COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 272

ON THE APPLICATIONS OF A RECENT COMBINATORIAL ALGORITHM

Terje Hansen and Herbert Scarf

April 14, 1969

ON THE APPLICATIONS OF A RECENT COMBINATORIAL ALGORITHM

by

Terje Hansen* and Herbert Scarf**

I. Introduction

In a recent series of papers [7, 8, 9], one of the authors has described an algorithm for a class of problems in mathematical economics and programming which had not previously been examined from a computational point of view. The present paper is meant to serve several purposes: to summarize those applications of the algorithm which are presently known, to suggest some computational simplifications for an important special case and to compare the algorithm with similar procedures suggested by Hansen [3], Cohen [1], Kuhn [5], and Shapley [10].

The algorithm is based upon the combination of two apparently unrelated considerations. One is the remarkable technique devised by Lemke and Howson [6] for the numerical calculation of Nash equilibrium points for a two person non-zero sum game. The second component involves replacing the familiar notions of a feasible basis and a pivot step, as they are used in linear programming, with an alternative construction known as a primitive set.

---

* Norwegian School of Economics and Business Administration.

**The research described in this paper was carried out under grants from the National Science Foundation and from the Ford Foundation. We would like to acknowledge our appreciation for a number of stimulating conversations with T.C. Koopmans, H.W. Kuhn, and Donald Richter.

In order to review the definition of a primitive set, let $\Pi = (\pi^1, \ldots \pi^k)$ be a collection of vectors in n dimensional Euclidian space. The first n members of Π are assumed to have the form

$$\pi^1 = (0, M_1, \ldots M_1)$$

$$\pi^2 = (M_2, 0, \ldots M_2)$$

$$\vdots$$

$$\pi^n = (M_n, M_n, \ldots 0)$$

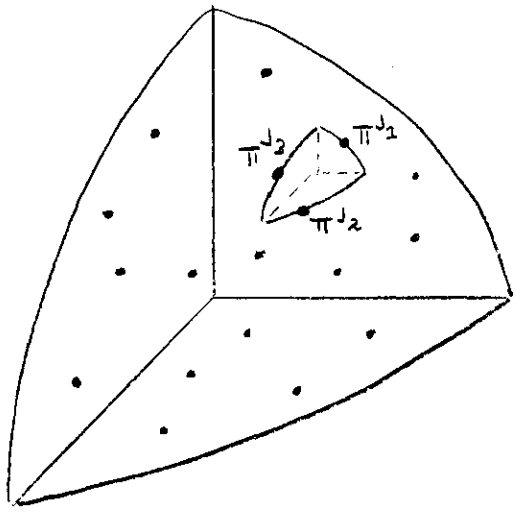
and the remaining vectors are arbitrary aside from the stipulation that $0 \leq \pi_i^j \leq M_i$ for $i = 1, \ldots n$, and for all $j > n$. As we shall see in our subsequent discussion the first n vectors in Π play a role analogous to the slack variables of a linear programming problem.

In formulating the definition of a primitive set it is convenient to adopt the notation that if $x, y, \ldots z$ are vectors, then $\min(x, y, \ldots z)$ is the vector each of whose coordinates is the minimum of the corresponding coordinates of $x, y, \ldots z$.

Definition: A set of n distinct vectors $\pi^{j_1}, \ldots \pi^{j_n}$ is defined to be a primitive set if there are no vectors $\pi^j \in \Pi$ with $\pi^j > \min(\pi^{j_1}, \ldots \pi^{j_n})$.

The definition has an obvious geometrical interpretation as the following figure indicates. In this example the vectors π^{j_1}, π^{j_2},

and π^{j_3} form a primitive set since there are no vectors in Π interior to that translate of the non-negative orthant whose origin is shifted to the vector $\min(\pi^{j_1}, \pi^{j_2}, \pi^{j_3})$.



The basic combinatorial theorem, whose proof will be reviewed in Section III, may now be stated.

Theorem 1. Let

$$A = \begin{bmatrix} 1 & 0 \ldots 0 & a_{1,n+1} & \cdots & a_{1,k} \\ 0 & 0 \ldots 1 & a_{n,n+1} & \cdots & a_{n,k} \end{bmatrix}$$

be a matrix and $b = (b_1, \ldots b_n)'$ a non-negative vector, such that the j^{th} column of A is associated with the j^{th} vector in Π. Assume that the set of non-negative vectors x, satisfying $Ax = b$, is bounded.

Then there exists a primitive set $\pi^{j_1}, \ldots \pi^{j_n}$, so that the columns $j_1, \ldots j_n$ form a feasible basis for $Ax = b$.

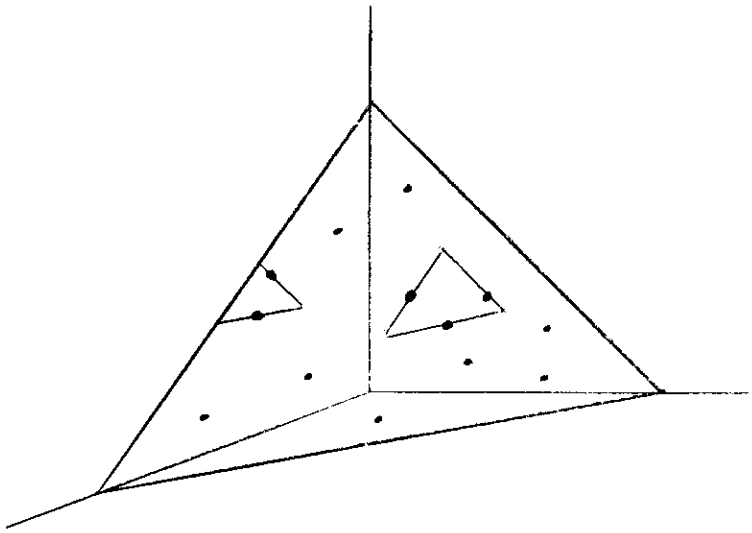
II. Some Applications of Theorem 1

In this section, six distinct applications of the algorithm underlying Theorem 1, will be described. Each of these examples involves selecting a set Π and establishing a correspondence between the vectors π^j and the columns of the matrix A in a way consistent with the goal of the problem.

In the first five examples the vectors π^j for $j = n+1, \ldots k$ will be assumed to lie on the unit simplex $S = \{\pi | \sum_1^n \pi_i = 1, \pi_i \geq 0\}$, a specialization which permits a geometrical interpretation somewhat easier to visualize than the general case. As the following figure indicates, a primitive set then corresponds to a subsimplex of S with sides parallel to the coordinate hyperplanes, each passing through a specific vector in the primitive set, and with no vectors in Π interior to the subsimplex. In the event that a vector π^j with $j \leq n$ is one of the members of the primitive set, the corresponding side of the subsimplex lies in that coordinate hyperplane whose j^{th} coordinate is equal to zero.

Moreover, in these five examples, the only property of primitive sets to be used in the application of Theorem 1, is that if k is large, and if the vectors in Π are distributed with some regularity throughout

the simplex S, then the n vectors in a primitive set must be close to each other, and have their j^{th} coordinates close to zero if π^j, for $j \leq n$, is a member of the primitive set. Both the sixth application, and



the algorithm underlying Theorem 1, make use of a variety of other properties associated with this concept.

1. Let $f_1(\pi), \ldots f_n(\pi)$, with $f_i(\pi) \geq 0$, and $\sum_1^n f_i(\pi) \equiv 1$, be the image of π under a continuous mapping of the simplex S into itself. In order to determine an approximate fixed point of the mapping, the vectors π^j, with $j > n$, are selected with some regularity throughout the simplex, possibly with a higher density in a region in which an approximate fixed point is expected to lie.

The column of A associated with π^j, for $j > n$, will consist of $n-1$ zeros, and a single one, with the entry one located in some row i with $f_i(\pi^j) \geq \pi_i^j$. The vector b is an arbitrary positive vector.

The hypotheses of Theorem 1, are clearly satisfied and we conclude that there exists a primitive set $\pi^{j_1}, \ldots \pi^{j_n}$ such that the columns $j_1, \ldots j_n$ form a feasible basis for $Ax = b$. Since for this example, the columns of a feasible basis form a unit matrix, if T represents the collection of indices referring to slack vectors in the primitive set then for $i \notin T$, there is a vector π^j in the primitive set with $f_i(\pi^j) \geq \pi_i^j$. In other words for every index i, there is some vector in the subsimplex corresponding to the primitive set, whose i^{th} coordinate is non-decreasing under the mapping. Any vector in the geometric subsimplex will therefore serve as an approximate fixed point, in the sense that it is close to its image. The degree of the approximation may be improved by selecting k very large, or by taking a local linear approximation to the mapping.

Brouwer's theorem, itself, follows by a simple passage to the limit. It is interesting to note, from a pedagogical point of view, that this proof avoids the tedious construction of a simplicial subdivision of the simplex.

2. For our second application, let $\varphi(\pi)$ be an upper semi-continuous, convex, point to set mapping of the simplex S into itself. Kakutani's theorem states that there exists a vector π^* which is a fixed point in the sense that it is contained in $\varphi(\pi^*)$.

In order to obtain an approximate fixed point by means of Theorem 1, we select the set Π as before. The column of the matrix A associated with a vector π^j, with $j > n$ will be given by

$$(q_1^j - \pi_1^j + 1, \ \ldots \ q_n^j - \pi_n^j + 1)' ,$$

where $q^j = (q_1^j, \ \ldots \ q_n^j)$ is an arbitrary vector in $\varphi(\pi^j)$. If the vector $b = (1, \ \ldots \ 1)'$, the hypotheses of Theorem 1 are clearly satisfied, and we may conclude that there exists a primitive set $\pi^{j_1}, \ \ldots \ \pi^{j_n}$, such that the columns $j_1, \ \ldots \ j_n$ form a feasible basis for $Ax = b$. The algorithm underlying Theorem 1 will therefore provide us with non-negative x_j's satisfying

$$\Sigma(q_i^j - \pi_i^j + 1)x_j \leq 1 ,$$

with strict equality unless π^i belongs to the primitive set, and with all of the π^j's corresponding to positive x_j's members of the primitive set.

Any vector in the geometric subsimplex corresponding to this primitive set will serve as an approximate fixed point in the sense that it will be close to a vector π which is in turn close to its image $\varphi(\pi)$. Rather then providing precise bounds for the degree of approximation, let us pass to the limit by considering an ever finer sequence of grids on the simplex and by selecting a subsequence for which the vectors in the primitive

set, the q's associated with these vectors, and the corresponding x's, all converge. The non-slack vectors of the primitive set will tend to a single vector π^*, and the upper-semi-continuity of the mapping guarantees that the limiting vectors q^j associated with the limiting positive x_j's are all in $\varphi(\pi^*)$. The equations $Ax = b$ become,

$$\sum_j (q_i^j - \pi_i^* + 1)x_j \leq 1 ,$$

with equality if $\pi_i^* > 0$.

Kakutani's theorem will follow, if we can demonstrate that $\Sigma x_j = 1$. To see this let us first add those equations for which $\pi_i^* > 0$, and equality holds, obtaining

$$\sum_j \sum_{\pi_i^*>0} (q_i^j - \pi_i^* + 1)x_j = N ,$$

with N the number of positive coordinates in π^*. But

$$\sum_{\pi_i^*>0} (q_i^j - \pi_i^* + 1) = \sum_{\pi_i^*>0} q_i^j + N - 1$$

$$\leq N , \quad \text{so that}$$

$$N\sum_j x_j \geq N \quad \text{and} \quad \sum_j x_j \geq 1 .$$

To obtain the reverse inequality, let us notice that if all $\pi_i^* > 0$, the above inequalities are in fact equalities and produce the

desired result. If, on the other hand some component of π^* is zero, say the first component, then the inequality

$$\sum_j (q_1^j - \pi_1^* + 1)x_j \leq 1$$

becomes

$$\sum_j (q_1^j + 1)x_j \leq 1 \ , \quad \text{from which}$$

we immediately derive the statement $\Sigma x_j \leq 1$.

If we now define $q^* = \sum_j x_j q^j$, then $q^* \in \varphi(\pi^*)$ from the convexity of this set, and moreover $q^* \leq \pi^*$, so that $q^* = \pi^*$, thereby demonstrating Kakutani's theorem.

Since the selection of a convergent subsequence is not possible in an actual application of Theorem 1, we must have recourse to approximate techniques, which have in fact worked very well in practice. It is interesting to note, from a theoretical point of view, that Theorem 1 provides a proof of Kakutani's theorem based on a single limiting process, rather than the double limit which is to be found in the customary proof.

3. Our third application of Theorem 1 involves a technique used by Debreu [2] and others to demonstrate the existence of a competitive equilibrium. Let $\varphi(\pi)$ be an upper semi continuous convex mapping of the simplex S into some bounded set Z , and assume that for each π and each $q \in \varphi(\pi)$ we have $\pi \cdot q \leq 0$. The theorem then concludes that there

exist π^* and $q^* \in \varphi(\pi^*)$ such that $q^* \leq 0$.

This result can be demonstrated by an appeal to Kakutani's theorem, so that the technique of the previous example may be used. But in this proof, Kakutani's theorem is applied in a space of dimension $2n$, which increases the computational burden substantially if an actual approximation is desired. On the other hand, Theorem 1 may be applied directly to this problem without increasing the dimensionality of the space.

Take Π as in the previous examples, and let b be a positive vector such that $b + Z > 0$. Then associate with the vector π^j for $j > n$, a column in the matrix A given by $(q_1^j + b_1, \ldots q_n^j + b_n)'$ with q^j an arbitrary vector in $\varphi(\pi^j)$. Again, the hypotheses of Theorem 1 are satisfied and we obtain a primitive set $\pi^{j_1}, \ldots \pi^{j_n}$ such that the columns $j_1, \ldots j_n$ form a feasible basis for $Ax = b$.

As in the previous example, the grid size is successively refined and a subsequence is selected so that the vectors in the primitive sets, the feasible bases and their associated activity levels, and the vectors q^j associated with the columns of the feasible basis all converge. We denote by π^* the common limit of the non-slack vectors in the primitive set. As a consequence of the upper semi-continuity of the mapping all of the limiting q^j's associated with non-slack vectors in the limiting feasible basis will be in $\varphi(\pi^*)$. The equations $Ax = b$, become

$$\Sigma(q_i^j + b_i)x_j \leq b_i$$

with equality if $\pi_i^* > 0$. If we multiply the i^{th} inequality by π_i^* and sum over i we obtain

$$\Sigma\Sigma\pi_i^*(q_i^j + b_i)x_j = \Sigma\pi_i^* b_i \quad \text{and}$$

since $\Sigma\pi_i^* q_i^j \leq 0$ by assumption we have

$$\Sigma\Sigma\pi_i^* b_i x_j \geq \Sigma\pi_i^* b_i \quad \text{or}$$

$$\Sigma x_j \geq 1 \ .$$

The inequalities

$$\Sigma(q_i^j + b_i)x_j \leq b_i \ , \quad \text{therefore}$$

imply

$$\Sigma_j q_i^j x_j \leq b_i(1 - \Sigma x_j) \leq 0 \ ,$$

and

$$q^* = \Sigma_j q^j x_j / \Sigma x_j \ ,$$

which is in $\varphi(\pi^*)$, will satisfy the conclusions of the theorem.

4. The next application of Theorem 1, described in detail in [9], involves the approximation of equilibrium prices in a general equilibrium model with an activity analysis formulation of production. It is well known that the existence of equilibrium prices can be demonstrated

either by Kakutani's theorem or by the theorem of the previous example. Both of these techniques, however, require the calculation of the market demand functions on each iteration of the basic algorithm, a step which may be expensive in terms of computer time, and which is avoided for a majority of the iterations if the procedure to be described is followed. On the other hand, Kakutani's theorem need only be applied in a space of dimension equal to the number of commodities actually appearing in the consumers' demand functions, and if this is a relatively small proportion of the total number of commodities, one of the previous techniques may be preferable. Some comparisons are examined in Hansen's thesis [3].

Let the stock of commodities prior to production be given by the vector w, for convenience, assumed to be positive. The market demand functions $\xi_1(\pi), \ldots \xi_n(\pi)$ are non-negative, continuous on the simplex S, and satisfy the identity $\pi \cdot \xi(\pi) \equiv \pi \cdot w$ for all price vectors.

Production will be described by an activity analysis matrix

$$B = \begin{bmatrix} -1 & \cdots & 0 & \cdots & b_{1j} & \cdots \\ \vdots & & & & \vdots & \\ 0 & \cdots & -1 & & b_{nj} & \cdots \end{bmatrix}$$

in which each column represents a possible production activity, inputs being represented by negative numbers and outputs by positive numbers. We make the assumption that $\{x | x \geqq 0,\ Bx + w \geqq 0\}$ is bounded an assumption quite similar to one customarily made in this area of study, in order to guarantee the impossibility of producing without inputs.

A competitive equilibrium is a price vector π and a set of activity levels x, such that

a. $\xi(\pi) = w + Bx$

b. for every column b_j of B, $\pi \cdot b_j \leq 0$, with equality if $x_j > 0$.

In order to apply Theorem 1 to the determination of approximate equilibrium prices, let Π be as before, and associate with π^j, for $j > n$, a column of the matrix A in the following fashion. First find an activity b_ℓ in B which maximizes $\pi^j \cdot b_\ell$. If this maximum is positive, then the j^{th} column of A is given by $-b_\ell$. On the other hand if the maximum profit at the prices π^j is ≤ 0, the j^{th} column will be $(\xi_1(\pi), \ldots \xi_n(\pi))'$. Finally we define the vector b to be equal to w.

It is easy to see that Theorem 1 is applicable, and that we may find a primitive set $\pi^{j_1}, \ldots \pi^{j_n}$ such that the columns $j_1, \ldots j_n$ correspond to a feasible basis for $Ax = b$. We have, in other words, a non-negative solution to the equations

$$\Sigma y_j \xi_i(\pi^j) = w_i + \Sigma x_j b_{ij},$$

with the x_j and $y_j > 0$ only for those columns corresponding to the vectors in the primitive set. (Here we are using y_j to refer to those com-

ponents of the x vector corresponding to a column of demands.) Again we pass to the limit, obtaining a common price π^*, and a non-negative solution to the equations

$$y \cdot \xi_i(\pi^*) = w_i + \Sigma x_j b_{ij} .$$

If we recall the construction of the matrix A, we see that $\Sigma \pi_i^* b_{ij} \geq 0$ for those $x_j > 0$, and moreover $y > 0$ implies that $\Sigma \pi_i^* b_{i\ell} \leq 0$ for <u>all</u> productive techniques. We cannot have $y = 0$, since then $0 = \pi^* \cdot w + \Sigma x_j \pi^* \cdot b_j > 0$, which is impossible. Therefore $\pi^* \cdot b_j = 0$ for those j with $x_j > 0$. Finally if we multiply the above equations by π_i^* and sum we have

$$y \, \Sigma \pi_i^* \xi_i(\pi^*) = \Sigma \pi_i^* w_i ,$$

so that $y = 1$. The price vector π^* and the activity levels x therefore represent a competitive equilibrium.

5. Our fifth application involves the approximation of the optimal solution of a concave programming problem given by

$$\max \quad g(\pi)$$

$$\text{subject to} \quad f_k(\pi) \leq 0 , \quad k = 1, \ldots m ,$$

$$\pi \geq 0 .$$

$g(\pi)$ is assumed to be concave and the functions $f_k(\pi)$ are assumed to be

convex. All functions are assumed to be twice differentiable. For technical reasons that will become apparent later π_1 is a dummy variable such that

$$\frac{\partial g}{\partial \pi_1} = \frac{\partial f_1}{\partial \pi_1} = \ldots = \frac{\partial f_m}{\partial \pi_1} = 0$$

for any π .

We further assume that the feasible region is bounded in the sense that no non-negative vector π with $\pi_1 = 0$ and whose coordinates sum to 1 is in the feasible region. Furthermore we assume that there exists a non-negative vector π^0 whose coordinates sum to 1 satisfying $f_k(\pi^0) < 0$ for all k .

Now a necessary and sufficient condition for π^* to represent the optimal solution of the above programming problem is that there exists a non-negative vector λ^* such that

$$\frac{\partial g}{\partial \pi_i^*} - \Sigma \lambda_k^* \frac{\partial f_k}{\partial \pi_i^*} \leq 0 ,$$

with equality if $\pi_i^* > 0$, and

$$f_k(\pi^*) \leq 0 ,$$

with equality if $\lambda_k^* > 0$.

In order to apply Theorem 1 to the determination of an approximate solution of the above programming problem we associate with π^j , for $j > n$, a column of the matrix A in the following way. If $f_k(\pi^j) \leq 0$

for all k, then the j^{th} column of A is given by $\left(\frac{\partial g}{\partial \pi_1^j} + 1, \ldots, \frac{\partial g}{\partial \pi_n^j} + 1 \right)'$. If $f_k(\pi^j) > 0$ for some k then the j^{th} column of A will be $\left(-\frac{\partial f_k}{\partial \pi_1^j} + 1, \ldots, -\frac{\partial f_k}{\partial \pi_n^j} + 1 \right)'$. Finally we define the vector b to be equal to $(1, \ldots, 1)'$.

Since the first row of A consists of 1's (aside from the columns referring to slack vectors) the x's are bounded and Theorem 1 is obviously applicable. There is therefore a primitive set $\pi^{j_1}, \ldots, \pi^{j_n}$ such that the columns $j_1, \ldots, j_n$ form a feasible basis for $Ax = b$. We have consequently a non-negative solution to the equations

$$y_i + \Sigma y_j \left(\frac{\partial g}{\partial \pi_i^j} + 1 \right) + \Sigma x_j a_{ij} = 1$$

with a_j a column of A associated with the derivatives of the f functions and with y_i, y_j and $x_j > 0$ only for the columns corresponding to the vectors in the primitive set. (Here we are using y_i to refer to that component of the x vector corresponding to the i^{th} slack vector and similarly y_j for those components associated with the derivatives of the objective function.) Again we pass to the limit obtaining a common vector π^*, and a non-negative solution to the equations

$$y_i + y \left(\frac{\partial g}{\partial \pi_i^*} + 1 \right) + \Sigma x_k \left(-\frac{\partial f_k}{\partial \pi_i^*} + 1 \right) = 1 .$$

If we recall the construction of the matrix A we have that $y > 0$ implies that the vector π^* satisfies all the constraints and that $x_k > 0$ implies that $f_k(\pi^*) \geq 0$. We shall now argue that $y > 0$ and that $y + \Sigma x_k = 1$; the desired result will then follow.

If $y = 0$, and we multiply the i^{th} equation by $\pi_i^0 - \pi_i^*$ and sum, we obtain

$$\Sigma y_i(\pi_i^0 - \pi_i^*) + \Sigma x_k \Sigma(\pi_i^0 - \pi_i^*)\left(-\frac{\partial f_k}{\partial \pi_i^*}\right) = \Sigma(\pi_i^0 - \pi_i^*) = 0 .$$

This will lead to a contradiction if we can show that both terms on the left hand side are non-negative, with the second strictly positive. But the first term is surely non-negative, since if $y_i > 0$ then $\pi_i^* = 0$. In order to see that the second term is positive, we observe that if $x_k > 0$ (which must be true for at least one k), then $f_k(\pi^*) \geq 0$, and from the convexity assumption

$$0 > f_k(\pi^0) - f_k(\pi^*) \geq \Sigma(\pi_i^0 - \pi_i^*)\frac{\partial f_k}{\partial \pi_i^*} .$$

This demonstrates that $y > 0$, and as a consequence, that all of the constraints are satisfied by π^*.

To complete the argument observe that $\Sigma x_k + y = 1$ unless $y_1 > 0$. If $y_1 > 0$, however, $\pi_1^* = 0$ and by assumption π^* does not satisfy all of the constraints, a contradiction.

If we define $\lambda_k^* = x_k/y$, then

$$\frac{\partial g}{\partial \pi_i^*} - \Sigma \lambda_k^* \frac{\partial f_k}{\partial \pi_i^*} \leq 0 ,$$

with equality if $\pi_i^* > 0$ and

$$f_k(\pi^*) \leq 0$$

with equality if $\lambda_k^* > 0$.

6. The final application to be discussed was the first to be discovered and originally motivated the development of Theorem 1. In this example the algorithm provides a set of sufficient conditions for the core of an n person game to be non empty, and a procedure for determining an approximate vector in the core. While it is possible to demonstrate the sufficiency of these conditions by the use of Kakutani's theorem, the proof is very awkward and the resulting algorithm quite inefficient, when compared to a direct application of Theorem 1.

For each subset S of the set of integers $N = (1, 2, \ldots n)$, let E^S denote that Euclidean space of dimension equal to the cardinality of S and whose coordinates correspond to the elements of S. If $u \in E^N$, its projection to E^S will be denoted by u^S.

A cooperative n person game is described by associating with each coalition S, a set of utility vectors V_S, which can be achieved by that coalition. The sets V_S are assumed to have the following properties:

1. V_S is a closed, non empty subset of E^S.
2. If $u \in V_S$ and $v \in E^S$ with $v \leq u$, then $v \in V_S$.
3. For each individual i, $V_{(i)} = \{u_i | u_i \leq 0\}$.
4. The set of non-negative vectors in V_N is bounded.

A utility vector $u \in E^N$ is said to be "blocked" by the vector $v \in V_S$, if $u^S < v$. The core of the game consists of those vectors in V_N which are blocked by no vectors in any V_S.

In order for the core to be non-empty some supplementary conditions must be assumed to hold for the game. A natural set of conditions, which apply to a large class of n person games, may be formulated in terms of the concept of a balanced collection of coalitions. A collection $T = \{S\}$ is said to be balanced if the equations $\sum_{S=\{i\}} x_S = 1$, for $i = 1, \ldots n$, have a non-negative solution with $x_S = 0$ for those S not in T. And the game is said to be balanced if for every balanced collection T, a

vector u whose projections u^S are in V_S for all $S \in T$, must also be in V_N.

Theorem 1 may be applied to determine an approximation to a vector in the core of a balanced n person game. We construct the set Π with the first n vectors as before. Then for every coalition S, consisting neither of a single player nor all of the players, we take a large number of vectors π^j whose projection into E^S is in V_S, and whose coordinates corresponding to players not in S, are arbitrary but large numbers. The vectors in Π may therefore be divided into a number of distinct groups, each corresponding to a specific coalition. If a vector corresponds to a coalition S, the column in the matrix A, associated with this vector will have a one in row i if $i \in S$, and zeros otherwise. The vector b is taken as $(1, 1, \ldots 1)'$.

If Theorem 1 is applied we obtain a primitive set $\pi^{j_1}, \ldots \pi^{j_n}$, such that the columns $j_1, \ldots j_n$ form a feasible basis for $Ax = b$. These columns will be associated with a distinct set of coalitions $S_{j_1}, \ldots S_{j_n}$ which clearly form a balanced collection. If $u = \min(\pi^{j_1}, \ldots \pi^{j_n})$, then $u \in V_N$ since the game is balanced and $u^{S_j} \in V_{S_j}$ for $j = 1, \ldots n$. But u cannot be blocked by any of the vectors used in forming Π, since this would require $\pi^j > u$, and contradicts the definition of a primitive set. The vector u will therefore serve as an approximation to a point in the core of the game, with the approximation becoming better the more vectors are selected in Π.

III. The Proof of Theorem 1

The algorithm which underlies Theorem 1 is based on the observation that primitive sets permit a replacement operation analogous to that of a pivot step in linear programming. In general, if a specific vector π^{j_α} in a primitive set $\pi^{j_1}, \ldots \pi^{j_n}$ is removed, there will be a unique replacement $\pi^j \in \Pi$ so that the new collection of vectors also forms a primitive set. In order to make this statement precise, the following assumption, similar to the non-degeneracy assumption of linear programming, is required.

Non-Degeneracy Assumption: No two vectors in Π have the same i^{th} coordinate for any i.

This assumption, while necessary for the algorithm, does cause difficulty in practice, since for several applications the most natural selection of the set Π is one in which this assumption is not satisfied. Section IV will be devoted to a discussion of one technique for resolving this problem.

Given the assumption of non-degeneracy, the following theorem may be demonstrated (see [8], for the details of a proof).

Theorem 2. Let $\pi^{j_1}, \ldots \pi^{j_n}$ be a primitive set and π^{j_α} a specific member. Then aside from one exceptional case there is a unique vector $\pi^j \in \Pi$ which yields a primitive set when it replaces π^{j_α}. The exceptional case arises when the primitive set consists of $n-1$ vectors

from the first n vectors of Π, one vector π^j with $j > n$, and we are attempting to remove the latter vector. In this case no replacement is possible.

The algorithm for Theorem 1 alternates between the operation of replacing a vector in a primitive set, and a pivot step for the equations $Ax = b$. Since the set $\{x | x \geq 0, Ax = b\}$ is assumed to be bounded, an arbitrary column outside of a feasible basis can be brought into the basis by a pivot step, and if we make the standard non-degeneracy assumption of linear programming, a unique column will be removed.

Let us begin with a primitive set consisting of the vectors $\pi^2, \ldots \pi^n$ and a single vector π^j with $j > n$. In order for this to yield a primitive set π^j must be that vector in Π (other than the first n members of Π), with the largest first coordinate. On the other hand the columns $1, 2, \ldots n$ form a feasible basis for $Ax = b$, since the vector b is assumed to be non-negative.

In each iteration the algorithm will typically be in a position similar to this. The primitive set will contain the vectors $\pi^{j_1}, \ldots \pi^{j_n}$ with none of them equal to π^1, and the feasible basis for $Ax = b$ will be given by the columns $(1, j_2, \ldots j_n)$. The two sets of indices, which we wish to make identical in all coordinates, will in fact be equal in $n-1$ coordinates, and differ in the remaining one. At each position other than the first one there will be two operations which lead to a similar state. One possibility is to remove the vector π^{j_1} from the primitive set, and

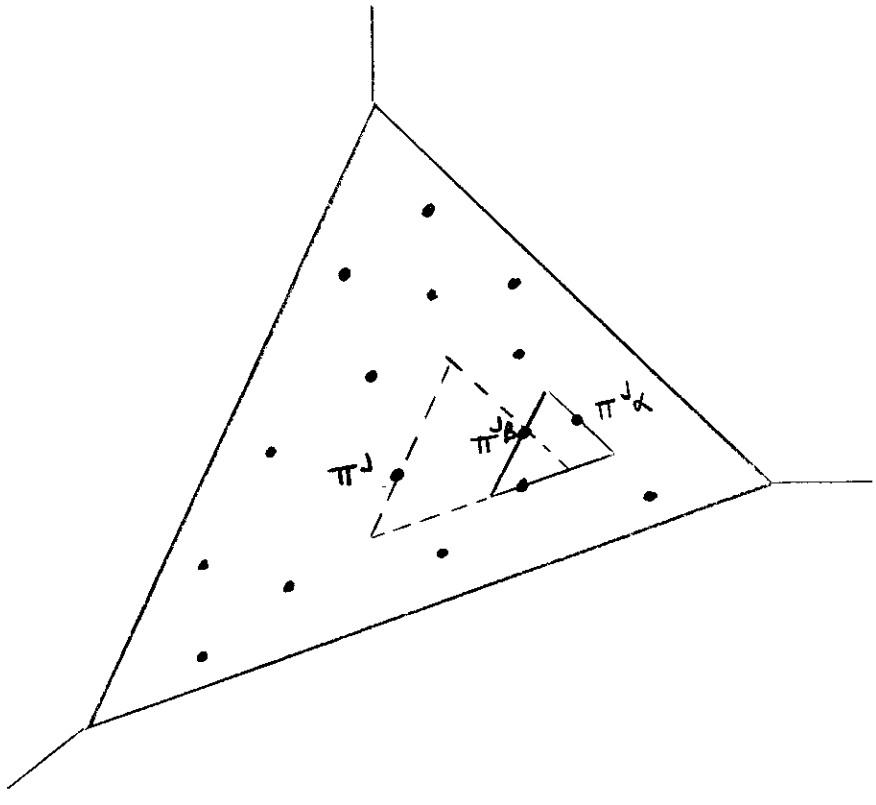
either to terminate if π^1 is introduced into the primitive set, or to be lead to a new position in which $n-1$ of the coordinates are identical. The other possibility is to introduce column j_1 into the feasible basis for $Ax = b$, and either to terminate if column 1 is removed, or again to be lead to a new position with $n-1$ identical coordinates. In the original position only one of these two operations can be carried out since the other alternative is the exceptional case referred to in Theorem 2. In any subsequent position we select that alternative other than the one used in arriving at that position.

The Lemke-Howson argument demonstrates that the algorithm cannot cycle, since if the first position to be repeated occurs in the middle of the algorithm there would necessarily be three alternatives available at that step rather than two. And if the first position to be repeated is the original one, there would necessarily be two alternatives, rather than one. Since the number of positions is finite, the algorithm must terminate, and it can only do so with a solution to the problem. This concludes the proof of Theorem 1.

In order to carry out the algorithm it is necessary to describe the specific procedure to determine the replacement for a vector in a primitive set. We begin by remarking that as a consequence of the non-degeneracy assumption, each column of a primitive set will have precisely one coordinate equal to the corresponding coordinate of $\pi = \min(\pi^{j_1}, \ldots \pi^{j_n})$. If a specific vector π^{j_α} is removed, and the minimum π' calculated for the remaining $n-1$ vectors, then one of these vectors, say π^{j_β}, will

have two of its coordinates equal to those of π'. Let the index i^* be such that $\pi^{j_\beta}_{i^*} = \pi'_{i^*} = \pi_{i^*}$. The replacement for π^{j_α} is then determined by examining all vectors π^j in Π with $\pi^j_i > \pi'_i$ for all $i \neq i^*$ and selecting that one with the largest value of $\pi^j_{i^*}$.

The argument that this rule produces the unique replacement for π^{j_α} may be found in the references previously cited. It should be remarked that the rule has a geometric interpretation which is particularly simple if the vectors in Π, other than the first n members, are all located on the simplex. In this case a primitive set is described by a subsimplex of S, with sides parallel to the coordinate hyperplanes, with each side containing precisely one member of the primitive set, and with no vectors in Π interior to the subsimplex.



To remove a specific vector π^{j_α}, we continuously reduce the subsimplex by moving the face on which π^{j_α} is situated into the subsimplex until it first touches another vector in the primitive set, π^{j_β}. We then enlarge the new subsimplex by moving the face on which π^{j_β} was originally situated until we first touch a vector $\pi^j \in \Pi$, which them becomes the replacement for π^{j_α}.

IV. A Procedure for Resolving Degeneracy

As we have seen in Section II, most of the applications of Theorem 1 currently known, involve a set Π whose members, aside from the first n, are on the simplex S. While it is reasonable to expect other applications of the general theorem to be developed in the future, this special case will clearly continue to be of considerable importance, and to warrent further study.

If we have no *a priori* information which emphasizes certain regions of the simplex it would seem useful, in implementing the algorithm, to select the non-slack vectors in Π distributed regularly over the entire simplex. And in the interest of accuracy the number of vectors in Π should be extremely large; examples have involved as many as 10^{30} such vectors.

The apparent problem raised by such a large number of vectors is that the replacement operation described in the previous section seems to require the examination of all vectors in Π, so as to determine that

vector which maximizes a specific coordinate subject to inequality restrictions on the remaining coordinates. If, however, the non-slack vectors in Π are selected with sufficient regularity, it seems reasonable to hope that this search procedure can be converted into a simple algebraic calculation.

We shall assume that the non-slack vectors in Π comprise all of those positive vectors on the simplex S, whose coordinates are rational numbers with a given denominator, i.e. all vectors of the form $(k_1/D, \ldots k_n/D)$, with k_i positive intergers summing to D.

The problem immediately raised is that this definition of Π does not satisfy the non-degeneracy assumption, since for any i there will be many vectors in Π with the same i^{th} coordinate. Some systematic procedure must be introduced for breaking ties and deciding which of two vectors with identical i^{th} coordinates should be considered to have the larger one. The algorithm will be successful as long as the tie-breaking rules satisfy the customary properties of a complete ordering for each coordinate.

In this section we shall use the following procedure for resolving degeneracy.

DEFINITION: Let $x = (x_1, \ldots x_n)$ and $y = (y_1, \ldots y_n)$. We define x_i to be larger than y_i (written $x_i \underset{i}{>} y_i$) if and only if the vector $(x_i, \ldots x_n, x_1, \ldots x_{i-1})$ is lexicographically larger than $(y_i, \ldots y_n, y_1, \ldots y_{i-1})$.

This definition, which produces a different tie-breaking rule for each coordinate, can never result in a tie between two vectors in any coordinate, unless the two vectors are identical. Moreover, the rule produces an extremely simple characterization of a primitive set, identical with that found in Hansen's thesis, and the replacement operation can be carried out very rapidly without searching through all of the vectors in Π. The algorithm which results is identical with an algorithm subsequently proposed by Kuhn [5] for the case of Brouwer's theorem, and has a simple geometric interpretation intimately related to the work of Cohen [1], and Shapley [10].

Given any n non-slack vectors in Π let us form a matrix K whose columns are the numerators of these vectors.

$$K = \begin{bmatrix} k_{11} & k_{12} \cdots k_{1n} \\ \vdots & \\ k_{n1} & k_{n2} \cdots k_{nn} \end{bmatrix}$$

The following theorem gives necessary and sufficient conditions for such a matrix to represent a primitive set. The case in which some of the first n vectors in Π are in the primitive set will be examined later.

Theorem 3. Let K be an $n \times n$ matrix with positive integral entries, and whose column sums are identical. The columns represent the vectors of a primitive set, if and only if there is a permutation $I(j)$ of the integers $(1, 2, \ldots n)$, and a rearrangement of the columns of K such that the j^{th} column of K is identical with column $j-1$, excepting the entry in row $I(j)$ which is one unit smaller and the entry in

row $I(j) - 1$ which is one unit larger. If $j = 1$, $j-1$ is to be interpreted as n, and similarly for $I(j)$.

As an example, the columns of

$$\begin{bmatrix} 10 & 10 & 10 & 11 & 11 \\ 20 & 20 & 21 & 20 & 20 \\ 30 & 31 & 30 & 30 & 30 \\ 10 & 9 & 9 & 9 & 10 \\ 30 & 30 & 30 & 30 & 29 \end{bmatrix},$$

with $D = 100$ satisfy the conditions of the Theorem if the permutation is given by

j	I(j)
1	1
2	4
3	3
4	2
5	5

In order to prove Theorem 3, let us begin by demonstrating a series of lemmas whose conclusion will show the necessity of these conditions. Let us assume that the columns of K are rearranged so that the first row is strictly increasing according to the lexicographical ordering described above. Since the columns, after division by D, form a primitive set, each column will have precisely one entry which is the smallest element in its row, according to the ordering described above. The row in which this entry is to be found in column j defines the permutation $I(j)$.

Lemma 1. No row of K consists of identical elements.

Let us assume, without loss of generality, that all of the elements in the first row are identical, and let $I(j^*) = 2$. By definition, the second coordinate of $(k_{1j^*}, k_{2j^*}, \ldots k_{nj^*})'$, is lexicographically smaller than the second coordinate of $(k_{11}, k_{21}, \ldots k_{n1})'$. But this must also be true of the first coordinate since $k_{11} = k_{1j^*}$, which contradicts the assumption that the columns are arranged in increasing lexicographical order.

Lemma 2. In any specific row of K, no two elements can differ by more than 1.

Without loss of generality we again restrict our argument to the first row, and assume that $k_{1n} \geq k_{11} + 2$. But then the vector $(k_{1n} - 1, k_{2n}, \ldots k_{nn} + 1)'$ has its i^{th} component lexicographically larger than the lexicographic minimum of the elements in the i^{th} row, for every i. This contradicts the definition of a primitive set.

Lemma 3. The elements in the second row of K are also lexicographically increasing, in the sense that $k_{2j^*} \underset{2}{<} \ldots \underset{2}{<} k_{2n} \underset{2}{<} k_{2,1} \underset{2}{<} \ldots \underset{2}{<} k_{2,j^*-1}$, for some j^*.

Since the top row is lexicographically increasing we must have $k_{1,1} = \ldots = k_{1,j^*-1}$ and $k_{1,j^*-1} + 1 = k_{1,j^*} = \ldots = k_{1,n}$ for some j^*.

But then $k_{21} \underset{2}{<} \ldots \underset{2}{<} k_{2,j^*-1}$, and $k_{2j^*} \underset{2}{<} \ldots \underset{2}{<} k_{2n}$. In order to prove Lemma 3, we need only show that $k_{2n} \underset{2}{<} k_{21}$. If the reverse inequality were true, the vector $(k_{1,n} - 1,\ k_{2,n},\ \ldots\ k_{n,n} + 1)'$ would have all of its components strictly larger than the lexicographic minima of the corresponding rows.

Lemma 3 may be applied, by induction, to show that every row is lexicographically increasing, as we read from its minimum to the right, reaching the last column, and then beginning with the first column and continuing to the right. In this cyclic fashion the elements in row i begin in that column j for which $I(j) = i$, with some constant, increase by one unit at some point and then stay constant. In the first row the increase comes at that column where $I(j^*) = 2$, and by induction a similar phenomenon

$$\begin{bmatrix} k_1 \ldots\ k_1 & k_1+1 \ldots k_1+1 \\ \ldots\ k_2+1 & k_2 \ \ldots \\ & \end{bmatrix},$$

occurs in the remaining rows. This demonstrates the necessity of the conditions described in Theorem 3.

In order to show that the conditions are sufficient we shall proceed by induction on the size of the matrix. Let K satisfy the conditions of the theorem and $k = (k_1,\ k_2,\ \ldots\ k_n)$ be a vector with positive integral components summing to the column sums of K and such that $(k_i,\ \ldots\ k_n,$

$k_1, \ldots k_{i-1})$ is lexicographically larger than $(k_{ij}, \ldots k_{nj}, k_{1j}, \ldots k_{i-1,j})$ for $i = 1, \ldots n$ and j such that $I(j) = i$.

Let us remark that for some row i, we must have $k_i = \min_j (k_{ij})$. For otherwise $k_i \geq 1 + \min_j (k_{ij}) \geq k_{i1}$ with strict inequality for at least one i, contradicting $\Sigma k_i = \Sigma k_{i1}$.

There is no loss in generality in assuming that $k_1 = \min_j (k_{1j}) = k_{11}$. But then $(k_i, \ldots k_n, k_1 + k_2, \ldots k_{i-1})'$ is lexicographically larger than $(k_{ij}, \ldots k_{nj}, k_{1j} + k_{2j}, \ldots k_{i-1,j})'$ for $i = 3, \ldots n$ and $I(j) = 1$. This is easy to see for if the tie is broken by one of the first $n-i$ elements of these vectors it will be broken as before. Since $k_1 \leq k_{1j}$, if the tie is not broken by one of the first $(n-i)$ elements we must have $k_1 = k_{1j}$, and if the tie is originally broken by k_2, it will be broken in the same way here. A similar remark holds if the tie is broken by some element after k_2.

The same argument shows that $(k_1 + k_2, k_3 \ldots k_n)'$ is lexicographically larger than $(k_{11} + k_{21}, k_{31} \ldots k_{n1})'$. Therefore the i^{th} entry in $(k_1 + k_2, \ldots k_n)'$ is strictly larger, in our ordering, than the lexicographic minimum in the i^{th} row of the $(n-1) \times n$ matrix

$$\begin{bmatrix} k_{11} + k_{21} & \cdots & k_{1j} + k_{2j} & \cdots & k_{1n} + k_{2n} \\ k_{31} & \cdots & k_{3j} & \cdots & k_{3n} \\ \vdots & & & & \\ k_{n1} & \cdots & k_{nj} & \cdots & k_{nn} \end{bmatrix}.$$

If $I(j^*) = 2$, then the $j^*\text{-}1^{st}$ and j^{*th} columns of this matrix are identical, and if one of them is deleted, the remaining columns satisfy the conditions of Theorem 3. This contradicts the induction assumption and verifies that the conditions are sufficient for K to represent a primitive set.

Theorem 3 provides a very compact description of those primitive sets in which no slack variables appear. It also permits a remarkably simple description of the replacement for a given vector in a primitive set.

Theorem 4. Let K represent a primitive set with no slack vectors. Then the vector $(k'_{1j}, \ldots k'_{nj})'$, defined by

$$k'_{ij} = k_{ij} + 1 \quad \text{for} \quad i = I(j), \; I(j+1) - 1$$

$$k'_{ij} = k_{ij} - 1 \quad \text{for} \quad i = I(j) - 1, \; I(j+1)$$

and $$k'_{ij} = k_{ij} \qquad \text{otherwise,}$$

is the unique replacement for column j, unless one of the components of k' is zero, in which case a slack vector replaces column j. If $I(j) = I(j+1) - 1$, then $k'_{ij} = k_{ij} + 2$ for this common value of i, and if

$I(j) - 1 = I(j+1)$, $k'_{ij} = k_{ij} - 2$ for the common row.

It is sufficient to show that this replacement generates a new matrix satisfying the conditions of Theorem 3, since the non-degenerate ordering guarantees that the replacement must be unique.

Columns $j-1$, j and $j+1$ (modulo n) are all identical except for the four elements in rows $I(j) - 1$, $I(j)$, $I(j+1) - 1$, and $I(j+1)$. Assuming, for the moment that these four rows are distinct, the submatrix consisting of these four rows and three columns is given by

$$\begin{array}{c} \\ I(j) - 1 \\ I(j) \\ I(j+1) - 1 \\ I(j+1) \end{array} \begin{array}{c} \begin{array}{ccc} j-1 & j & j+1 \end{array} \\ \begin{bmatrix} a & a+1 & a+1 \\ b+1 & b & b \\ c & c & c+1 \\ d+1 & d+1 & d \end{bmatrix} \end{array},$$

with a , b , c and d positive integers.

The suggested replacement for column j , will produce a new submatrix given by

$$\begin{array}{c} \begin{array}{ccc} j-1 & j & j+1 \end{array} \\ \begin{bmatrix} a & a & a+1 \\ b+1 & b+1 & b \\ c & c+1 & c+1 \\ d+1 & d & d \end{bmatrix} \end{array},$$

which clearly satisfies the conditions of Theorem 3, with a new permutation I' defined by $I'(j) = I(j+1)$ and $I'(j+1) = I(j)$, and equal to I otherwise.

In the case in which $I(j) = I(j+1) - 1$, the three columns differ only in three elements and the appropriate submatrix of K is

$$\begin{array}{c} \\ I(j)-1 \\ I(j) \\ I(j+1) \end{array} \begin{array}{c} \begin{array}{ccc} j-1 & j & j+1 \end{array} \\ \begin{bmatrix} a & a+1 & a+1 \\ b+1 & b & b+1 \\ c+1 & c+1 & c \end{bmatrix} \end{array} .$$

The replacement rule produces a submatrix

$$\begin{array}{c} \begin{array}{ccc} j-1 & j & j+1 \end{array} \\ \begin{bmatrix} a & a & a+1 \\ b+1 & b+2 & b+1 \\ c+1 & c & c \end{bmatrix} \end{array}$$

again satisfying the conditions of Theorem 3, with the same new permutation as before.

The third possibility, $I(j+1) = I(j) - 1$, presents us with

$$\begin{array}{c} \\ I(j+1)-1 \\ I(j+1) \\ I(j) \end{array} \begin{array}{c} \begin{array}{ccc} j-1 & j & j+1 \end{array} \\ \begin{bmatrix} a & a & a+1 \\ b & b+1 & b \\ c+1 & c & c \end{bmatrix} \end{array} ,$$

and after replacing column j we obtain

$$\begin{array}{c} \begin{array}{ccc} j-1 & j & j+1 \end{array} \\ \begin{bmatrix} a & a+1 & a+1 \\ b & b-1 & b \\ c+1 & c+1 & c \end{bmatrix} \end{array} .$$

Again we satisfy the conditions of Theorem 3, with the same modified permutation, unless $b = 1$. In this case the i^{th} slack vector will replace column j, where $i = I(j)$. This demonstrates Theorem 4.

The following examples of the replacement operation may be useful. If the second column is removed in

$$\begin{bmatrix} 10 & 10 & 10 & 11 & 11 \\ 20 & 20 & 21 & 20 & 20 \\ 30 & 31 & 30 & 30 & 30 \\ 10 & 9 & 9 & 9 & 10 \\ 30 & 30 & 30 & 30 & 29 \end{bmatrix},$$

we obtain

$$\begin{bmatrix} 10 & 10 & 10 & 11 & 11 \\ 20 & 21 & 21 & 20 & 20 \\ 30 & 29 & 30 & 30 & 30 \\ 10 & 10 & 9 & 9 & 10 \\ 30 & 30 & 30 & 30 & 29 \end{bmatrix}$$

and if the first column is subsequently removed from this latter matrix we have

$$\begin{bmatrix} 11 & 10 & 10 & 11 & 11 \\ 21 & 21 & 21 & 20 & 20 \\ 29 & 29 & 30 & 30 & 30 \\ 10 & 10 & 9 & 9 & 10 \\ 29 & 30 & 30 & 30 & 29 \end{bmatrix}.$$

These operations can easily be programmed for a high speed computer, and carried out with considerable rapidity, so that even those problems requiring as many as 10,000 iterations present no serious difficulty.

In order to complete the version of the algorithm being developed in this section, we need to characterize those primitive sets which contain some slack vectors, and to describe the replacement operation when slack vectors are involved.

Consider a primitive set which contains the slack vectors $\pi^{i_1}, \ldots \pi^{i_m}$, and $n-m$ non-slack vectors $\pi^{j_1}, \ldots \pi^{j_{n-m}}$. Let us form an $n \times (n-m)$ matrix whose columns are the numerators of the non-slack vectors.

$$K = \begin{bmatrix} k_{1,1} & \cdots & k_{1,n-m} \\ \vdots & & \\ k_{n,1} & \cdots & k_{n,n-m} \end{bmatrix}$$

The following theorem extends Theorem 3, and characterizes those matrices which represent primitive sets containing some slack vectors.

Theorem 5. Let K be an $n \times (n-m)$ matrix with positive integral entries and whose column sums are identical. The columns of K, along with the slack vectors $\pi^{i_1}, \ldots \pi^{i_m}$ represent the vectors of a primitive set if and only if

1. The entries in rows $i_1, i_2, \ldots i_m$ are all ones, and

2. The $(n-m) \times (n-m)$ square submatrix of K obtained by deleting rows $i_1, i_2, \ldots i_m$, represents a primitive set of dimension $n-m$.

The proof of this theorem may be obtained by arguments virtually identical with those of Theorem 3 and will be omitted. As an example the columns of

$$\begin{bmatrix} 1 & 1 & 1 \\ 49 & 49 & 50 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \\ 15 & 14 & 14 \end{bmatrix}$$

describe three vectors which form a primitive set in conjunction with the first and fourth slack vectors.

The replacement operation is essentially identical with that given in Theorem 4, with minor modifications to accomodate the slack vectors. If a non-slack vector is removed from K the replacement operation is carried out on the square submatrix representing a primitive set of dimension $n-m$. A slack vector will be introduced only if the incoming column has a zero component, and in this case the number of columns in K shrinks by one. This occurs for example if the middle column is removed in the above matrix, yielding a new matrix

$$\begin{bmatrix} 1 & 1 \\ 49 & 50 \\ 1 & 1 \\ 1 & 1 \\ 15 & 14 \end{bmatrix}$$

whose two columns form a primitive set along with the first, third and fourth slack vectors.

If, on the other hand, a slack vector is removed from a primitive set the number of columns in K increases by one. The operation is the reverse of deleting the middle column in the case

$$\begin{bmatrix} a & a & a+1 \\ 1 & 2 & 1 \\ b+1 & b & b \end{bmatrix},$$

and this determines precisely the location of the incoming non-slack column.

In the above example if the first slack vector is deleted we obtain

$$\begin{bmatrix} 1 & 1 & 2 \\ 49 & 50 & 49 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 15 & 14 & 14 \end{bmatrix},$$

and if the third slack vector is then deleted the resulting matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 49 & 49 & 50 & 49 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 15 & 14 & 14 & 14 \end{bmatrix}.$$

This completes the discussion of primitive sets and the replacement operation, when the vectors in Π are ordered by the cyclic lexicographical ordering. A Frotran program for that part of the algorithm involving the replacement operation is given in Section VI.*

V. Some Geometric Considerations

In this section we shall continue with the assumption that the vectors in Π, aside from the first n, consist of all of those positive vectors on the simplex S, whose coordinates are rational numbers with a given denominator D. Theorem 3, provides us with a simple characterization of primitive sets, if the cyclic lexicographic ordering is used to resolve degeneracy.

As we have seen, one of the applications of Theorem 1 is the approximation of a fixed point of a continuous mapping of the simplex into itself. As a consequence, a proof of Brouwer's theorem may be obtained

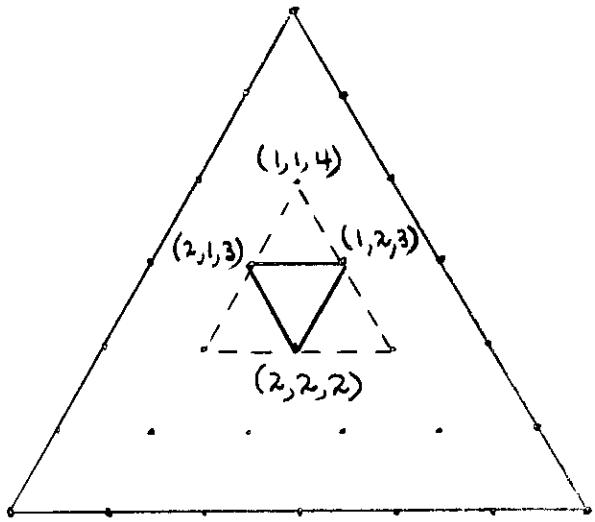
---

*For convenience in programming, the FORTRAN statement treats slack in a slightly different fashion from that described in this section.

by a suitable limiting process. The customary proof of Brouwer's theorem makes use of Sperner's Lemma, a combinatorial theorem concerned with a labeling of the vertices of a simplicial subdivision of the simplex.

In a paper published in 1960, Kuhn [4] examined the particular subdivision of the simplex consisting of those subsimplices whose vertices have non-negative rational coordinates with a given denominator D. It is by no means trivial to determine when a given set of n vectors represent the vertices of a subsimplex in this subdivision; Kuhn's paper contains a theorem fully answering this question. In our terminology, Kuhn's theorem may be described as saying that n such vectors represent the vertices of a subsimplex if and only if the n vectors form a primitive set.

As the following figure indicates, there is a slight difference in emphasis between a subsimplex in Kuhn's subdivision and the geometric objects which we have previously associated with primitive sets.



The three vectors $(1, 2, 3)^t$, $(2, 1, 3)^t$ and $(2, 2, 2)^t$ are the vertices of the heavily market subsimplex in the center of the figure, and the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{bmatrix}$$

represents a primitive set. But for us, the geometric object associated with a primitive set is a subsimplex such that the side parallel to the i^{th} coordinate hyperplane contains that vector with the smallest i^{th} coordinate, and is therefore the subsimplex represented by the dashed lines.

If the vector $(2, 2, 2)^t$ is removed from this primitive set, the new matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 4 \end{bmatrix}$$

is obtained, and both Kuhn's simplex and the one associated with a primitive set coincide.

In a recent paper [5], Kuhn has given an algorithm for approximating fixed points of a continuous mapping using this particular simplicial subdivision, and a replacement operation in which a vertex is removed from a subsimplex and replaced by a vertex of that unique adjacent subsimplex containing the remaining $n-1$ vertices. The determination of the vertex to be removed is based on the Lemke-Howson technique. It should be clear

that this algorithm is identical with that described in Section IV, aside from slight differences in the treatment of vectors on the boundary.

The algorithm may be placed in the setting of an arbitrary simplicial subdivision, rather than the particular one described above, with a possible decrease in computational efficiency. This point was realized in an earlier paper by Cohen [1], more concerned with a proof of Sperner's lemma than with numerical approximations, in Kuhn's paper, and in an unpublished manuscript by Shapley [10].

VI. A FORTRAN Version of the Main Program

```
      DIMENSION K(40, 40), IO(40)
      COMMON N, INDEX, K, JOUT
 1    READ M, N, MAXI
      INDEX = 1
      JOUT = 1
      DO 10  I = 1, N
      DO 5   J = 1, N
 5    K(I, J) = 1
10    CONTINUE
      DO 15  J = 1, N
      K(1, J) = M + 2 - N
      K(J, J) = K(J, J) - 1
      IO(J) = J
15    CONTINUE
```

```
C       INPUT OVER.  THE SUBROUTINE IS CALLED TO DETERMINE JOUT.
16      CALL SUBROUTINE
        IF (JOUT. EQ. 0) GO TO 500
C       THE SUBROUTINE GIVES THE COLUMN OF THE PRIMITIVE SET TO BE REMOVED.
          TERMINATION IS INDICATED BY RETURNING A VALUE OF JOUT = 0 TO THE
          MAIN PROGRAM.
        JP = JOUT + 1
        IF(JOUT. EQ. N) JP = 1
        LOO = IO(JOUT)
        L10 = IO(JOUT) - 1
        IF (L10. EQ. 0)L10 = N
        L01 = IO(JP)
        L11 = IO(JP) - 1
        IF (L11. EQ. 0)L11 = N
        K(LOO, JOUT) = K(LOO, JOUT) + 1
        K(L11, JOUT) = K(L11, JOUT) + 1
        K(L01, JOUT) = K(L01, JOUT) - 1
        K(L10, JOUT) = K(L10, JOUT) - 1
        IO(JOUT) = L01
        IO(JP) = LOO
        INDEX = INDEX + 1
        IF (INDEX. EQ. MAXI) GO TO 500
        GO TO 16
500     WRITE...
        END
```

REFERENCES

[1] Cohen, D.I.A., "On the Sperner Lemma," Journal of Combinatorial Theory, 2 (1967).

[2] Debreu, G., The Theory of Value, Wiley (1959).

[3] Hansen, T., "On the Approximation of a Competitive Equilibrium," Ph.D. Dissertation, Yale University.

[4] Kuhn, H.W., "Some Combinatorial Lemmas in Topology," IBM J. Res. Develop., 4 (1960).

[5] _____, "Simplicial Approximation of Fixed Points," Proceedings of the National Academy of Sciences, 61 (1968).

[6] Lemke, C.E. and J.T. Howson, "Equilibrium Points of Bi-Matrix Games," SIAM Journal, 12 (1964).

[7] Scarf, H., "The Core of an N person Game," Econometrica, 35 (1967).

[8] _____, "The Approximation of Fixed Points of a Continuous Mapping," SIAM Journal of Applied Math., 15 (1967).

[9] _____, "On the Computation of Equilibrium Prices," Ten Economic Studies in the Tradition of Irving Fisher, Wiley (1967).

[10] Private communication.