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Summary

In this paper we discuss a theory of consumer's choice, based on the idea of utility maximization, subject to constraints in expenditures. A description is given of those utility functions for which future tastes are independent of past consumption, and an even smaller class is described for which tastes are stationary in time. It is shown that the utility function will have these properties only if it is a discounted sum of functions pertaining to consumption in the separate periods. The discount factor leads to the notion of a personal interest rate, which may be constant, or a function of the consumption level.

The question of whether a decreasing ratio of expenditure to total wealth is compatible with such a utility function is discussed. The case of constant personal interest is examined in detail and it is shown that no stationary utility function permits a decreasing ratio of expenditure to wealth for a sufficiently wide range of market interest rates. On the other hand, an example is given of a utility indicator with a variable personal interest rate, which predicts a decreasing ratio of expenditure to wealth for all market interest rates.

I. Introduction

In this paper we shall discuss several mathematical problems that arise in the construction of a dynamic theory of consumer's choice. During the last several years there have been at least two mathematical approaches to the problem of consumer's choice over time, that of Friedman (3) and that of Modigliani and Brumberg (5). Our discussion will be similar in spirit, though not in detail, to that offered by Modigliani and Brumberg, in the sense that these authors view the problem specifically as one of utility maximization, subject to constraints on expenditures.

The problem faced by a consumer is first of all, that of expressing preferences among alternative streams of consumption bundles. Let $x_1, x_2, x_3, \ldots$ be such a stream of consumption bundles, where x_t is a vector whose j^{th} component represents the number of units of commodity j to be consumed in time period t. We shall begin by assuming that the preferences of the typical consumer are expressed by a utility function

$$U(x_1, x_2, \ldots),$$

such that when alternative consumption streams are presented to the consumer, he will attempt to secure that stream with the highest utility.

Actually we shall assume somewhat more; in the general case, the consumer does not select levels of consumption for all future time periods, but rather he takes a series of actions which lead to consumption levels, perhaps depending randomly upon the actions taken. For this reason the consumer should, ideally at least, be able to express preferences among probability distributions of consumption streams.[1] This consideration leads to the fact that $U(x_1, x_2, \ldots)$ is a Bernoulli utility indicator, with preferences being ranked according to expected utility. Of course, if the Bernoulli utility indicator is left in a perfectly general form, there is very little that can be said about the actual behavior of the consumer. There is an enormous variety of behavior consistent with a general utility function, and if we are interested in predicting specific patterns of consumption and saving over time, some restrictions must be made as to the form of the utility function.

Our restriction, which seems quite plausible as a first approximation, may be described by saying that the future tastes are independent of past consumption. As we shall show in Section II, this implies that the utility function U may be written in the following form:

$$U(x_1, x_2, x_3, \ldots) = \varphi_1(x_1) + \beta_1(x_1)\, \varphi_2(x_2) + \beta_1(x_1)\, \beta_2(x_2)\, \varphi_3(x_3) + \ldots, \text{ with } \beta_n > 0 \,. \tag{1}$$

At this point we are being deliberately vague as to whether the number of time periods under consideration is finite or infinite. The reasoning

[1] No distinction will be made in this paper between subjective and objective probability distributions.

which produces (1) is valid for both of these cases. The function β_n may be looked upon as the personal discount factor which is operative in time period n . β_n is a function rather than a constant, and therefore the personal discount factor depends on the level of consumption during the period. The personal interest rate would, of course, be defined by $\frac{1-\beta_n}{\beta_n}$ and is in no way connected with the market rate of interest.

To be somewhat realistic, it should probably be assumed that for large n, $\beta_n \equiv 0$, that is consumption at a time exceeding the possible life-time of the consumer should be discounted completely. However, it is occasionally useful to introduce as an additional restriction on the Bernoulli indicator U , the assumption that future tastes are not only independent of past consumption, but are also stationary over time. This assumption will be introduced primarily for the purposes of mathematical simplification; it requires that the argument of the Bernoulli indicator be an infinite sequence of consumption vectors. As a consequence of the assumption of stationary tastes it will be shown in Section II, that the utility indicator may be written in the form

$$(2) \qquad U(x_1, x_2, \dots) = \varphi(x_1) + \beta(x_2)\,\varphi(x_2) + \beta(x_1)\,\beta(x_2)\,\varphi(x_3) + \cdots \quad ,$$

again with $\beta > 0$.

It should be remarked that there are corresponding expressions if the consumption takes place continuously over time, rather than at discrete time intervals. In particular (2) is replaced by an expression of the form

$$U(x(\cdot)) = \int_0^\infty e^{-\int_0^t i(x(\xi))d\xi} \varphi(x(t))dt \;, \tag{3}$$

where $x(t)$ is the vector valued rate of consumption, $i(x)$ is the personal interest rate (rather than discount factor) corresponding to a consumption rate x (it is generally > 0, but may actually be negative) and $U(x(\cdot))$ is the utility associated with the consumption plan $x(t)$ $(0 \leq t \leq \infty)$. The function φ appearing underneath the integral sign in (3) is analogous to the function φ appearing on the <u>right</u> hand side of (2). We shall find it convenient to work with the continuous version (3), rather than the discrete version (2).

In Section III we shall discuss the behavior of the consuming unit when guided by a utility indicator of the form (3). For simplicity we shall assume that there is a single item of consumption so that the vector function $x(t)$ appearing in (3) will actually be a scalar, and that the price of this item remains constant over time (the price will be taken as 1). This general item of consumption will be assumed to be completely non-durable, so that no inventories are carried. The income rate of the consumer will be assumed to be a known function of the time $I(t)$ (random fluctuations of this income will be examined in a future paper.) Strictly speaking we mean

$I(t)$ to be the income component which is independent of returns on investments and interest payments on loans. Any expenditure plan will result in excesses or deficits of expenditures over assets and any disparity between assets and expenditures is meant to be converted into investments or borrowings at the market rate of interest j. The only constraints that we shall impose on the expenditure plan are

$$
(4) \quad
\begin{aligned}
&\text{a.} \quad x(t) \geq 0 \quad \text{and} \\
&\text{b.} \quad \int_0^\infty e^{-jt}\, x(t)dt \leq M_o + \int_0^\infty e^{-jt}\, I(t)dt = M\,,
\end{aligned}
$$

where M_o represents assets at time $t=0$. (We shall always use M in this sense, and talk of it as the assets, meaning the discounted value of current plus future assets.) The latter constraint is a very weak one; it is equivalent to assuming that the present value of indebtedness at time t tends to zero as $t \to \infty$. Other constraints, such as a limit on the actual indebtedness at any time are also possible, but we shall not discuss them in this paper.

In Section III we determine the expenditure plan which maximizes (3) under the constraints (4), when the personal interest rate i is a constant, and φ is an increasing concave function. The solution is as follows

Theorem I.

1. If $i < j$ then the optimal consumption plan $x(t;j,M)$ consists of (possibly) an interval $(0, t_o)$ in which $x=0$, and in (t_o, ∞) $x(t)$

satisfies the differential equation

$$\frac{dx}{dt} = (i-j)\,\frac{\varphi'(x)}{\varphi''(x)}. \tag{5}$$

t_o is taken as the smallest possible value such that the solution satisfies (4).

2. If $i > j$, then $x(t)$ consists of (possibly) an interval (t_o, ∞) in which $x = 0$, and in $(0, t_o)$ $x(t)$ satisfies (5). t_o is taken as the largest possible value such that the solution satisfies (4).

3. If $i = j$, then $x(t) = jM$.

The general features of the solution are clear: if $i < j$, then expenditures are increasing steadily over time. If M is thought of as the capital of the consumer at $t = 0$, then the capital at time t, $M(t)$, will be an increasing function of the time,[2/] and expenditures are proceeding at a rate which is less than the interest on the capital. It is probably fair to call this type of consumer a saver.

On the other hand if $i > j$, the situation is reversed, and the consumer is dissaving.

2/ $M(t)$ satisfies the equation

$$\frac{dM}{dt} = jM(t) - x(t).$$

We should remark that it is possible for there to be no optimal consumption plan. It is possible to construct examples in which any consumption plan can be improved. Theorem 1 only applys to the case in which an optimal consumption plan exists.

Actual data on consumption patterns seem to indicate that the ratio of consumption to income falls as income rises. This would seem to be reflected in our theory by the statement that $\frac{x(0;j,M)}{M}$ should be strictly decreasing in M. In Section IV we shall take a look at this topic. It will be shown that as long as the personal interest rate i is constant, there is no utility function φ which has the property that $\frac{x(0;j,M)}{M}$ is decreasing in M for all j. The only utility indicators which are in any way close to fulfilling this condition are $\varphi(x) = \log x$ or $\varphi(x) = x^a$,[3/] and for these utility functions $\frac{x}{M}$ is actually independent of M, for each fixed j. This seems to indicate, in no uncertain terms, that the personal interest rate should depend on the rate of consumption. In Section V an example of a utility indicator with a variable personal interest rate is given, for which $\frac{x(0;j,M)}{M}$ is strictly decreasing for all j.

We should like to thank K.J. Arrow, L. Hurwicz, and S. Karlin for a number of stimulating conversations on these topics.

3/ Linear transformations are possible.

II. The Independence Assumption.

There are several excellent discussions in the literature of Bernoulli utility indicators, and we shall not repeat these discussions here (6,1). The main points are the following: Consider a set consisting of a number of alternatives, and with a preference relationship which permits one to compare an arbitrary pair of probability distributions for these alternatives. If the preference relationship is assumed to have a number of simple and intuitively plausible properties, then it may be shown that there is a utility indicator U, defined on the set of alternatives, with the property that one probability distribution is preferred to another if its expected utility is larger. Moreover, the function U is unique up to a linear transformation, in the following sense: if $U' = a+bU$ with $b > 0$, then preferences ranked according to the expectation of U are the same as preferences ranked according to the expectation of U', and if U' and U are not related in this linear fashion, then they express different preferences for probability distributions over the alternatives. We shall take it for granted, in the remainder of this paper, that all preference relationships will be such as to imply the existence and uniqueness of a Bernoulli utility indicator in this sense.

Now let us turn our attention to the independence assumption described in Section I. We consider, as the space of possible alternatives, all sequences of consumption vectors, $x_1, x_2, \ldots$ (These may be finite sequences, if we wish, or else infinite.), and along with this a Bernoulli utility indicator $U_1(x_1, x_2, \dots)$. Preferences over probability distributions are obtained by ranking the expectations of this function. In particular, all probability distributions for which consumption in the first period is not random, but actually equal to $\bar{x}_1$, may be compared. This implies that if we consider all consumption streams which are equal to $\bar{x}_1$ in the first period, then preferences are determined by ranking the expectations of $U_1(\bar{x}_1, x_2, x_3, \dots)$, and therefore $U_1(\bar{x}_1, x_2, x_3, \dots)$ as a function of $x_2, x_3, \dots$ is a Bernoulli utility indicator for consumption from period two onward, given that the consumption in period 1 is $\bar{x}_1$. But according to our assumption about future tastes being independent of past consumption, we see that as $\bar{x}_1$ ranges over all possible values, the family of functions of $x_2, x_3, \dots, U_1(\bar{x}_1, x_2, x_3, \dots)$ should be equivalent Bernoulli indicators, and therefore should be linked by a linear relationship. This implies that

$$(6) \qquad U_1(x_1, x_2, \dots) \equiv \varphi_1(x_1) + \beta_1(x_1)\, U_2(x_2, \dots),$$

for some $\varphi_1, \beta_1 > 0$ and U_2. In order to obtain (1), we apply the same argument to the function U_2, etc.

If in addition to the independence of future tastes and past consumption we also require that tastes be stationary over time, the preceding argument implies that $U_1(\bar{x}_1, x_2, x_3, \dots)$ and $U_1(x_2, x_3, \dots)$ are equivalent Bernoulli indicators for each fixed $\bar{x}_1$, and therefore,

(7) $$U_1(x_1, x_2, \dots) \equiv \varphi(x_1) + \beta(x_1)\, U_1(x_2, \dots) \ .$$

We may therefore drop the calendar date from the utility indicator and by the iteration of (7) we obtain (2).

It is interesting to note that unless the personal discount factor β is constant, the function φ should not be interpreted as a utility indicator, in the sense that if φ is changed by a linear transformation the over-all tastes of the consumer are actually modified.

If the consumer is depicted as consuming continuously over time, at a rate $x(t)$, then the Bernoulli utility indicator would associate a number with each such function. We may arrive at such a number, by a limiting operation on (2), by letting the periods between successive consumption choices approach zero. If the personal interest rate is considered to be proportional to the time between successive consumption decisions, then (2) becomes

$$U(x(\cdot)) \sim \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{1}{1+i(x(k\Delta t))\Delta t} \right) \varphi(x(n\Delta t)) \ .$$

If we multiply (normalizing) U by Δt, and pass to the limit, we obtain

(8) $$U(x(\cdot)) = \int_0^{\infty} e^{-\int_0^t i(x(\xi))d\xi} \varphi(x(t))dt \ ,$$

where $i(x)$ may be interpreted as the personal interest rate corresponding consumption rate x. This formula, for the case of stationary independent tastes, may also be derived directly by considerations similar to those of the discrete time period model. We shall find it convenient to work with the continuous time model rather than the discrete model.

III. The Optimal Consumption Plan for Constant Personal Interest Rate.

In this section we shall discuss the optimal consumption plan when the consumer is guided by a utility function of the form (8), and with a constant personal interest rate. In this case φ may be considered as a proper utility function, and we shall assume that as a function of the rate of consumption it is both increasing and concave.

The only constraints that we shall impose on the possible consumption plans $x(t)$ are

$$\text{(9)} \qquad \begin{aligned} &a; \quad x(t) \geq 0 \qquad \text{and} \\ &b; \quad \int_0^\infty e^{-jt}\, x(t)dt \leq M \;. \end{aligned}$$

(j is the market rate of interest.)

The problem is to determine the function $x(t)$ which maximizes (8), subject to the constraints (9). The mathematical technique is similar to that used by Karlin in (4), and Karlin and Arrow in (3).

Let us begin by assuming the existence of an optimal consumption plan $x^*(t)$, and deduce several conditions that must by satisfied by this plan.

As was remarked in the introduction, it is possible for there to be no optimal consumption plan. For the moment, let us consider the case where the personal interest rate i is smaller than the market rate j. If $x(t)$ is any other consumption plan satisfying (9), then for any θ between zero and one $\theta x^*(t) + (1-\theta)x(t)$ is a consumption plan which satisfies the constraints. The utility of this latter plan is given by

$$J(\theta) = \int_0^\infty e^{-it}\, U(\theta x^*(t) + (1-\theta)x(t))dt \; . \tag{10}$$

$J(\theta)$ is concave in θ, and therefore, a necessary and sufficient condition that it assumes its maximum at $\theta=1$ is that $J'(1) \geq 0$. But

$$\begin{aligned} J'(1) &= \int_0^\infty e^{-it}\, U'(x^*(t))\;(x^* - x)dt \\ &= \int_0^\infty e^{(j-i)t}\, U'(x^*(t))\;(e^{-jt}\, x^*(t) - e^{-jt}\, x(t))dt. \end{aligned} \tag{11}$$

Let us integrate this last expression by parts, defining $C(t)$ by the expression

$$C(t) = \int_0^t e^{-j\xi}\, x(\xi)d\xi \; , \tag{12}$$

and $C^*(t)$ in a corresponding way. Since $C(\infty) = C^*(\infty) = M$, (we are comparing the optimal plan with other plans that utilize all resources.) the integration by parts yields

(13) $$0 \leq \int_0^\infty \frac{d}{dt} \left\{ e^{(j-i)t} U'(x^*(t)) \right\} \left\{ C^*(t) - C(t) \right\} dt$$

and therefore, $C^*(t)$ maximizes the integral

(14) $$\int_0^\infty \frac{d}{dt} \left\{ e^{(j-i)t} U'(x^*(t)) \right\} C(t)dt \quad ,$$

for all choices of $C(t)$ arising from a feasible consumption plan $x(t)$.

This remark permits us to deduce some simple properties satisfied by the optimal consumption plan $x^*(t)$. First of all, let us suppose that there is some interval of t-values such that

(15) $$\frac{d}{dt} \left\{ e^{(j-i)t} U'(x^*(t)) \right\} > 0 \ , \qquad \text{say,}$$

t in (a,b). We shall show that this is impossible, unless $C^*(t)$ is actually constant in this interval. For, assume that it is not constant, so that $C^*(b) > C^*(a)$. Then we can construct a consumption plan $x(t)$ such that $C(a) = C^*(a)$, and $C(b) = C^*(b)$, and $C(t) > C^*(t)$ for t between a and b . (A simple graph will show that this is possible.) This clearly improves the integral (14) so that if (15) holds, we must have $x^*(t) = 0$ in (a,b). But since $j < i$ this contradicts (15) and therefore (15) can never hold.

We have shown that $\frac{d}{dt} \ e^{(j-i)t} U'(x^*(t))$ must be everywhere ≤ 0. An argument similar to that of the preceding paragraph shows that if this function is ever strictly less than zero in an interval, then $x^*(t) = 0$

in this interval. These remarks may be summarized in the following lemma.

Lemma 1: The optimal consumption plan $x^*(t)$ is composed of pieces, in which $x^*(t)$ is either identically zero or given by the solution of the differential equation

$$\frac{d}{dt}\left\{e^{(j-i)t}\, U'(x^*(t))\right\} = 0 \ . \tag{16}$$

The next step is to show that when $j < i$ the only place where $x^*(t)$ can be identically zero is (possibly) in an interval connected to the origin, and that if this interval is given by $(0, t_o)$ with $t_o > 0$, then $x^*(t_o + 0) = 0$.

Let us begin by assuming that the optimal strategy contains an interval of zero consumption which is not connected to the origin. The function $C^*(t)$ will then have the appearance of the solid line in Figure 1.

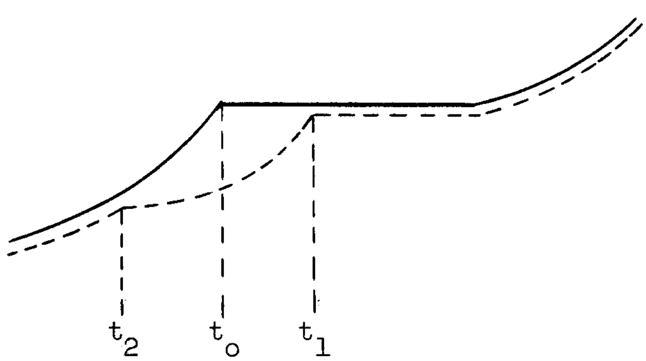

FIGURE 1.

We mean to compare the optimal policy with one whose $C(t)$ function has the form of the dashed line in Figure 1. This comparison policy may be obtained in the following way: pick a point t_2 slightly to the left of t_o. t_2 is in an interval in which $x^*(t)$ satisfies the differential equation (16). It may be shown directly from this equation, since $j < i$ and φ is increasing and concave, that $x^*(t)$ is increasing, and therefore, if t_2 is close to t_o, $x^*(t_2) > 0$. We change the x^* policy by beginning at t_2 with a slightly lower value of $x(t_2 + 0)$, and continuing by means of the differential equation until the two C functions intersect, as is shown in Figure 1. There will always be such an intersection point if $x(t_2 + 0)$ is taken sufficiently close to $x^*(t_2)$.

If $x^*(t)$ is, indeed, optimal then the integral appearing in (11) must be ≥ 0, when x is the comparison policy described above. Let us show that this is false; x^* and x agree everywhere, except in the interval from t_2 to t_1, and we may therefore write (11) as

$$(17) \quad \int_{t_2}^{t_o} e^{-it}\, \varphi'(x^*(t))(x^* - x)dt + \int_{t_o}^{t_1} e^{-it}\, \varphi'(x^*(t))(x^* - x)dt \; .$$

Consider the first integral in (17). In the interval (t_2, t_o) $x^*(t)$ satisfies the differential equation (16). This implies that

$$(18) \qquad e^{(j-i)t}\varphi'(x^*(t)) = e^{(j-i)t_o}\, \varphi'(x^*(t_o - 0)),$$

and therefore the first integral is equal to

$$e^{(j-i)t_o}\varphi'(x^*(t_o-0))\int_{t_2}^{t_o} e^{-jt}(x^*-x)\,dt. \tag{19}$$

Now consider the second integral. For this range of t values, $x^*(t) = 0$, and therefore the second integral is

$$\begin{aligned}
&-\varphi'(0)\int_{t_o}^{t_1} e^{-it}\,x(t)dt \\
&= -\varphi'(0)\int_{t_o}^{t_1} e^{-jt}e^{(j-i)t}\,x(t)dt \\
&< -\varphi'(0)e^{(j-i)t_o}\int_{t_o}^{t_1} e^{-jt}\,x(t)dt \\
&< -\varphi'(x^*(t_o-0))e^{(j-i)t_o}\int_{t_o}^{t_1} e^{-jt}\,x(t)dt,
\end{aligned}$$

since $x^*(t_o-0) > 0$ and φ' is a strictly decreasing function of its argument.

Therefore (17), which is meant to be ≥ 0 if x^* is optimal, is actually

$$\begin{aligned}
&< \varphi'(x^*(t_o-0))\,e^{(j-i)t_o}\left\{\int_{t_2}^{t_o} e^{-jt}(x^*-x)\,dt - \int_{t_o}^{t_1} e^{-jt}x\,dt\right\} \\
&= \varphi'(x^*(t_o-0))\,e^{(j-i)t_o}\left\{\int_{t_2}^{t_1} e^{-jt}(x^*-x)\,dt\right\} = 0,
\end{aligned}$$

which shows that x^* is not optimal if it contains an interval of zero consumption which is not connected to the origin.

On the other hand let us assume that $x^*(t)$ indicates zero consumption for the interval $(0,t_o)$. We shall show that this policy is only optimal if $x^*(t_o+0) = 0$. For suppose that $x^*(t_o+0) > 0$. In this case let us compare, by means of (11), the x^* policy with one whose $C(t)$ function is given by the dashed line in Fig. 2. The solid line is meant to be $C^*(t)$.

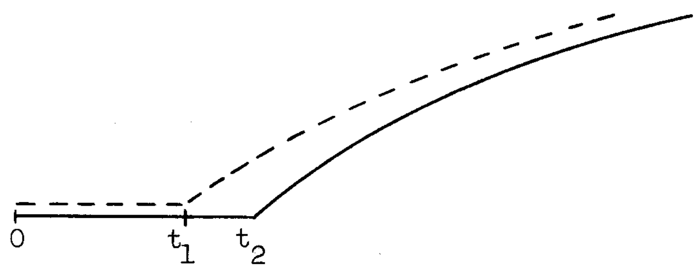

FIGURE 2.

The dashed policy indicates no consumption in $(0,t_1)$ and consumption according to the differential equation in (t_1,∞), with $\int_o^\infty e^{-jt}x(t)dt = M$. Let us show that such a policy can be constructed when $x^*(t_o+0) > 0$ and t_1 is close to t_o. For consider the consumption curve which follows the differential equation in (t_1,∞) and for which $x(t_1+0) = 0$. For t_1 close to t_o, $C(\infty)$ for this policy, will be close to the $C(\infty)$ for the policy which follows the differential equation from t_o onwards and for which $x(t_o+0) = 0$. But for such a policy $C(\infty) < M$, since $\int e^{-jt}x(t)dt$ is monotone in the initial conditions (because of the properties of φ' and φ''.) Therefore for t_1 close to t_o, discounted expenditure will be less than M if we start out with zero expenditure at

t_1. Raise the expenditure gradually at t_1, until discounted expenditure is actually equal to M and we get the dashed policy.

Now let us compute (11) where x is the dashed policy. This integral may be written as

$$(20) \qquad \int_{t_1}^{t_o} e^{-it}\varphi'(x^*(t))(x^*-x)dt + \int_{t_o}^{\infty} e^{-it}\varphi'(x^*(t))(x^*-x)dt.$$

In the first integral $x^* = 0$ and therefore it may be written as

$$-\varphi'(0) \int_{t_1}^{t_o} e^{-it} x dt$$

$$= -\varphi'(0) \int_{t_1}^{t_o} e^{-jt} e^{(j-i)t} x dt$$

$$< -\varphi'(0)\, e^{(j-i)t_1} \int_{t_1}^{t_o} e^{-jt} x dt\,.$$

Consider the second integral in (20) In this integral x^* satisfies the differential equation and therefore

$$e^{(j-i)t}\varphi'(x^*(t)) = e^{(j-i)t_o}\varphi'(x^*(t_o)),$$

so that the integral may be written as

$$e^{(j-i)t_o}\varphi'(x^*(t_o)) \int_{t_o}^{\infty} e^{-jt}(x^*-x)dt.$$

Now let us pick t_1 so close to t_o such that

$$\varphi'(0)\, e^{(j-i)t_1} > \varphi'(x^*(t_o))\, e^{(j-i)t_o}\,,$$

which we can do since $x^*(t_o) > 0$. Then (20) is less then

$$e^{(j-i)t_o}\varphi'(x^*(t_o))\left\{\int_{t_o}^{\infty} e^{-jt}(x^*-x)\,dt - \int_{t_1}^{t_o} e^{-jt}x\,dt\right\} = 0,$$

and this shows that x^* is not optimal if $t_o > 0$ and $x^*(t_o + 0) > 0$.

If these remarks are combined, they indicate the procedure for constructing the optimal consumption plan when $i < j$. Pick an initial consumption level $x^*(0) \geq 0$, and solve the differential equation (5) with this initial condition. If there is an initial condition such that the resulting solution gives $\int_0^{\infty} e^{-jt}x^*(t)dt = M$, then x^* is the optimal plan. On the other hand if all solutions give rise to an $x^*(t)$ with $\int_0^{\infty} e^{-jt}x^*(t)dt > M$, then we must have recourse to an initial interval of zero consumption, say $(0, t_o)$. But the point t_o is quite simple to determine. Solve the differential equation (5) with the initial condition $x^*(0) = 0$. Then

$$e^{jt_o}M = \int_0^{\infty} e^{-jt}x^*(t)\,dt, \tag{21}$$

and the actual optimal plan is given by $x^*(t-t_o)$ for $t \geq t_o$.

This disposes of the case when the personal rate of interest is less than the market interest rate. A similar set of remarks may be made when the personal interest rate i is larger than the market rate. By reasoning similar to that used above it is possible to show that the optimal consumption plan consists of at most two parts: a part governed by the solution of the differential equation (5) and a part of zero consumption. In this case however the zero consumption level, if it exists, must be connected to infinity, rather than to the origin. It is also possible to show that if such an interval (t_o, ∞) exists, then $x^*(t_o - 0) = 0$.

Therefore to solve the problem in the case where $i > j$, we select an initial condition $x^*(0) > 0$, and solve the differential equation (5) with this initial condition. If the solution becomes negative at some point t_o, then cut it off at this point and continue with zero consumption. In either event this defines a function $x^*(t)$. If for this function $\int_o^\infty e^{-jt} x^* dt < M$ take a larger initial consumption, if $\int_o^\infty e^{-jt} x^* dt > M$, take a smaller, until equality is reached. The point t_o may be found by solving the differential equation backwards with the initial condition of zero consumption.

These remarks constitute Theorem I of Section I.

It might be instructive to develope an example at this point. Let us consider $\varphi(x) = \log x$. Then the differential equation (5) becomes

$$\frac{dx}{dt} = (j-i)\,x, \qquad \text{whose solution is}$$

$$x(t) = x(0)\, e^{(j-i)t}, \qquad \text{and in order to have}$$

$$\int_o^\infty e^{-jt}\, x(t)dt = M, \qquad \text{we select } x(0) = iM, \text{ so that}$$

(22) $$x^*(t) = iMe^{(j-i)t}.$$

It is interesting to note that in this case we never have intervals of zero consumption.

If the optimal consumption plan is used the discounted assets at time $t, M(t)$, satisfies the differential equation

$$\frac{d\mathcal{M}}{dt} = j\,\mathcal{M} - x(t)\,,$$

from which we see that

$$\mathcal{M}\,(t) = \mathcal{M}\,e^{(j-i)t}\,,$$

and therefore the optimal consumption plan is to consume at a rate proportional to the assets, the proportionality constant being the personal interest rate. The utility functions $\varphi(x) = x^{\alpha}$ for $\alpha < 1$ also lead to consumption at a rate proportional to assets, but with different proportionally constants. In this later case, for certain relations among the parameters , it may be possible to construct an infinite sequence of consumption plans which lead in the limit to an infinite utility, whereas the plans themselves essentially defer consumption for longer and longer periods.

IV. <u>The Behavior of the Optimal Consumption Plan as $\mathcal{M}$ varies</u>.

As in the preceding section we consider a consumer whose utility function is composed of a constant personal interest rate i, and a single period utility function φ which is concave, increasing. When an optimal consumption plan exists it will depend on time, the market interest rate j and the value of the assets $\mathcal{M}$. We may therefore denote this optimal plan by $x(t; j, \mathcal{M})$.

Let us denote by $\mathcal{M}(t)$ the assets of the consumer at time t. Then it is easy to see that $\mathcal{M}(t)$ satisfies the differential equation

$$\frac{d\mathcal{M}}{dt} = j\,\mathcal{M}\,(t) - x(t), \tag{23}$$

with the initial condition $\mathcal{M}(0) = \mathcal{M}$.

Lemma 2. If the personal interest rate is less than the market interest rate, then $\mathcal{M}(t)$ is increasing and $x(t) \leq j\mathcal{M}(t)$. If on the other hand the personal interest rate is larger than the market rate then $\mathcal{M}(t)$ is decreasing and $x(t) \geq j\mathcal{M}(t)$.

The proof of this important proposition depends on the following observation. Let $x(t)$ be the optimal consumption plan, and $\mathcal{M}(t)$ the associated function measuring assets at time t. Let t^* be any point greater than zero and let us consider the problem of selecting a consumption plan in the interval from t^* to infinity, which maximizes utility subject to the constraint

$$\int_{t^*}^{\infty} e^{-j(t-t^*)}\, x(t)\, dt \leq \mathcal{M}(t^*).$$

It follows, from the assumption of the independence of future and past consumption , that the optimal plan for this problem is to continue, from t^* onwards, the optimal plan for the original problem. This observation implys that in order to demonstrate that $x(t) \leq j\mathcal{M}(t)$ (in the case $i < j$ say), it is sufficient to demonstrate this fact for $t = 0$. But this is quite easy, since according to Theorem 1, the consumption rate $x(t)$ is increasing when $i < j$. Therefore

$$\begin{aligned}\mathcal{M}(0) &= \int_0^{\infty} e^{-jt} x(t)\,dt \\ &\geq x(0) \int_0^{\infty} e^{-jt}\,dt \\ &= \frac{x(0)}{j}\,.\end{aligned}$$

It follows that $x(t) \leq j\mathcal{M}(t)$, and from the differential equation for $\mathcal{M}(t)$ we see that it is always increasing for $i < j$. For $i > j$ we use an argument based on the fact that $x(t)$ decreases.

In the process of this argument we observed the interesting fact that

$$(24) \qquad x(t;\, j, \mathcal{M}(0)) \equiv x(0;\, j, \mathcal{M}(t)),$$

as an identity. If we differentiate both sides of (24) with respect to t, we get

$$(25) \qquad \frac{dx(t;\, j, \mathcal{M}(0))}{dt} = \frac{\partial x(0;\, j, \mathcal{M})}{\partial \mathcal{M}} \frac{d\mathcal{M}}{dt}.$$

Since both $\frac{dx}{dt}$ and $\frac{d\mathcal{M}}{dt}$ have the same algebraic sign we may conclude that $\frac{\partial x(0;\, j, \mathcal{M})}{\partial \mathcal{M}} \geq 0$. More in fact may be said. Instead of (24) we may use the identity

$$(26) \qquad x(t+t_o;\, j, \mathcal{M}(0)) = x(t_o;\, j, \mathcal{M}(t)).$$

Differentiating both sides of this identity with respect to t, we obtain $\frac{\partial x(t_o;\, j, \mathcal{M})}{\partial \mathcal{M}} \geq 0$, for all t_o. We obtain

<u>Lemma 3</u>. The wealthier the consumer (in terms of present plus future assets) the larger his expenditure.

The type of behavior described in Lemmas 2 and 3 is of course what one would expect. It is interesting to see that these characteristics arise from a mathematical formulation of the problem, though hardly surprising. We shall now turn our attention to a more subtle type of behavior and see whether it may be accounted for in our model.

The quantity M is not in any way synonymous with income, though barring unusually large legacies etc., one would expect it to be high for people with high incomes and low for people with low incomes. There seems to be considerable evidence that people with high incomes, while they spend more than people with low incomes, actually spend a smaller fraction of their income than people with smaller incomes. In our model this might be described by saying that

$$(27) \qquad \frac{\partial \left\{ \dfrac{x(0; j, M)}{M} \right\}}{\partial M} \leq 0,$$

and we shall now examine the conditions under which this is true. We shall begin by establishing a sufficient condition for (27), and then investigate its necessity.

Theorem 2. If $\frac{d}{dx} \frac{\varphi'}{x\varphi''} \leq 0$, then

$$\frac{\partial \left\{ \dfrac{x(0; j, M)}{M} \right\}}{\partial M} \leq 0 \quad \text{for all} \quad j \geq i.$$

On the other hand if $\frac{d}{dx} \frac{\varphi'}{x\varphi''} \geq 0$, then $\dfrac{\partial \left\{ \dfrac{x(0; j, M)}{M} \right\}}{\partial M} \leq 0$ for all $j \leq i$. In both of these cases the assumption is made that an optimal consumption plan exists.

Let us demonstrate the first part of the theorem under the assumption that the market interest rate is larger than the personal interest rate. We begin by noticing that if $\frac{d}{dx} \frac{\varphi'}{x\varphi''} \leq 0$, then the optimal consumption plan requires a positive consumption rate when $t = 0$. To see this we first remark that since $\frac{\varphi'}{x\varphi''}$ is a negative decreasing function it must be finite when $x = 0$, and this implies that $\frac{\varphi'}{\varphi''}(x)$ is zero when $x = 0$. If there is an interval of zero consumption connected to the origin or if $x(0) = 0$, then the part of the optimal consumption plan which is non-zero would satisfy the equation

$$\frac{dx}{dt} = (i - j)\,\frac{\varphi'}{\varphi''}$$

with an initial condition equal to zero. But with this initial condition and $\frac{\varphi'}{\varphi''}(0) = 0$, the solution would be identically zero which is impossible. Therefore the consumption plan is always strictly positive, and given by the differential equation.

Now

$$\text{(28)} \qquad \frac{\partial\, \frac{x(0;\, j, M)}{M}}{\partial M}$$

has the same sign as

$$\text{(29)} \qquad M\,\frac{\partial x}{\partial M} - x$$

$$= \frac{M\,\frac{dx}{dt}}{\frac{dM}{dt}} - x$$

$$\text{(30)} \qquad \frac{M(i-j)}{jM - x}\,\frac{\varphi'}{\varphi''} - x\,,$$

where everything is evaluated at $t = 0$. If we take the formula

$$M = \int_0^\infty e^{-jt}\,x(t)\,dt\,,$$

and integrate by parts we obtain

$$\mathcal{M} = \frac{x}{j} + \frac{1}{j}\int_0^\infty e^{-jt}\,\frac{dx}{dt}\,dt$$

$$= \frac{x}{j} + \frac{(i-j)}{j}\int_0^\infty e^{-jt}\,\frac{\varphi'}{\varphi''}(x(t))dt$$

so that

(31) $$\frac{j\mathcal{M}-x}{i-j} = \int_0^\infty e^{-jt}\,\frac{\varphi'}{\varphi''}(x(t))dt.$$

But $\frac{\varphi'}{x\varphi''}$ is decreasing and $x(t)$ is increasing so that

$$\frac{1}{x(t)}\,\frac{\varphi'}{\varphi''}(x(t)) \le \frac{1}{x(0)}\,\frac{\varphi'}{\varphi''}(x(0)).$$

Using (31) we see that

(32) $$\frac{j\mathcal{M}-x}{i-j} \le \frac{\varphi'}{x\varphi''}(x)\int_0^\infty e^{-jt}\,x(t)dt$$
$$= \frac{\varphi'}{x\varphi''}(x)\,\mathcal{M}\,. \qquad (x = x(0;\ j,\mathcal{M}).)$$

The right hand side of (32) is, of course, negative and we multiply by its reciprocal we reverse the sense of the inequality, obtaining

(33) $$\frac{\mathcal{M}(i-j)}{j\mathcal{M}-x}\,\frac{\varphi'}{\varphi''} - x \le 0.$$

If we examine (30) we see that this is just what is needed to demonstrate the first part of the theorem. The second half of the theorem may be demonstrated in the same way. Some care must be taken because there may actually exist an interval of zero consumption connected to ∞.

Now let us turn our attention to the necessity of the conditions described in Theorem 2. As we shall see these conditions are not only sufficient but actually necessary if the optimal consumption plan is to have the property that the higher the assets, the smaller is the actual fraction of the assets spent. Let us begin by investigating the optimal consumption plan when the market rate of interest is close to the personal interest rate. Of course, when the two interest rates are the same the optimal consumption plan is $x(t) = i\mathcal{M}$. We shall find it convenient to assume that for j close to i there are no intervals of zero consumption, so that the consumption is governed by one differential equation. This assumption may be relaxed but the argument becomes more tedious.

The differential equation implies

$$(34) \qquad e^{(j-i)t}\varphi'(x(t;\, j,\mathcal{M})) = \varphi'(x(0;\, j,\mathcal{M}))$$

and if we differentiate both sides of this equation with respect to j and then set $j = i$, we obtain

$$\left.\frac{\partial x(0;\, j,\mathcal{M})}{\partial j}\right|_{j=i} = \left.\frac{\partial x(t;\, j\,\mathcal{M})}{\partial j}\right|_{j=i} + t\,\frac{\varphi'(i\mathcal{M})}{\varphi''(i\mathcal{M})}\,.$$

We may use this expression to obtain an explicit value for $\left.\frac{\partial x(0;\, j,\mathcal{M})}{\partial j}\right|_{j=i}$ by differentiating the relationship $\mathcal{M} = \int_0^\infty e^{-jt}x(t,j\,\mathcal{M})dt$ with respect to j, and then setting $j=i$, so that

$$(36) \qquad 0 = -i\mathcal{M}\int_0^\infty t e^{-it}dt + \int_0^\infty e^{-it}\left.\frac{\partial x(t;\, j,\mathcal{M})}{\partial j}\right|_{j=i} dt$$

$$\frac{\mathcal{M}}{i} = \int_0^\infty e^{-it}\left\{\left.\frac{\partial x(0;\, j,\mathcal{M})}{\partial j}\right|_{j=i} - t\,\frac{\varphi'(i\mathcal{M})}{\varphi''(i\mathcal{M})}\right\}dt = \frac{1}{i}\left.\frac{\partial x(0;\, j,\mathcal{M})}{\partial j}\right|_{j=i} - \frac{1}{i^2}\,\frac{\varphi'(i\mathcal{M})}{\varphi''(i\mathcal{M})}\,,$$

and as a consequence

$$\left.\frac{\partial x(0;j,M)}{\partial j}\right|_{j=i} = \frac{iM + \frac{\varphi'(iM)}{\varphi''(iM)}}{i} . \tag{37}$$

This is an explicit expression for the rate of change of initial consumption with respect to the market interest rate, M being held constant, and therefore we may represent the actual initial consumption $x(0;j,M)$ for market rates close to the personal interest rate i, by the Taylor series expansion

$$x(0;j,M) = iM + (j-i)\left(\frac{iM + \frac{\varphi'(iM)}{\varphi''(iM)}}{i}\right) + \ldots \tag{38}$$

The ratio of initial consumption to total assets is given by

$$\frac{x(0;j,M)}{M} = i + (j-i)\left\{1 + \frac{\varphi'(iM)}{iM\,\varphi''(iM)}\right\} + \ldots . \tag{39}$$

It is clear that if we require the left-hand side of (39) to be decreasing in M for all market rates j close to i and larger than i, then we must have $\frac{d}{dx}\left(\frac{\varphi'(x)}{x\varphi''(x)}\right) \leq 0$, and if we require the left-hand side to be decreasing for all market rates j close to i and less than i, then we must have $\frac{d}{dx}\left(\frac{\varphi'(x)}{x\varphi''(x)}\right) \geq 0$. These results form a converse to Theorem 2.

<u>Theorem 3</u>. If $\frac{x(0;j,M)}{M}$ is decreasing for $j \geq i$ then $\frac{d}{dx}\left(\frac{\varphi'}{x\varphi''}\right) \leq 0$, and if it is decreasing for $j \leq i$ then $\frac{d}{dx}\left(\frac{\varphi'}{x\varphi''}\right) \geq 0$.

It follows from Theorem 3 that the only utility functions φ for which we may have $\frac{x(0;j,M)}{M}$ decreasing for all market interest rates are the utility functions satisfying the condition $\frac{\varphi'}{\varphi''} = cx$ where c is a negative constant. Such a utility function, however, must be equivalent to either $\log x$ or some power of x, and for utility functions of this form if an optimal consumption plan exists then consumption is strictly proportional to total assets. The upshot of this argument is that the property of a strictly decreasing ratio of consumption to assets for all market interest rates cannot be explained by a constant personal interest. My own feeling on this matter is that that the personal interest rate is indeed dependant upon the consumption level, and in the next section we shall examine utility functions with variable personal interest to see whether a strictly decreasing ratio of consumption to assets may be obtained. It is , of course, possible for alternative explanations to be offered.

V. The Optimal Consumption Plan with Variable Personal Interest Rate.

In the previous section we arrived at the conclusion that a decreasing ratio of consumption to total assets for all market rates of interest could not be explained by a constant personal interest rate i. The question arises as to whether this phenomenon can be explained by a variable personal interest rate. In this section we shall exhibit a simple stationary utility function with a variable personal interest for which the ratio of consumption to assets is strictly decreasing, for all market interest rates. We are not contending that this utility function has much in the way of economic significance; it is merely meant to be an example of a variable interest rate and has been chosen primarily because of the simplicity of the calculation. We shall defer the general problem of a variable interest rate until a subsequent paper.

The general stationary utility function is

In the process of preparing this report the author became aware of the fact that the example given in this section is slightly artificial, in the sense that an infinite utility may be obtained for any J and M. If the personal interest rate $\frac{1}{x}$ is replaced by $1 + \frac{1}{x}$, this can no longer happen. The techniques of this section may be applied to this new interest rate with only minor changes.

$$i(x) = \frac{1}{x} \text{ and} \qquad (40)$$
$$\varphi(x) = 1.$$

This interest rate decreases as consumption increases, as it should. For small levels of consumption there is an exceptionally high interest rate, expressing a strong preference for immediate consumption, while for large consumption levels the personal interest rate is small expressing an indifference between present and future consumption.

The selection of $\varphi(x)$ as identically 1 may seem somewhat surprising. However, φ itself is not to be interpreted as a utility indicator. In fact, it is easily to see that

$$i(x) = \frac{1}{x} \text{ and}$$
$$\varphi(x) = \frac{a}{x} + b, \quad \text{with} \quad b > 0,$$

describes a preference pattern equivalent to (40) and this latter representation of the preference pattern may appear to be somewhat more reasonable.

Now let us turn our attention to the problem of maximizing

$$U(x(\cdot)) = \int_0^{\infty} e^{-\int_0^t \frac{d\xi}{x(\xi)}} dt \qquad (41)$$

with respect to the constraints

$$x(\xi) \geq 0, \quad \text{and} \quad \int_0^{\infty} e^{-jt} x(t)dt \leq M.$$

It is clear that an optimal consumption plan can never be zero on an interval, unless that interval is connected to ∞, and we shall only consider plans of this form. Let x^* be optimal, positive in $(0, T)$ except perhaps for isolated zeros and zero thereafter (T may be ∞), and x any other plan of the same type, satisfying the constraints. We define

(42) $J(\theta) = U(\theta x^* + (1-\theta) x)$. If x^* is optimal it is necessary that $J'(1) \geq 0$, and we obtain

$$0 \leq \int_0^T e^{-\int_0^t \frac{d\xi}{x^*(\xi)}} \int_0^t \frac{x^*(u) - x(u)}{\left[x^*(u)^2\right]} \, du \, dt \,. \tag{43}$$

We change the order of integration in (43) and obtain

$$0 \leq \int_0^T \frac{x^*(u) - x(u)}{x^*(u)^2} \int_u^T e^{-\int_0^t \frac{d\xi}{x^*(\xi)}} \, dt \, du$$

$$= \int_0^T e^{-ju} \left[x^*(u) - x(u) \right] \left[\frac{e^{ju} \int_u^T e^{-\int_0^t \frac{d\xi}{x^*(\xi)}} dt}{\left[x^*(u) \right]^2} \right] du$$

We define, as before, $C(u) = \int_0^u e^{-jt} x(t) dt$, and a corresponding function C^* so that $C(T) = C^*(T) = M$, and then integrate by parts. If we define

$$f(u) = \frac{d}{du} \left[\frac{e^{ju} \int_u^T e^{-\int_0^t \frac{d\xi}{x^*(\xi)}} dt}{\left[x^*(u)\right]^2} \right], \tag{45}$$

then (44) becomes

$$0 \leq \int_0^T \left[C^*(u) - C(u) \right] f(u) \, du,$$

so that $C^*(u)$ maximizes

$\int_0^\infty C(u)\, f(u)\, du$, subject to the constraints on the definition of C. At this point we may reason precisely as we did in Section III, to show that if $f(u)$ is different from zero in an interval, then x^* most be identically zero in that interval. But in our case this is impossible, and we may therefore conclude that $f(u) \equiv 0$, or

$$\left[x^*(u)\right]^2 = c\, e^{ju} \int_u^T e^{-\int_0^t \frac{d\xi}{x^*(\xi)}} dt, \qquad \text{for } 0 \leq u \leq T,$$

where c is a constant depending on j and M. But this is equivalent to

$$(46) \qquad \left[x^*(u)\right]^2 = c\, e^{ju} \int_u^\infty e^{-\int_0^t \frac{d\xi}{x^*(\xi)}} dt, \qquad \text{for all } u.$$

If we take the logarithmic derivative of both sides of (46) we obtain the somewhat more useful form

$$(47) \qquad 2\,\frac{d \log x^*(u)}{du} = j - \frac{e^{-\int_0^u \frac{d\xi}{x^*(\xi)}}}{\int_u^\infty e^{-\int_0^t \frac{d\xi}{x^*(\xi)}} dt} .$$

By differentiating (47) we see that x^* satisfies the differential equation

$$(48) \qquad 2\,\frac{d^2 \log x^*(u)}{du^2} = \frac{1}{x}\left(j - 2\,\frac{d \log x^*}{du}\right) - \left(j - 2\,\frac{d \log x^*}{du}\right)^2 ,$$

which we shall also find useful.

We shall now turn our attention to showing that the optimal consumption plan requires a decreasing ratio of consumption to assets for all market interest rates.

<u>Lemma 4.</u> The optimal consumption plan is unique.

We shall not include in this paper a proof of the actual existence of the optimal consumption plan, but such a proof may be given. Suppose that there is a value $\mathcal{M}_o > 0$, for which there are two distinct optimal consumption plans, x_1^* and x_2^*. If we examine (47) when $u = 0$, and realize that $\int_o^\infty e^{-\int_o^t \frac{d\xi}{x^*}} dt$ is the optimal utility which must be the same for both plans, we see that $\frac{d \log x^*(u)}{du}$, evaluated at $u = 0$, is the same for both plans. Let us also show that $x^*(0)$ is the same for both plans. For if this were not so, then we could find a consumption problem which begins at time $t = -\Delta$, with a value of $\mathcal{M}(-\Delta)$, such that the optimal consumption plan with these conditions gives $\mathcal{M}(0) = \mathcal{M}_o$. But we have two alternative ways of continuing from $t = 0$, and they cannot both satisfy (46) for $t > -\Delta$ unless $x^*(0)$ is the same for both plans. Therefore $\log x_1^*(u)$ and $\log x_2^*(u)$ are both solutions of the same second order differential equation (48), with the same initial conditions, and they must be identical. This proves Lemma 4.

The next lemma is the crucial one in demonstrating the decreasing ratio of consumption to assets.

<u>Lemma 5.</u> Let $\mathcal{M}(t)$ be non-decreasing at the point t. Then $\frac{d^2 \log x^*(u)}{du^2} \geq 0$, at this point. If $\mathcal{M}(t)$ is non-increasing at t then $\frac{d^2 \log x^*(u)}{du^2} \leq 0$ at the point t.

Let us consider the increasing case. It is clearly sufficient to consider $t = 0$ only. As in the case of constant personal interest rate we have (see (26)),

$$x(t + t_o, j, \mathcal{M}(0)) \equiv x(t_o, j, \mathcal{M}(t)). \tag{49}$$

We take the logarithims of both sides of (49) and differentiate with respect to t_o, obtaining

$$\frac{\partial \log x(t + t_o, j, \mathcal{M}(0))}{\partial t_o} \equiv \frac{\partial \log(x(t_o), j, \mathcal{M}(t))}{\partial t_o}.$$

We now differentiate both sides of this expression with respect to t, obtaining

$$\frac{\partial^2 \log x(t + t_o, j, \mathcal{M}(0))}{\partial t^2} = \frac{\partial^2 \log(x(t_o), j, \mathcal{M}(t))}{\partial t_o \partial \mathcal{M}} \frac{d\mathcal{M}}{dt}.$$

When t_o and t are both zero, the expression

$$\left.\frac{\partial \log(x(t_o), j, \mathcal{M}(0))}{\partial t_o}\right|_{t_o = 0}$$

is equal to $\frac{j}{2} - \frac{1}{2(\text{optimal utility})}$ by (47), and this is certainly increasing in $\mathcal{M}(0)$. This proves Lemma 5.

We are now ready to prove the main theorem of this section.

Theorem 4. The optimal consumption plan for the utility function considered in this section always requires a decreasing ratio of expenditures to assets.

Let us begin by considering the special case where the optimal consumption plan gives rise to an non-decreasing $\mathcal{M}(t)$ for all $t \geq 0$. Then from Lemma 5, we have $\frac{d^2 \log x(u)}{du^2} \geq 0$. Now we are interested in

$\frac{\partial}{\partial \mathcal{M}} \left(\frac{x(0, j, \mathcal{M})}{\mathcal{M}} \right)$ which has the same sign as

(50) $$\frac{\mathcal{M}\frac{dx}{dt}}{j\mathcal{M} - x} - x .$$

Using the same argument as the one immediately following (30), we have

$$j\mathcal{M} - x = \int_0^\infty e^{-jt} \frac{dx}{dt}\, dt$$

$$= \int_0^\infty e^{-jt} x(t) \frac{d \log x(t)}{dt}\, dt.$$

But $\frac{d \log x(t)}{dt}$ is increasing, so that

(51) $$j\mathcal{M} - x > \left.\frac{d \log x(t)}{dt}\right|_{t=0} \mathcal{M}$$

$$= \left.\frac{dx}{dt}\right|_{t=0} \frac{\mathcal{M}}{x} .$$

But this is the same as saying that (50) is negative, which we wanted to prove. The proof when $\mathcal{M}(t)$ is decreasing for all t runs along the same lines. We run into a possible problem only when $\frac{d\mathcal{M}}{dt}$ is equal to zero for some t with $\mathcal{M}(t) > 0$. As we shall show this can only occur when $x^*(t)$ is constant for all t. For suppose that $\mathcal{M}'(0) = 0$ (we take $t = 0$, with no loss in generality). Then $x(0) = j\mathcal{M}(0)$, and from Lemma 5, we have $\frac{d^2 \log x(u)}{du^2} = 0$, at $t = 0$. But from (48) since $j > 2 \frac{d \log x}{du}$ (see (47)), we obtain

(52) $$\frac{1}{x(0)} = j - 2\,\frac{d \log x}{du} \;.$$

This tells us, however, (see (47)) that the optimal utility is equal to $x(0) = j\mathcal{M}(0)$. But it is easy to see that this utility may be obtained by the policy $x^*(u) \equiv j\mathcal{M}(0)$, and since this is the optimal utility, this implies that $x^*(u) = j\mathcal{M}(0)$ is the optimal consumption plan for this initial condition. It also is true, from (52) that

$\frac{1}{x(0)} = j$, so that $\frac{1}{\mathcal{M}(0)} = j^2$.

It is easy to show from these observations that no optimal plan other than a constant consumption level ever gives rise to a zero of $\mathcal{M}'(t)$. For if this were to occur the optimal consumption plan past that zero would be constant, and equal to $j\mathcal{M}(t)$ at the zero. However the differential equation may be solved backwards and yields the same constant consumption plan for all time; a plan for which $\mathcal{M}'(t) \equiv 0$. This demonstrates Theorem 4 in the following way. If $\mathcal{M} < \mathcal{M}_0 = \frac{1}{j^2}$ then the consumption ratio decreases, and similarly if $\mathcal{M} > \mathcal{M}_0$. At $\mathcal{M} = \mathcal{M}_0$ it is easy to show by a separate argument that $\frac{x}{\mathcal{M}_0}$ is larger than the corresponding consumption ratios for larger $\mathcal{M}$ and smaller then the corresponding ratios for smaller $\mathcal{M}$.

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