at least one π in Π with $(\pi,a) \geq 0$. Then there will be a π in Π such that $(\pi,a) \geq 0$ for <u>all</u> a in A .

Assume that the conclusion of the theorem is not correct. Then if A^+ represents the dual cone to A , that is the set of t such that $(t,a) \geq 0$ for all a in A , we see that A^+ and Π will necessarily be disjoint. Since the two sets are disjoint and closed, and one is bounded, it follows that we may find $c_1, \ldots, c_m$, not all zero and $b > d$, with

$$(t,c) \geq b \quad \text{for all } t \text{ in } A^+ ,$$

and $(\pi,c) \leq d$ for all π in Π . Since A^+ is a cone, containing the origin, it follows that $b \leq 0$, and therefore $d < 0$. But on the other hand we must have $(t,c) \geq 0$ for all t in A^+ . For if there were a t_0 in A^+ with $(t_0,c) < 0$, then since A^+ is a cone, λt_0 would also be in A^+ , and by taking λ sufficiently large we would not have $(t,c) \geq b$ for all t in A^+ . Since $(t,c) \geq 0$ for all t in A^+, c must be in A . We have therefore constructed a point c in A , such that $(\pi,c) < 0$ for all π in Π , and this contradicts the assumptions of the lemma.

In the application of the lemma to our problem, we define A to be the convex cone consisting of all points of the form $\sum_1^N \delta_j(I^j - x^j)$, with $\delta_j \geq 0$, and Π the set of separating hyperplanes defined before. It follows from lemma 5, that there is at least one π in Π such that

$$\sum_1^N \delta_j(\pi, I^j - x^j) \geq 0 ,$$

for <u>all</u> $\delta_j \geq 0$, and this certainly implies that $(\pi, I^j - x^j) \geq 0$ for all j . This completes the proof that the allocation $x^1, \ldots, x^N$ is a competitive equilibrium. We may also clear up the technical point of the last argument as to whether $(\pi, x^j) = 0$. From what we have just done we have $(\pi, x^j) = (\pi, I^j)$; and if we assume that all of the components of I^j are strictly positive, it follows that $(\pi, x^j) > 0$.

It is perhaps worthwhile to point out the role played in the above argument by the increasing number of consumers in the market. The existence of prices π such that the commodity bundles x^j maximize preferences at these prices is correct even for markets of finite size and depends merely on the observation that an allocation in the core is certainly Pareto optimal. It then follows from theorems of Arrow and Debreu [1,6] which state that any Pareto optimum allocation of social resources may be achieved by means of fixed prices. The important point, however, is that, in general, redistribution of income is necessary prior to maximization at the fixed set of prices. In other words it will generally not be true that

$$(\pi, x^j - I^j) \leq 0 ,$$

for an arbitrary Pareto optimum allocation x^j . In fact, the initial holdings of the various consumers, as distinct from the total market supply, never enter the discussion of Pareto optimality at all . It is only by means of concepts such as the contract surface or the core that the holdings of individual consumers and groups of consumers become relevant.

On the other hand if we do not let the number of consumers become infinite, the core will generally contain allocations other than competitive ones. In a technical sense the passage to the limit is similar to a process of differentiation. It is only by means of the device of an infinite number of consumers that we were able to show that the Pareto optimum prices (or more specifically at least one set of Pareto optimum prices), would in addition satisfy the inequalities

$$(\pi, x^j - I^j) \leq 0 .$$

Let us summarize the main result of this section as follows.

Theorem 4. Consider a market with a finite number of types of consumers, all consumers of the same type having identical tastes and initial holdings. Let there be an infinite number of consumers of each type. Consider an allocation $x^1,..,x^N$ with $\sum_1^N x^j = \sum_1^N I^j$, such that every consumer of type j receives the commodity bundle x^j. If this allocation cannot be improved upon by any finite collection of consumers, on the basis of their own initial holdings, then it is a competitive equilibrium.

4. The Case of Different Allocations For Consumers of the Same Type.

The result of the previous section can be understood in two possible ways. One interpretation is that we are considering a sequence of markets composed of consumers of the same types, and such that the numbers of repetitions of each basic type are increasing in a regular fashion. An alternative interpretation, which was stressed in a statement of Theorem 4, is that the market consists of an infinite number of consumers, and that no passage to the limit is required. Mathematically the two interpretations are identical since we considered only those allocations which assign the same commodity bundle to consumers of the same type. In the present section we would like to consider the somewhat more general case in which different commodity bundles may be assigned to consumers of the same type, and it will be more convenient for us to favor the second interpretation.

We shall assume, as before, that the market consists of a finite number of types of consumer, but now with an infinite number of each type. The commodity bundle assigned to the i^{th} consumer of type j will be denoted by x^{ij}. Our purpose will be to show that if allocacations of this sort are in the core, then of necessity all consumers of the same type receive the same commodity bundle, and that the assignment is a competitive equilibrium.

The concept of the core requires that we consider allocations of the total market supply, which is of course infinite, if we adopt the interpretation of an infinite number of consumers. The condition that we shall impose on the allocations to be considered in this section is that

$$\lim_{n \to \infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{N} x^{ij} - n \sum_{j=1}^{N} I^j \right) = 0 .$$

It should be clear that a condition of this sort is necessary, rather than the weaker type of condition that

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} x^{ij}}{n} = \sum_{j=1}^{N} I^j$$

For if only the latter condition were to be assumed, no allocation would even be Pareto optimal; an improvement could always be found merely by increasing the value of the bundle assigned to a specific consumer while leaving the remaining consumers unchanged.

In order to obtain the result that all consumers of the same type receive the same commodity bundle, I have found it necessary to impose two additional requirements on the preferences of the various consumers.

1. A version of strict convexity to the effect that if $x \geq y$ and x and y are different commodity bundles, then $\alpha x + (1-\alpha) y > y$ if $0 < \alpha < 1$.

2. A condition which essentially says that the indifference surfaces of the various consumers do not pass through the coordinate planes. More specifically we assume that any commodity bundle which contains a zero level for any commodity will be indifferent to the commodity bundle $(0,\ldots,0)$. This is an excessively strong assumption which it would be desirable to eliminate.

The argument of this section will again be based on a collection of functions $g_j(z)$, but they will of necessity be different from the functions of the previous section. Consider sets S consisting of n individuals of type j, and the associated g functions $g(S,z)$. As S varies these functions will, for any fixed z, be bounded from above since x^{ij} will certainly be preferred to I^j. Let

$$g_j^n(z) = \sup g(S,z)$$

as S varies over all sets of n individuals of type j. It is a simple matter to show, by means of lemma 4, that $ng_j^n(z)$ is decreasing in n, and will therefore approach a limit as $n \to \infty$. We define $g_j(z) = \lim_{n \to \infty} ng_j^n(z)$.

As we see from the following theorem, these functions will play a role somewhat similar to the g_j functions of the previous section.

Theorem 5. Let x^{ij} be in the core and satisfy

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{N} x^{ij} - n \sum_{j=1}^{N} I^j = 0 .$$

Then

$$\max_{\sum_1^N (t^j - I^j)=0} \ \min_j g_j(t^j) = 1 .$$

There is one difference between this theorem and theorem 3. That is, the present theorem refers only to the case where all $\rho_j = 1$. There is a corresponding result for the ρ_j's different from each other, but since we shall not use it in this section, it has been left out of the statement of theorem 5.

The proof of theorem 5 is quite direct. We shall first demonstrate that if

$$\sum_1^N t^j = \sum_1^N I^j , \quad \text{then} \quad \min_j g_j(t^j) \leq 1 .$$

We know from theorem 1, that if S_j consists of n consumers of type j, then

$$\min_j g(S_j, nt^j) \leq 1 ,$$

and therefore

$$\min_j ng_j^n(t^j) \leq 1 .$$

Letting n tend to infinity we obtain the desired inequality.

Now let us turn our attention to the inequality

$$\max_{\Sigma t^j = \Sigma I^j} \ \min_j g_j(t^j) \geq 1 .$$

Since $\lim_{m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{N} x^{ij} - n \sum_{j=1}^{n} I^j = 0$, it follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{N} x^{ij} \leq (1+\epsilon) n \sum_{j=1}^{N} I^j$$

for each commodity, when n is sufficiently large. (We are using the assumption that there is a positive supply of every commodity.) If S denotes the set consisting of the first n consumers of each type it follows that

$$g(S, \sum_{j=1}^{N} I^j) \geq \frac{1}{n(1+\epsilon)} ,$$

and therefore

$$\underset{\Sigma t^j = \Sigma I^j}{\text{Max}} \quad \text{Min}\ ng(S_j, t^j) \geq \frac{1}{1+\epsilon} ,$$

with S_j the set of the first n consumers of type j . Let $\bar{t}^j(n)$ maximize, so that

$$\underset{j}{\text{Min}}\ ng_j^n(\bar{t}^j(n)) \geq \frac{1}{1+\epsilon} \quad .$$

Now let $\bar{t}^j$ be a limit point of $\bar{t}^j(n)$. Let us assume for the moment that $\bar{t}^j$ has no components equal to zero. Then we may find a sequence of n's , tending to infinity such that $\bar{t}^j(n) \leq (1+\epsilon)\ \bar{t}^j$ for all j and for each commodity in the vector. For such n's we have

$$\underset{j}{\text{Min}}\ ng_j^n(\bar{t}^j) \geq \frac{1}{(1+\epsilon)^2} \quad ,$$

and letting n tend to infinity we obtain

$$\underset{j}{\text{Min}}\ g_j(\bar{t}^j) \geq \frac{1}{(1+\epsilon)^2} \quad .$$

If we then let ϵ tend to zero ($\bar{t}^j$ does not depend on ϵ) , the conclusion of theorem 5 is obtained.

We have only to clear up the point that <u>all</u> of the components of $\bar{t}^j$ are different from zero. If this were not the case then we would have $\bar{t}^j$ equivalent to $(0,\ldots,0)$, and by the assumption of continuity it would follow that

$$a\bar{t}^j(n) \underset{j}{\leq} I^j ,$$

for any specific a , for a sequence of n's tending to infinity. But if n is large

$$ng(S_j,\ \bar{t}^j(n)) \geq \frac{1}{1+\epsilon} ,$$

and therefore we may write

$$n\bar{t}^j(n) \;=\; \sum_{i=1}^{n} y^i$$

with

$$(1+\epsilon)\ y^i \underset{j}{\geq} x^{ij} \underset{j}{\geq} I^j \ .$$

It follows from the convexity of the preferences that

$$(1+\epsilon)\ \bar{t}^j(n) \underset{j}{\geq} I^j \quad ,$$

which is a contradiction if $a > (1+\epsilon)$. This concludes the proof of theorem 5.

The specific values of $\bar{t}^j$ obtained in the proof of this theorem will be of some importance for us. In fact we will show that $\bar{t}^j = x^{ij}$ for all i and j. First of all let us remark that since $\underset{j}{\mathrm{Min}}(g_j(\bar{t}^j)) = 1$, we have $ng_j^n(\bar{t}^j) \geq 1$ for all n and j. We wish to show that, in fact, $ng_j^n(\bar{t}^j) = 1$ for all n and j. Suppose that $g_j^1(\bar{t}^j) > 1$. Then there will be an $0 < \alpha < 1$ such that $g_j^1(\alpha\bar{t}^j) \geq 1$. Let us define $z = \sum_{k \neq j} \bar{t}^k + \alpha\bar{t}^j$. It follows that we may find a set S, consisting of precisely one consumer of each type, for which for which

$$g(S,z) \geq 1 - \epsilon .$$

But

$$\sum_{j=1}^{N} I^j = z + (1 - \alpha)\bar{t}^j$$

and therefore

$$\begin{aligned} g(S, \sum_1^N I^j) &\geq g(S,z) + (1 - \alpha)\, g(S,\bar{t}^j) \\ &\geq 1 - \epsilon + (1 - \alpha)\, g(S,\bar{t}^j) . \end{aligned}$$

However, since $\bar{t}^j$ has no components equal to zero, we may find a $\lambda > 0$ such that $\bar{t}^j \geq \lambda \sum_1^N I^j$ for all components. This implies that $g(S,\bar{t}^j) \geq \lambda\, g(S, \sum_1^N I^j)$, and combining this inequality with the previous one we obtain

$$(1 - \lambda(1 - \alpha))\, g(S, \sum_1^N I^j) \geq 1 - \epsilon .$$

If we take ϵ sufficiently small this would provide us with a set S consisting of n consumers of each type for which $g(S, \sum_1^N I^j) > 1$, which violates the condition that x^{ij} be in the core. We have therefore verified that $g_j^1(\bar{t}^j) = 1$ for all j.

It is an immediate consequence of this fact that $\bar{t}^j \underset{j}{\leq} x^{ij}$ for <u>all</u> i and j. But it is then an easy matter to finish the analysis of this section and show that $\bar{t}^j = x^{ij}$ for all i and j. We shall demonstrate, first of all, that $\bar{t}^1,\ldots,\bar{t}^N$ represents a Pareto optimum allocation of the commodity bundle $\sum_{j=1}^{N} I^j$ in a market consisting of a single consumer of each type. For suppose that this were not the case. If the assumptions of continuity and monotonicity are used, it can be shown that there is a small positive ϵ, and an allocation of the market supply $\sum_{j=1}^{N} y^j = \sum_{j=1}^{N} I^j$, such that

$$y^j \underset{j}{>} \frac{\bar{t}^j}{1 - \epsilon} .$$

Then since $g_j^1(\bar{t}^j) = 1$, it follows that for our ϵ, there is a consumer of type j (with commodity bundle x^{ij}) such that $\frac{\bar{t}^j}{1-\epsilon} \underset{j}{\geq} x^{ij}$, and therefore $y^j \underset{j}{>} x^{ij}$. This would provide us with a finite coalition which blocks the underlying allocation, and therefore $\bar{t}^1, \ldots, \bar{t}^N$ is Pareto optimal.

But it is then an immediate consequence of the theorems of Arrow and Debreu that there exists a set of prices $\pi_1, \ldots, \pi_m$ such that $\bar{t}^j$ minimizes (π, x) for all $x \underset{j}{\geq} \bar{t}^j$. Recall that we have already demonstrated that $\bar{t}^j \underset{j}{\leq} x^{ij}$. By our hypothesis of strict convexity it follows that $(\pi, \bar{t}^j) \leq (\pi, x^{ij})$, with strict inequality if a single $\bar{t}^j$ is different from x^{ij}. But we have assumed that

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{N} x^{ij} - n \sum_{j=1}^{N} I^j = 0 ,$$

and therefore

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{N} (\pi, x^{ij}) - n \sum_{j=1}^{N} (\pi, \bar{t}^j) = 0 ,$$

since $\sum_{j=1}^{N} \bar{t}^j = \sum_{j=1}^{N} I^j$. If $\bar{t}^j$ is not identical with x^{ij} for all i and j we would have a contradiction. This implies that all of the commodity bundles assigned to a consumer of a particular type are identical, and we may then return to the argument of the previous section to show that they represent a competitive allocation.

REFERENCES

[1] Arrow, Kenneth J., "An Extension of the Basic Theorems of Classical Welfare Economics," Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, J. Neyman, ed., (University of California Press, 1951) pp. 507-532.

[2] Arrow, K. J., and L. Hurwicz, "On the Stability of the Competitive Equilibrium," Econometrica, vol. 22, 1958, pp. 522-552.

[3] Aumann, R. J., "The Core of a Cooperative Game without Side Payments," Trans. Amer. Math. Soc., vol. 98 (1961) pp. 539-552.

[4] Aumann, R. J., and B. Peleg, "Von-Neumann Morgenstern Solutions to Cooperative Games without Side Payments," Bull. Amer. Math. Soc., vol. 66 (1960) pp. 173-179.

[5] Debreu, Gerard, "The Coefficient of Resource Allocation," Econometrica, vol. 19, 1951, pp. 273-292.

[6] Debreu, Gerard, The Theory of Value, Cowles Foundation Monograph 17, (New York, John Wiley and Sons, 1959).

[7] Edgeworth, F. Y., Mathematical Psychics, (London, 1881).

[8] Shapley, L. S., Markets as Cooperative Games, The RAND Corporation, p.629, Mar.7, 1955.

[9] Shapley, L. S., The Solutions of a Symmetric Market Game, The RAND Corporation, p. 1392, June 5, 1958.

[10] Shapley, L. S., and M. Shubik, "Solutions of n-person Games with Ordinal Utilities," (abstract), Econometrica, vol. 21 (1953) p. 348.

[11] Shubik, Martin, "Edgeworth Market Games," Contribution to the Theory of Games IV, R. D. Luce and A. W. Tucker, eds. (Princeton: Princeton U. Press, 1958).

[12] Stigler, George J., The Theory of Competitive Price, (New York, Macmillan, 1946).