ON CORES AND INDIVISIBILITY*

Lloyd Shapley
and
Herbert Scarf

1. INTRODUCTION

This paper has two purposes. To the reader interested in the mathematics of optimization, it offers an elementary introduction to n-person games, balanced sets, and the core, applying them to a simple but nontrivial trading model. To the reader interested in economics, it offers what may be a new way of looking at the difficulties that afflict the smooth functioning of an economy in the presence of commodities that come in large discrete units.

The core of an economic model, or of any multilateral competitive situation, may be described as the set of outcomes that are

* This research was supported by National Science Foundation grant GS-31253. The authors are indebted to Professors David Gale and Bezalel Peleg for several helpful comments.

"coalition optimal," in the sense that they cannot be profitably upset by the collusive action of any subset of the participants, acting by themselves. There is no reason, *a priori*, that such outcomes must exist; the core may well be empty. But it has been shown that important classes of economic models do have nonempty cores. In fact, whenever a system of *competitive prices* exists (prices under which individual optimization decisions will lead to a balance of supply and demand), the resulting outcome is in the core.* The core may also exist in the absence of competitive prices. It is of some interest, therefore, to relax one or more of the classical "regularity" assumptions that, taken together, ensure the existence of competitive prices—such as convexity of preferences, perfect divisibility of commodities, constant returns to scale in production, and absence of externalities—and to ask under what conditions the resulting system will have a core. There is already a considerable literature in this area.**

We stress that the core is a general game-theoretic concept, definable without reference to any market model.*** Moreover, its existence for the classical exchange economies can be established, if we wish, without making use of the idea of competitive prices.****

In this paper we consider the case of a commodity that is inherently indivisible, like a house. We formulate a class of markets in which a consumer never wants more than one item, but has ordinal preferences among the items available. We then prove that a core always exists for this model, making use of the concept of "balanced sets." The competitive prices for this model are next determined by a separate argument, providing an alternative and more constructive proof of the existence.

A counterexample is then considered, showing that if more complex schemes of preferences among the indivisible goods are al-

---

* See [5].

** For nonconvexity, see [12, 19, 25]; for indivisibility, see [11, 12, 22, 24]; for nonconstant returns, see [14, 16]; for externalities, see [7, 21]. Not all of these refer directly to the core.

*** See [1, 3, 15, 17, 18].

**** See [15, 18, 20].

lowed, the core may disappear, even though all of the classical conditions except perfect divisibility are satisfied.

In the final section, for perspective, we review a series of other models involving indivisible commodities that have been discussed in the literature from the viewpoint of the core.

2. THE MODEL

Let there be n traders in the market, each with an indivisible good to offer in trade (e.g., a house). The goods are freely transferable, but we shall assume that a trader never has use for more than one item. There being no money or other medium of exchange, the only effect of the market activity is to redistribute the ownership of the indivisible goods, in accordance with the (purely ordinal) preferences of the traders. We shall describe these preferences with the aid of a square matrix: $A = (a_{ij})$, where $a_{ij} > a_{ik}$ means that trader i prefers item j to item k, and $a_{ij} = a_{ik}$ means that he is indifferent.* Owning no items, we assume, is ranked below all else, and owning several items is ranked only equal to the maximum of their separate ranks. Although only ordinal comparisons are involved in the model, it will be convenient to think of A as a matrix of real numbers.

The final effect of any sequence of transfers can be described by another matrix $P = (p_{ij})$, called an *allocation*, in which $p_{ij} = 1$ if trader i holds item j at the end of trading, and $p_{ij} = 0$ otherwise. In the interesting cases, P will be a permutation matrix, i.e., a zero-one matrix with row-sums and column-sums all equal to 1. In any case, the column-sums of P will be equal to 1.

Let N denote the set of all traders, and let $S \subseteq N$. By an *S-allocation* P_S, we shall mean an n-by-n zero-one matrix containing one 1 in each column indexed by a member of S, but containing only zeros in the rows and columns indexed by members of $N - S$. An S-allocation describes a distribution of goods that the "submarket" S could effect. An S-allocation with no row-sum greater

* By "item j" we mean the good brought to the market by trader j.

than 1 is called an *S-permutation.* It is clear that for every S-allocation that is not an S-permutation, there is an S-permutation that is at least as desirable to every member of S, and more desirable to at least one member of S.

An allocation will be said to be a *core allocation* if there is no submarket S that could have done better for all its members. A core allocation, therefore, is one that cannot be improved upon by "recontracting" in the sense of Edgeworth [6]. Our aim is to show that *every market of the kind described possesses at least one core allocation.*

3. GAMES AND CORES

First, let us recast the problem in a game-theoretic form. Let E^N denote the n-dimensional Euclidean space with coordinates indexed by the elements of the set N, and similarly E^S for $S \subset N$. For x, $y \in E^N$ and $S \subseteq N$, we define $y \geqq_S x$ to mean that $y_i \geqq x_i$, all $i \in S$. The notations $>_S$ and $=_S$ are defined similarly.

A "cooperative game without side payments" [1, 3, 15, 18] will be identified with its "characteristic function." This is a function V from the nonempty subsets of the "player space," N, to the subsets of the "payoff space," E^N, satisfying the following conditions for each S, $\emptyset \subset S \subseteq N$:

(a) $V(S)$ is closed.
(b) If $x \in V(S)$ and $x \geqq_S y$, then $y \in V(S)$.
(c) $[V(S) - \bigcup_{i \in S} \operatorname{int} V(\{i\})] \cap E^S$ is bounded and nonempty.

Here "int" denotes "interior of."* Note that property (b) implies that each $V(S)$ is a *cylinder* (i.e., the Cartesian product of E^{N-S} with a subset of E^S). Intuitively, the projection of $V(S)$ on E^S is supposed to represent the payoffs that the members of S, acting cooperatively, can achieve (or exceed) without outside help.

This interpretation suggests a fourth property, namely, *super-*

* Condition (c) implies, in particular, that none of the $V(S)$ is either empty or equal to E^N.

additivity, which may be expressed as

$$\text{(d)} \quad V(S \cup T) \supseteq V(S) \cap V(T) \qquad \text{if} \quad S \cap T = \phi.$$

Although this condition often holds in practice, it will not be required in the definition of a game.

Of particular interest is a class of games in which the $V(S)$ are generated by *finite* sets $Y(S) \subseteq E^N$, as follows:

$$V(S) = \{x : y \geqq_S x \text{ for some } y \in Y(S)\}.$$

Assuming that there are no superfluous generators, each payoff $y \in Y(S)$ identifies what might be called a "corner" of the cylindrical set $V(S)$.* Finitely generated games arise often in applications; they also figure prominently in the proof of the main theorem of [15].**

The *core* of the game may be defined as the set

$$V(S) - \bigcup_{\phi \subset S \subseteq N} \text{int } V(S).$$

In other words, the core is the intersection of $V(N)$ with the closures of the complements of *all* of the $V(S)$, including $V(N)$ itself. The core is a closed subset of the boundary of $V(N)$, possibly empty but certainly bounded, and every point in the core is (weakly) Pareto optimal.

Intuitively, the core consists of those outcomes of the game that are *feasible* (i.e., are in $V(N)$), and that cannot be *improved upon* by any individual or coalition of individuals (i.e., are not interior to any $V(S)$). It is of considerable interest in the analysis of any cooperative game to determine whether its core is nonempty.

Finally, we need the notion of a "balanced" game. Let us call a family T of nonempty subsets of N *balanced* if the system of equa-

---

* That is, a vertex of the projection of $V(S)$ on E^S.

A glance ahead to Figs. 1, 2, or 4 may aid the reader in visualizing the definitions of this section.

** But not in the alternative proof given in [18].

tions

$$\sum_{S:j\in S} \delta_S = 1, \qquad j \in N,$$

has a nonnegative solution with $\delta_S = 0$ for all S not in T.* A partition of N is a simple example of a balanced family. The numbers δ_S are called *balancing weights* for T; it has been shown [12] that they are unique for T if and only if no proper subfamily of T is balanced.

The game V is said to be *balanced* if the following inclusion statement:

(e) $\bigcap_{S\in T} V(S) \subseteq V(N)$,

holds for *all* balanced families T. Property (e) is obviously related to property (d) via the partitions of N, but neither condition directly implies the other. A fundamental theorem states that the core of a balanced game is not empty.** We shall apply this theorem to the market described in Sec. 2.

4. THE CORE OF THE MARKET

We now return to the market model of Sec. 2. First, we must determine the characteristic function. The sets $V(S)$ are finitely generated by the S-permutations, since all other S-allocations are dominated by S-permutations in the sense of (b) above. It will be convenient to express $V(S)$ with the aid of a zero-one matrix $B_S(x)$, defined for each $x \in E^N$ as follows:

$$b_{S|ij}(x) = \begin{cases} 1 & \text{if} \quad a_{ij} \geqq x_i \quad \text{and} \quad i \in S, \\ 0 & \text{if} \quad a_{ij} < x_i \quad \text{or} \quad i \notin S. \end{cases}$$

Thus, $B_S(x)$ tells us, for each trader i in S, exactly which items he ranks *at or above* his utility level x_i. We can then define the

* See [10, 13, 17, 18].

** See [4, 15, 17, 18] or the survey article [3].

game V as follows:

$$V(S) = \{x : B_S(x) \geqq P_S \text{ for some } S\text{-permutation } P_S\}.$$

THEOREM: *V is a balanced game; hence the market in question has a nonempty core.*

Proof: It is immediately evident that V satisfies conditions (a), (b), and (c) of Sec. 3. (We could also show without difficulty that V satisfies (d).) It remains to be shown that V satisfies (e).

Let T be any balanced family of coalitions, and let $x \in \bigcap_{S \in T} V(S)$. Let $\{\delta_S\}$ be balancing weights for T. Then we have

$$B_N(x) = \sum_{S \in T} \delta_S B_S(x).$$

By definition of $V(S)$, there exists an S-permutation P_S, for each $S \in T$, such that $B_S(x) \geqq P_S$, and so

$$\sum_{S \in T} \delta_S B_S(x) \geqq \sum_{S \in T} \delta_S P_S.$$

Call the matrix on the right D; then we have

$$B_N(x) \geqq D.$$

The crucial fact about D is that it is doubly stochastic; that is, it is nonnegative and has all row- and column-sums equal to 1. This follows directly from the definition of balancing weights; thus, the ith row sum is

$$\sum_{j \in N} \sum_{S \in T} \delta_S p_{S|ij} = \sum_{S \in T} \delta_S \sum_{j \in N} p_{S|ij}$$
$$= \sum_{S \in T} \delta_S \begin{cases} 1 & \text{if } \; i \in S \\ 0 & \text{if } \; i \notin S \end{cases} = \sum_{S : i \in S} \delta_S = 1,$$

and the argument for the column sums is the same.

The next step will be to change D into a permutation matrix P_N, that is, to eliminate any fractional entries without changing the row or column sums—and to do so without disturbing any entries that are already 0 or 1. Since all entries of $B_N(x)$ are 0 or 1, we will thereby ensure that $B_N(x) \geqq P_N$.

Since a fraction cannot occur alone in a row or column, either D is already a permutation matrix or there is a closed loop of fractional entries:

$$\begin{array}{ccccc}
d_{i_1 j_1} & \rightarrow & d_{i_1 j_2} & & \\
 & & \downarrow & & \\
 & & d_{i_2 j_2} & \rightarrow & d_{i_2 j_3} \\
 & & & & \downarrow \\
 & & & & \cdots \quad \cdots \\
 & & & & \downarrow \\
d_{i_p j_1} & \longleftarrow & & & d_{i_p j_p}
\end{array}$$

Alternately adding and subtracting a fixed number ϵ to the elements of this loop will clearly preserve row and column sums. If ϵ is too large, then negative entries will be created, but making ϵ as large as possible consistent with nonnegativity will produce a new doubly stochastic matrix D' that has at least one more zero than D, and hence *fewer* fractional entries. If D' is not yet doubly stochastic, we can repeat the operation. Eventually we must obtain what we want—a permutation matrix P_N such that $B_N(x) \geqq P_N$. Hence x is in $V(N)$. Hence $\bigcap_{S \in T} V(S) \subseteq V(N)$. Hence the game is balanced. Q.E.D.

5. AN EXAMPLE TO ILLUSTRATE THE THEOREM

Let $n = 3$, and let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This means that the first and third traders want only item 2, while the second trader wants either item 1 or item 3, indifferently.

The characteristic function, being finitely generated, can be described in terms of its "corners" as follows:*

$$\begin{aligned} V(\{1\})&: & (0, -, -),\\ V(\{2\})&: & (-, 0, -),\\ V(\{3\})&: & (-, -, 0),\\ V(\{1, 2\})&: & (1, 1, -),\\ V(\{1, 3\})&: & (0, -, 0),\\ V(\{2, 3\})&: & (-, 1, 1),\\ V(\{1, 2, 3\})&: & (1, 1, 0) \text{ and } (0, 1, 1). \end{aligned}$$

As shown in Fig. 1, this game has an L-shaped core, with successive vertices (1, 1, 0), (0, 1, 0), (0, 1, 1).

It is a curious fact that *all* outcomes in the core of this example are "weakly" improvable, in the sense that one member of an effective coalition can do better while the other member does no worse. The reader may verify that {2, 3} can weakly improve upon any point in the core except (0, 1, 1), while {1, 2} can weakly improve upon that point as well as any other point in the core except (1, 1, 0). This illustrates the fact that the "strict" improvability implicit in the definition of the core cannot be dispensed with, unless we are willing to give up the existence theorem for balanced games.

It should not be overlooked that what we are calling "outcomes" of the game do not always correspond to actual trades in the market. Indeed, there are only finitely many ways in which the goods can be reallocated. Only if one allowed some sort of free disposability, permitting the traders to diminish at will the value of the goods,** would it be possible to realize an arbitrary payoff

---

* See Sec. 3. Here, for example, (1, 1, −) means the set $\{(1, 1, x_3): x_3$ arbitrary$\}$. Note that since $V(N)$ happens to require more than one generator in this example it is a nonconvex set.

** The reader with some experience in the paradoxes of bargaining will recognize that it is not inconceivable that deliberately damaging one's goods might change the overall pattern of trade in a way that returns an advantage to the damager.

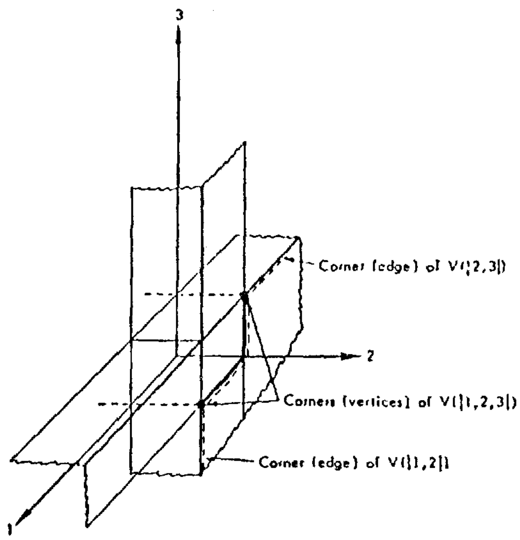

FIG. 1

vector in the "feasible" set $V(N)$. This observation does not affect the *existence* of core allocations in the market when a core exists for the game, since every "unrealizable" point in $V(N)$ is majorized by a "realizable" point. In the present example, of course, only the two tips of the "L" represent actual trades in the market.

6. COMPETITIVE PRICES

After the proof in Sec. 4 had been discovered, David Gale pointed out to the authors a simple constructive method for finding competitive prices in this market, and hence a point in its core. The following is based on his idea.

Let $R \subseteq N$, and define a *top trading cycle for R* to be any set S, $\phi \subset S \subseteq R$, whose s members can be indexed in a cyclic order:

$$S = \{i_1, i_2, \cdots, i_s = i_0\},$$

in such a way that each trader i_ν likes the $i_{\nu+1}$'st good at least as well as any other good in R. It is evident that every nonempty $R \subseteq N$ has at least one top trading cycle, for we may start with any trader in R and construct a chain of best-liked goods that eventually must come back to some earlier element.*

Using this idea, we can partition N into a sequence of one or more disjoint sets:

$$N = S^1 \cup S^2 \cup \cdots \cup S^p,$$

by taking S^1 to be any top trading cycle for N, then taking S^2 to be any top trading cycle for $N - S^1$, then taking S^3 to be any top trading cycle for $N - (S^1 \cup S^2)$, and so on until N has been exhausted. We can now construct a payoff vector x by carrying out the indicated trades within each cycle. That is, if $i = i_\nu^j \in S^j$, then x_i is i's utility for the good of trader $i_{\nu+1}^j$. We assert (1) that x, so constructed, is in the core, and (2) that a set of competitive prices exists for x.

To establish (1), let S be any coalition. Consider the first j such that $S \cap S^j \neq \phi$. Then we have

$$S \subseteq S^j \cup S^{j+1} \cup \cdots \cup S^p = N - (S^1 \cup \cdots \cup S^{j-1}).$$

Let $i \in S \cap S^j$. Then i is already getting in x the highest possible payoff available to him in S. No improvement is possible for him, unless he deals outside of S. Hence S cannot strictly improve, and it follows that x is in the core.

To establish (2), we merely assign arbitrary prices

$$\pi^1 > \pi^2 > \cdots > \pi^p > 0$$

to the goods belonging to the respective cycles $S^1, S^2, \cdots, S^p$. Then trader i in S^j can sell his own item for π^j "dollars." He cannot afford any items from $S^1, \cdots, S^{j-1}$, and so his utility is maximized if he buys the item of his cyclic successor in S^j. This, of course, costs him precisely π^j "dollars" and yields him the payoff x_i.

Not surprisingly, given the discreteness of the model, the condi-

---

* A top trading cycle may consist of a single trader!

tions on the prices are purely ordinal. To the extent that there may be different ways of constructing top trading cycles, nonuniqueness may occur in the final outcome as well as in the price ordering. It is easily seen, however, that there are no other competitive prices beyond those constructed in the above fashion, except that when two or more *disjoint* top trading cycles exist, at any stage of the construction, they may be assigned equal prices.

In the example of Sec. 5, either {1, 2} or {2, 3} will serve as the first top trading cycle S^1, so that we can use either $\pi_1 = \pi_2 > \pi_3$ or $\pi_3 = \pi_2 > \pi_1$ for the competitive prices. The corresponding competitive outcomes are the two tips of the L-shaped core (Fig. 1). Note that $\pi_1 = \pi_2 = \pi_3$ would *not* be competitive, as the demand for item 2 would then exceed the supply. Thus, we see from this example that the set of competitive prices is not necessarily closed.

7. ANOTHER EXAMPLE

It may be wondered in models of this type whether the core is really more general than the competitive solution. In the previous example the core of the *game* contained many "outcomes" in addition to the competitive outcome, but none of them was realizable in the *market*, given the indivisibility and undamageability of the goods. One might suspect that every *core allocation* is necessarily competitive, after all.

A counterexample is provided, however, by the following preference matrix:

goods

	0	1	2
traders	1	0	−1
	−1	1	0

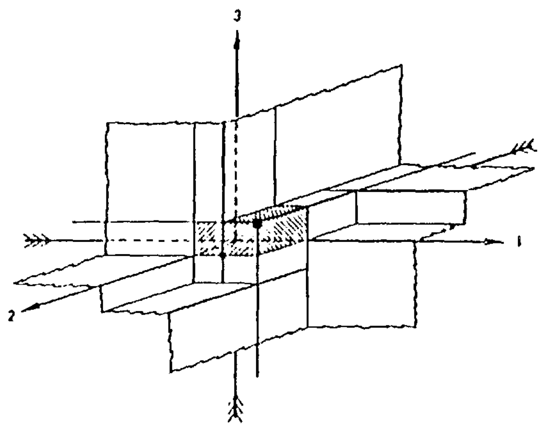

FIG. 2

The characteristic function is depicted in Fig. 2; the point marked ■ is the unique "corner" of $V(N)$. Here there is a unique top trading cycle, namely N itself, so the competitive prices are all equal and the unique competitive payoff is (2, 1, 1), resulting from the N-permutation

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This is of course the point ■. The core of the game evidently consists of the three shaded rectangles grouped around that point, and we see that it contains the point ● = (1, 1, 0) that results from the N-permutation

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is *not* a competitive allocation.

This example is not completely satisfying, since the noncompetitive outcome ● in the core is weakly majorized by the competitive outcome ■. Perhaps a better example exists, with more traders. However, as we have already remarked, we cannot base the definition of the core on weak majorization without losing the fundamental existence theorem.

8. MORE COMPLEX PREFERENCES: A COUNTEREXAMPLE

Let three traders have symmetric holdings in a tract of nine houses, as shown in Fig. 3. (Thus, trader 1 owns houses 1, 1′, and 1″.) For reasons inscrutable, each trader wants to acquire three houses in a row, including exactly one of his original set. Moreover, each prefers the long row that meets this condition to the short row.

We shall show that this example of a slightly more general trading game than the preceding is not balanced, and indeed has no core—thereby dispelling any idea that the core might prove a universal remedy for market failure due to indivisibility.

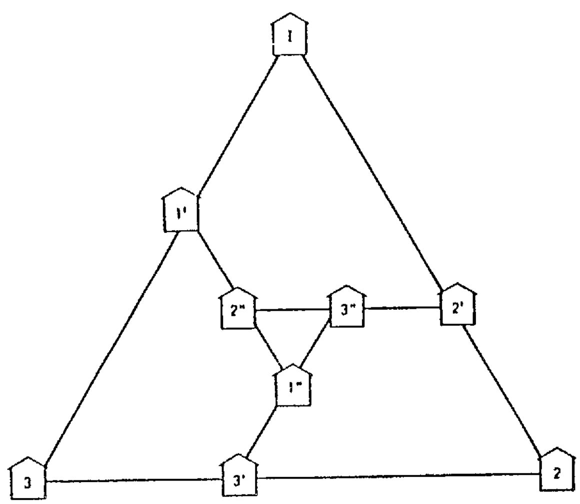

FIG. 3

The configuration of the tract is such that any two traders can make a profitable exchange. For example, a swap of 1′ and 1″ for 2 and 2′ gives trader 1 his long row and trader 2 his short row. Let us assign numerical values 2, 1, and 0 to the possession of the long row, the short row without the long row, and neither row, respectively. Then the two-person coalitions have single "corners":

$$V(\{1, 2\}): \quad (2, 1, -),$$
$$V(\{1, 3\}): \quad (1, -, 2),$$
$$V(\{2, 3\}): \quad (-, 2, 1),$$

as shown in Fig. 4. The three-person coalition cannot improve upon these pairwise exchanges; so its characteristic function is generated

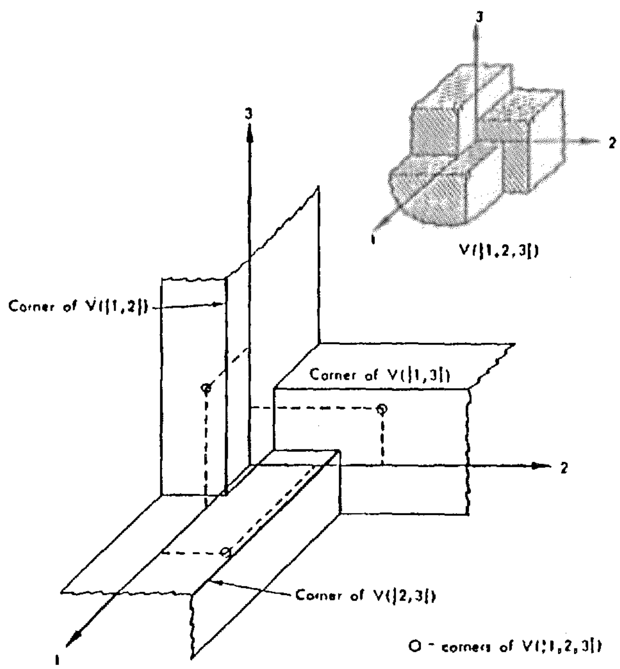

Fig. 4

by three "corners":

$$V(\{1, 2, 3\}): \quad \begin{cases} (2, 1, 0), \\ (1, 0, 2), \\ (0, 2, 1), \end{cases}$$

as shown in the inset. Of course, the three singleton coalitions can achieve only 0.

It is easy to see that this game is *not* balanced (condition (e) in Sec. 3), since the point (1, 1, 1) is in $V(\{1, 2\}) \cap V(\{1, 3\}) \cap V(\{2, 3\})$, but not $V(\{1, 2, 3\})$. It is equally easy to see that the game has no core,* since each of the generators of $V(\{1, 2, 3\})$ is interior to one of the $V(\{i, j\})$. In other words, the set $V(\{1, 2, 3\})$ is completely hidden by the union of the $V(\{i, j\})$, as suggested by the broken lines in the main figure.

It might be asked whether the absence of a core in this example might not arise from some intrinsic nonconvexity in the preference sets themselves that has nothing to do with indivisibility. The answer is no; to see this, define

$$U^1(x) = \min[2, \min(x_1, x_2, x_{2'}) + 2 \min(x_{3''}, x_{1''}, x_{3'})],$$

with $U^2(x)$ and $U^3(x)$ similar. These utility functions are concave in the nine real variables $x_1, \cdots, x_{3''}$ (since taking a "min" preserves concavity), and so generate convex preference sets. Moreover, if we restrict the variables to be 0 or 1 we obtain just the payoffs used above. Thus, the actual preference sets are nonconvex *only* because of indivisibility.

9. OTHER INDIVISIBILITY MODELS

Several other trading models with indivisible goods have been considered in the literature from the point of view of the core. They range from fairly general situations for which positive results can be obtained to specific illustrative counterexamples.

---

* We remind the reader that balancedness is sufficient, but not necessary, for a nonempty core.

In the "marriage market" or "dance floor" of [9], there are two *types* of traders. The members of each type rank those of the other type in order of preference as partners; then they pair off. There are generally many core allocations for this situation, that is, arrangements into pairs such that no two individuals of opposite type could do better. Curiously enough, it may be that none of these core allocations gives anyone his (or her) first choice. (In contrast, the "top trading cycle" construction of Sec. 6 obviously ensures that at least one trader gets his first choice.) A simple "courtship" algorithm is described in [9] for reaching a point in the core.* In fact, two extremal core allocations are reached: one gives every member of the first type the best outcome possible within the core; the other does the same for the second type. These two allocations coincide only in the case of a one-point core.**

It does not appear to be possible to set up a conventional market for this model, in such a way that a competitive price equilibrium will exist and lead to an allocation in the core.

A very similar model is the "problem of the roommates," also discussed in [9]. The difference is that there are no types; each trader ranks *all* the others as potential partners before they pair off. In this case, a very simple four-person example shows that there need not be a core.***

---

* The deferred acceptance algorithm of [9] is easily described: (1) Let each boy propose to his favorite girl; (2) let each proposed-to girl keep her best suitor waiting and reject all others; (3) let each rejected boy propose to his next-best girl; (4) repeat steps (2) and (3) until either there are no rejected boys left or every rejected boy has exhausted the list of girls. The resulting pairing-off corresponds to a point in the core, and no boy can do better at any other point in the core.

See also *The New Yorker*, Sept. 11, 1971, p. 94.

** Similar results hold for the more general "college admissions market," in which each trader of the first type can accommodate a large number of traders of the second type; see [9].

*** Let A rank $B > C > D$; let B rank $C > A > D$; let C rank $A > B > D$; and let D rank arbitrarily. Then no pairing is stable, in the sense of the core. For example, (AB) (CD) can be improved upon by the coalition $\{B, C\}$.

In the "treasure hunt" of [23],* a party of explorers finds a big cache of treasure chests in the desert, momentarily exposed during a sandstorm, but so heavy that it takes two men to carry each chest to high ground. If we consider the bargaining game that determines how the profits are to be divided, it is not difficult to see that the existence of a core depends on whether the size of the party is even or odd. Similarly, in the "Bridge game" economy of [24], an exact multiple of four players is required, if the card party is to have a core.

In most of these examples, the real issue is the indivisibility of the participants themselves, rather than the indivisibility of some more or less tangible economic commodity that is owned and is transferable. The individual is required to participate fully and exclusively in a single activity in order to have any effect. Thus, these examples could also be regarded as instances of increasing returns to scale in the labor inputs to certain production processes, or even as nonconvexities in the preferences for certain forms of consumption.**

In the "assignment game" of [22],*** there are again two types of traders, namely, *sellers* and *buyers*. The first have houses, say, and the second have money. Preferences are not merely ordinal, but are expressed as monetary evaluations of the different houses. Since money is fully transferable and infinitely divisible, the sets $V(S)$ are not finitely generated; instead they are half-spaces, and the market reduces to a "game with side payments." Competitive prices exist and are closely related to the linear-programming solution to the problem of maximizing the total monetary value of the allocation, which turns out to be the familiar "optimal assignment" problem. These prices are usually not unique. The core in this case is exactly the set of competitive allocations, in contrast to what we

* Inspired by the Huston-Traven classic *Treasure of the Sierra Madre*, whose plot turns on the coalitional instability of a party of *three* prospectors in a rather similar predicament.

** Compare the "gin and tonic" example in [19], where again the existence of a core depends on the parity of the number of traders.

*** See also [8], pp. 160–162.

found in Sec. 7 above. As in the marriage market, two extremal allocations can be distinguished in the core: a "high-price" corner, which is best possible for every seller, and a "low-price" corner, which is best possible for every buyer. If the houses happen to be all alike, then the core reduces to a line segment and at any given point in the core all houses have the same price.

It would be interesting if a general framework could be found that would unify some or all of these scattered results.

REFERENCES

1. Aumann, R. J., and B. Peleg, "Von Neumann-Morgenstern solutions to cooperative games without side payments," *Bull. Am. Math. Soc.*, 66 (1960), 173–179.
2. Billera, L. J., "Some theorems on the core of an n-person game without side payments," *SIAM J. Appl. Math.*, 18 (1970), 567–579.
3. ———, "Some recent results in n-person game theory," *Math. Programming*, 1 (1971), 58–67.
4. Bondareva, O. N., "Some applications of linear programming methods to the theory of cooperative games," (Russian), *Problemy Kibernet.* 10 (1963), 119–139.
5. Debreu, G., and H. E. Scarf, "A limit theorem on the core of an economy," *Intern. Econ. Rev.*, 4 (1963), 235–246.
6. Edgeworth, F. Y., *Mathematical Psychics*, London: Kegan Paul, 1881.
7. Foley, D. K., "Lindahl's solution and the core of an economy with public goods," *Econometrica*, 38 (1970), 66–72.
8. Gale, David, *The Theory of Linear Economic Models*, New York: McGraw-Hill, 1960.
9. Gale, D., and L. S. Shapley, "College admissions and the stability of marriage," *Amer. Math. Monthly*, 69 (1962), 9–15.
10. Graver, J. E., "Maximum depth problem for indecomposable exact covers," (mimeographed, Syracuse University), paper presented at the *International Colloquium on Infinite and Finite Sets*, June 1973, Keszthely, Hungary.
11. Henry, Claude, "Indivisibilités dans une économie d'échanges," *Econometrica*, 38 (1970), 542–558.
12. ———, "Market games with indivisibilities and non-convex preferences," *Rev. Econ. Studies*, 39 (1972), 73–76.
13. Peleg, Bezalel, "An inductive method for constructing minimal balanced collections of finite sets," *Naval Res. Logist. Quart.*, 12 (1965), 155–162.
14. Rader, J. T., "Resource allocation with increasing returns to scale," *Am. Econ. Rev.*, 60 (1970), 814–825.

15. Scarf, H. E., "The core of an n-person game," *Econometrica*, 35 (1967), 50–69.
16. ———, "Notes on the core of a productive economy," unpublished.
17. Shapley, L. S., "On balanced sets and cores," *Naval Res. Logist. Quart.*, 14 (1967), 453–460.
18. ———, "On balanced games without side payments," in *Mathematical Programming* (T. C. Hu and S. M. Robinson, eds.), New York: Academic Press, 1973, pp. 261–290.
19. Shapley, L. S., and M. Shubik, "Quasi-cores in a monetary economy with nonconvex preferences," *Econometrica*, 34 (1966), 805–827.
20. ———, "On market games," *J. Econ. Theory*, 1 (1969), 9–25.
21. ———, "On the core of an economic system with externalities," *Am. Econ. Rev.*, 59 (1969), 678–684.
22. ———, "The assignment game I: The core," *Intern. J. Game Theory*, 1 (1972), 111–130.
23. ———, *Competition, Welfare, and the Theory of Games*, unpublished.
24. Shubik, Martin, "The 'Bridge Game' economy: An example of indivisibilities," *J. Pol. Econ.*, 79 (1971), 909–912.
25. Starr, R. M., "Quasi-equilibria in markets with nonconvex preferences," *Econometrica*, 37 (1969), 25–38.