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Approximate Solutions to a Simple Multi-Echelon Inventory Problem

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1. Introduction

The multi-echelon inventory problem is concerned with the determination of optimal purchasing and trans-shipment rules for a single item which may be stocked at several installations and whose use is subject to stochastic variability. The problem is a sequential decision problem whose solution may be found, at least theoretically, by a sequence of dynamic programming calculations. The calculations, however, involve functions of at least as many variables as there are installations to be considered, and possibly even more. It is the computational burden imposed by working with functions of several variables that makes it necessary to resort either to approximations or to simplifying assumptions.

In a previous paper [2], a list of assumptions was presented that reduced the problem of calculating optimal policies to the recursive calculation of functions of a single variable for one very restrictive type of multi-echelon situation, the arrangement of the various installations in series: $\mathbf{n} \to \mathbf{n} - \mathbf{1} \to \cdots \to \mathbf{2} \to \mathbf{1}$. A procedure, which again involved functions of only one variable, was also suggested for the more general multi-echelon situation in which a single installation was capable of shipments to several alternative installations, and reasons were given for our belief that this procedure, while not optimal, represented a good approximation.

Admittedly, work remains to be done in the refinement of our method and in the examination of alternative approaches for a general multi-echelon arrangement. However, even for the extremely simple echelon structure described above, one of the assumptions of our previous paper has been criticized, and it is the purpose of this paper to weaken this assumption.

The assumption in question is assumption 2 [2, p. 478], that the cost of shipping stock from installation 2 to installation 1, or more generally from any installation other than the top one to its immediate successor, is proportional to the quantity of stock shipped, with no setup or reorder cost.

To demonstrate the relevance of this assumption, we shall refer to the example that is considered in detail in [2]. (The reader is advised to consult that paper for notation, for the other assumptions which we are not describing explicitly, and for background material.)

Consider a situation with two installations, $2 \to 1$, with demand arising at installation 1 according to a probability distribution with density φ. The stock at installation 1 immediately prior to ordering is x_1, and the stock at installation 1 plus the stock at installation 2 plus the stock in transit is denoted by x_2. To be specific, we shall assume that the lead time in delivery from installation 2 to installation 1 is two periods, and that deliveries to installation 2 are effected in a single period. We shall also denote by w_1 the stock in transit to installation 1 marked for delivery next period.

The state of the system is described by the three variables x_1, w_1, x_2, and the optimal purchase and shipment policy may be found by the recursive calculations of functions $C_n(x_1, w_1, x_2)$, defined as the minimum discounted expected cost if there are n periods remaining in the program and if the current state is (x_1, w_1, x_2). Of course, it is precisely this calculation that we wish to avoid.

The procedure described in [2] begins by neglecting the second installation and computing the optimal policies at installation 1, disregarding the possibility that there may be no stock at installation 2 to implement these policies. It is assumed that excess demand at each echelon is backlogged. For this reason the optimal policy at the lower installation, computed by itself, will depend on stock on hand plus stock on order [1]; moreover, the optimal policy may be found by means of the recursive calculation

$$(1) \qquad f_n(u) = \min_{y \geq u} \left\{ c_1(y-u) + \alpha^2 \iint L(y - \xi_1 - \xi_2)\varphi(\xi_1)\varphi(\xi_2)\,d\xi_1\,d\xi_2 + \alpha \int_0^\infty f_{n-1}(y-\xi)\varphi(\xi)\,d\xi \right\},$$

with c_1 the unit cost of shipping stock from installation 2 to installation 1, $L(y)$ the expected holding and shortage cost per period as a function of stock on hand at the beginning of the period, and α the discount factor. In the calculation, f_1 and f_2 are set equal to zero. The relationship between $f_n(u)$ and the minimum cost functions $C_n(x_1, w_1)$ associated with the lower level alone is

$$(2) \qquad C_n(x_1, w_1) = L(x_1) + \alpha \int L(x_1 + w_1 - \xi)\varphi(\xi)\,d\xi + f_n(x_1 + w_1)\,.$$

(See [1], or [2, p. 477].)

Now if $L(y)$ is convex, as it is generally assumed to be, the optimal policy will be defined by a sequence of critical numbers $\bar{x}^3, \bar{x}^4, \cdots$ such that if $x_1 + w_1 < \bar{x}^n$ at the beginning of period n, an order is placed for the difference; if $x_1 + w_1 \geqq \bar{x}^n$, no order is placed.

The problem is to integrate this sequence of decisions, which would be optimal if installation 1 were considered by itself, with the constraint imposed by the possibility that the total system stock x_2 may itself be less than $\bar{x}^*$ making it impossible to raise the stock on hand plus the stock on order at the lower installation to this desired level.

In our previous paper we demonstrate that if $x_2 < \bar{x}^*$, all the stock at the higher installation should be shipped to the lower, and on the other hand, if $x_2 > \bar{x}^*$, just that part should be shipped which is sufficient to raise stock on hand plus stock on order at the lower installation to $\bar{x}^*$. This specifies part of the system-wide optimal policy.

There remains the problem of determining the appropriate purchasing decisions for echelon 2. If these decisions were to be made independently of their influence on the lower level, insufficient stock would be procured. The important feature of the optimal system policy as demonstrated in [2] is that purchasing decisions for the entire system may be made on the basis of the consideration of *system stock alone* if the "natural" echelon holding and shortage costs are augmented by an additional shortage cost function that penalizes the system for its inability to deliver the required amount of stock to the lower level.

The additional shortage cost function may be computed in the following intuitive way: If $x_2 > \bar{x}^*$, there is no constraint on the delivery of stock to the lower level, and therefore the additional shortage cost will be zero. On the other hand, if $x_2 < \bar{x}^*$, stock on hand plus stock on order at the lower level can be raised to at most x_2. If $x_1 + w_1 = u < \bar{x}^*$, then

$$f_n(u) = c_1(\bar{x}^* - u) + \alpha^2 \iint L(\bar{x}^* - \xi_1 - \xi_2)\varphi(\xi_1)\varphi(\xi_2)\, d\xi_1\, d\xi_2 + \alpha \int f_{n-1}(\bar{x}^* - \xi)\varphi(\xi)\, d\xi ,$$

with $f_n(u)$ the minimum discounted cost for the future of the program (exclusive of those costs beyond our immediate control). We cannot achieve this cost, since it is impossible to raise u to the level $\bar{x}^*$; at best we can raise it to x_2, thereby incurring a cost of

$$c_1(x_2 - u) + \alpha^2 \iint L(x_2 - \xi_1 - \xi_2)\varphi(\xi_1)\varphi(\xi_2)\, d\xi_1\, d\xi_2 + \alpha \int f_{n-1}(x_2 - \xi)\varphi(\xi)\, d\xi .$$

The difference between these two expressions is the additional cost to be charged to the system because of its inability to meet requests from below; it may be written

$$\Delta_n(x_2) = c_1(x_2 - \bar{x}^*) + \alpha^2 \iint [L(x_2 - \xi_1 - \xi_2) - L(\bar{x}^* - \xi_1 - \xi_2)]\varphi(\xi_1)\varphi(\xi_2)\, d\xi_1\, d\xi_2 + \alpha \int [f_{n-1}(x_2 - \xi) - f_{n-1}(\bar{x}^* - \xi)]\varphi(\xi)\, d\xi$$

when $x_2 < \bar{x}^*$, and $\Delta_n(x_2) = 0$ otherwise.

If this cost is added to the "natural" holding and shortage cost for the system [say $\bar{L}(x_2)$], then the optimal system policy is obtained by means of the standard functional equation

$$(3) \qquad g_n(x_2) = \min_{z\geq 0} \left\{ c(z) + \bar{L}(x_2) + \Lambda_n(x_2) + \alpha \int g_{n-1}(x_2 + z - \xi)\varphi(\xi)\, d\xi \right\}.$$

This intuitive argument summarizes the procedure for the determination of optimal policies given in our previous paper, which also contains a rigorous demonstration of its validity, based on the theorem

$$C_n(x_1, w_1, x_2) \equiv C_n(x_1, w_1) + g_n(x_2).$$

Our problem now is to attempt to incorporate a setup cost associated with the transportation of items from installation 2 to installation 1. It is therefore appropriate to ask, still on the intuitive level, for the part played by the assumption of no setup cost in the previous policy. First of all, the lack of a setup cost was responsible for the simple description of optimal policies at the lower level in terms of a sequence of single critical numbers $\bar{x}_3, \bar{x}_4, \cdots$. If a setup cost in transportation were included in the problem, the optimal policy would no longer be of this simple form. It is known, however, that if the customary assumptions are met (convexity of $L(x_1)$, backlogging of excess demand), then the optimal policies are of the (S, s) type, with a pair of numbers, S_n and s_n, relevant for each period [3]. These numbers are computed by means of a functional equation similar to (1), but written

$$(1') \quad f_n(u) = \min_{y\geq u} \left\{ K\cdot\delta(y-u) + c_1(y-u) + \alpha^2 \iint L(y - \xi_1 - \xi_2)\varphi(\xi_1)\varphi(\xi_2)\, d\xi_1\, d\xi_2 + \alpha \int f_{n-1}(u - \xi)\varphi(\xi)\, d\xi \right\},$$

where

$$\delta(x) = \begin{cases} 0 & x = 0, \\ 1 & x > 0. \end{cases}$$

Is it possible, in this case, to assign an additional shortage cost to the entire system, as a function of x_2 alone? This is the crucial point in the simplification described above, and we must see if this simplification is still possible when a setup cost is introduced.

Suppose that $x_1 + w_1 > s_n$. In this case no ordering is required, and it might seem reasonable to charge no additional shortage cost even if $S_n > x_2 \geqq x_1 + w_1 > s_n$. On the other hand, if $x_1 + w_1 < s_n$, the optimal policy would seem to be to request a shipment of size $S_n - (x_1 + w_1)$ from installation 2. If $x_2 < S_n$, it is impossible to meet this request, and it would seem reasonable to charge an additional shortage cost. Hence we are led to the conclusion that the appropriate shortage cost to be added when $x_2 < S_n$ seems to depend on whether $x_1 + w_1 > s_n$ or $x_1 + w_1 < s_n$, and is therefore *not* a function of x_2 alone. This conclusion precludes obtaining a simple form for the optimal system policy with a setup cost in transportation.

However, we can make use of a substantial amount of the preceding argument if we turn our attention from optimal policies to approximations. Instead of attempting to find the correct additional shortage cost Λ_n, we shall attempt to bound this function of x_2 and $x_1 + w_1$ from above and below by functions of x_2 alone; i.e., we shall establish lower and upper bounds $\underline{\Lambda}_n$ and $\bar{\Lambda}_n$, respectively, such that $\underline{\Lambda}_n \leqq \Lambda_n \leqq \bar{\Lambda}_n$. (We shall occasionally refer to $\bar{\Lambda}_n$ as the *upper* shortage cost, and $\underline{\Lambda}_n$ as the *lower* shortage cost.)

The upper bound $\bar{\Lambda}_n(x_2)$ will be obtained by charging an additional shortage cost whenever $x_2 < S_n$, regardless of the size of $x_1 + w_1$; in other words, by calculating as if $x_1 + w_1$ were less than s_n, in which case the true minimum cost would be

$$K + c_1(S_n - u) + \alpha^2 \iint L(S_n - \xi_1 - \xi_2)\varphi(\xi_1)\varphi(\xi_2)\, d\xi_1\, d\xi_2 + \alpha \int f_{n-1}(S_n - \xi)\varphi(\xi)\, d\xi \,,$$

with $u = x_1 + w_1$. We are constrained by having $x_2 < S_n$; if the stock at installation 1 and in transit to 1 is brought to x_2, the cost will be

$$K + c_1(x_2 - u) + \alpha^2 \iint L(x_2 - \xi_1 - \xi_2)\varphi(\xi_1)\varphi(\xi_2)\, d\xi_1\, d\xi_2 + \alpha \int f_{n-1}(x_2 - \xi)\varphi(\xi)\, d\xi \,.$$

We define $\bar{\Lambda}_n(x_2)$ as the difference between these two costs, charged for all values of $x_2 < S_n$.

$$\begin{aligned} (4) \qquad \bar{\Lambda}_n(x_2) = {} & c_1(x_2 - S_n) \\ & + \alpha^2 \iint [L(x_2 - \xi_1 - \xi_2) - L(S_n - \xi_1 - \xi_2)]\varphi(\xi_1)\varphi(\xi_2)\, d\xi_1\, d\xi_2 \\ & + \alpha \int [f_{n-1}(x_2 - \xi) - f_{n-1}(S_n - \xi)]\varphi(\xi)\, d\xi \qquad (x_2 \leqq S_n)\,, \end{aligned}$$

and $\bar{\Lambda}_n(x_2) = 0$ elsewhere.

The lower shortage cost $\underline{\Lambda}_n(x_2)$ will be defined as different from zero only for $x_2 \leqq s_n$; specifically,

$$\begin{aligned} (5) \qquad \underline{\Lambda}_n(x_2) = {} & -K + c_1(x_2 - S_n) \\ & + \alpha^2 \iint [L(x_2 - \xi_1 - \xi_2) - L(S_n - \xi_1 - \xi_2)]\varphi(\xi_1)\varphi(\xi_2)\, d\xi_1\, d\xi_2 \\ & + \alpha \int [f_{n-1}(x_2 - \xi) - f_{n-1}(S_n - \xi)]\varphi(\xi)\, d\xi \,. \end{aligned}$$

The upper and lower shortage costs may be used in two ways. First, they furnish us with a simple procedure, based on the calculation of functions of one variable, for estimating the true minimum cost function $C_n(x_1, w_1, x_2)$. The procedure is to define two sequences of functions, $\bar{g}_n(x_2)$ and $\underline{g}_n(x_2)$, by means of the functional equations

$$(6) \qquad \bar{g}_n(x_2) = \min_{z \geqq 0} \left\{ c(z) + \tilde{L}(x_2) + \bar{\Lambda}_n(x_2) + \alpha \int \bar{g}_{n-1}(x_2 + z - \xi)\varphi(\xi)\, d\xi \right\}$$

and

$$(7) \qquad \underline{g}_n(x_2) = \min_{z \geqq 0} \left\{ c(z) + \tilde{L}(x_2) + \underline{\Lambda}_n(x_2) + \alpha \int \underline{g}_{n-1}(x_2 + z - \xi)\varphi(\xi)\, d\xi \right\},$$

with $\bar{g}_1(x_2) = \underline{g}_1(x_2) = \tilde{L}(x_2)$.

If there were no setup cost at installation 1, these two sequences of functions would be identical and (3) would apply. In our case, with a positive setup cost, this is no longer true. However, as we shall demonstrate below, these functions furnish us with simple approximations to the minimum cost functions, namely

$$C_n(x_1, w_1) + \underline{g}_n(x_2) \leqq C_n(x_1, w_1, x_2) \leqq C_n(x_1, w_1) + \bar{g}_n(x_2) . \tag{8}$$

If the functions $\underline{g}$ and $\bar{g}$ are close, the bounds will be good, as computation of several examples has suggested. We shall see, moreover, that there is a specific policy associated with the computation of the functions $\bar{g}_n$, and that if this policy is adopted, the cost incurred is always less than $C_n(x_1, w_1) + \bar{g}_n(x_2)$. Therefore, if the two bounds are close, we have not only a good estimate for the true minimum cost, but also a simple policy whose cost is very close to the true minimum cost.

2. Verification of the Bounds

We next demonstrate the validity of the bounds given by equation (8). We shall examine first the upper bound and then the lower bound, which involves a slightly different argument.

The argument will be an inductive one; we shall assume that

$$C_{n-1}(x_1, w_1, x_2) \leqq C_{n-1}(x_1, w_1) + \bar{g}_{n-1}(x_2) , \tag{9}$$

and then verify that the corresponding statement is correct with $n-1$ replaced by n. We shall require the functional equation satisfied by the functions $C_n(x_1, w_1, x_2)$. Equation (14) of [2] may be modified by the introduction of a setup cost to read

$$\begin{aligned} C_n(x_1, w_1, x_2) = \min_{\substack{x_1+w_1 \leq y \leq x_2 \\ 0 \leq z}} \Big\{ & c(z) + K \cdot \delta(y - x_1 - w_1) \\ & + c_1(y - x_1 - w_1) + \tilde{L}(x_2) + L(x_1) \\ & + \alpha \int C_{n-1}(x_1 + w_1 - \xi, y - x_1 - w_1, x_2 + z - \xi) \varphi(\xi)\, d\xi \Big\} . \end{aligned} \tag{10}$$

We shall also assume in the following arguments that the reader is familiar with the notion of K-convexity defined in [3]. Specifically, a function $f(x)$ is said to be K-convex if for all $a, b \geqq 0$ and all x we have

$$f(x+a) - f(x) - a\left[\frac{f(x) - f(x-b)}{b}\right] + K \geqq 0 ,$$

or, in geometric terms, if whenever the secant line is drawn through two points of the function and extended to the right, the function never drops more than K units below this line. If $C_n(x_1, w_1)$ is written

$$L(x_1) + \alpha \int L(x_1 + w_1 - \xi) \varphi(\xi)\, d\xi + f_n(x_1 + w_1) ,$$

it is demonstrated in [3] that the functions $f_n(u)$ are each K-convex, with K equal to the setup cost at the lower installation. This result is useful in demonstrating the optimality of (S, s) policies and will also be of use in our argument.

Now let us turn to the inductive proof of the upper bound of (8). Since

$$C_1(x_1, w_1) = L(x_1)\,, \qquad \bar{g}_1(x_2) = \tilde{L}(x_2)\,, \qquad C_1(x_1, w_1, x_2) = L(x_1) + \tilde{L}(x_2)\,,$$

the result is correct when the index n is equal to 1. If it is correct for $n-1$, we may use (10) to conclude that

$$\begin{aligned} C_n(x_1, w_1, x_2) \leqq \min_{\substack{x_1+w_1 \leq y \leq x_2 \\ 0 \leq z}} \Big\{ & c(z) + K\cdot\delta(y - x_1 - w_1) + c_1(y - x_1 - w_1) \\ & + L(x_1) + \tilde{L}(x_2) + \alpha\int C_{n-1}(x_1 + w_1 - \xi, y - x_1 - w_1)\varphi(\xi)\,d\xi \\ & + \alpha\int \bar{g}_{n-1}(x_2 + z - \xi)\varphi(\xi)\,d\xi \Big\}\,. \end{aligned}$$

We next substitute for C_{n-1} its value in terms of f_{n-1} [equations (2) and (1′)]:

$$\begin{aligned} C_n(x_1, w_1, x_2) \leqq {} & L(x_1) + \alpha\int L(x_1 + w_1 - \xi)\varphi(\xi)\,d\xi + \tilde{L}(x_2) \\ & + \min_{\substack{x_1+w_1 \leq y \leq x_2 \\ 0 \leq z}} \Big\{ c(z) + \alpha\int \bar{g}_{n-1}(x_2 + z - \xi)\varphi(\xi)\,d\xi \\ & \quad + K\cdot\delta(y - x_1 - w_1) + c_1(y - x_1 - w_1) \\ & \quad + \alpha^2\iint L(y - \xi - \xi_1)\varphi(\xi)\varphi(\xi_1)\,d\xi\,d\xi_1 \\ & \quad + \alpha\int f_{n-1}(y - \xi)\varphi(\xi)\,d\xi \Big\}\,. \end{aligned} \tag{11}$$

Aside from the constraint $y \leqq x_2$, the optimal selection of y is to adopt the optimal (S_n, s_n) policy; i.e., if $x_1 \dotplus w_1 < s_n$, set $y = S_n$, and if $x_1 + w_1 > s_n$, set $y = x_1 + w_1$. If $x_2 > S_n$, this constraint will be inoperative and the right-hand side of (11) will become

$$\begin{aligned} & L(x_1) + \alpha\int L(x_1 - w_1 - \xi)\varphi(\xi)\,d\xi + f_n(x_1 + w_1) \\ & \quad + \min_{z \geq 0}\Big\{ c(z) + \tilde{L}(x_2) + \alpha\int \bar{g}_{n-1}(x_2 + z - \xi)\varphi(\xi)\,d\xi \Big\} = C_n(x_1, w_1) + \bar{g}_n(x_2)\,, \end{aligned}$$

since $\bar{\Lambda}_n(x_2) = 0$ when $x_2 > S_n$.

This demonstrates the result for values of $x_2 > S_n$. Now let us suppose that $x_2 < S_n$. We shall attempt to find the policy that minimizes

$$K\cdot\delta(y - u) + c_1(y - u) + \alpha^2\iint L(y - \xi - \xi_1)\varphi(\xi)\varphi(\xi_1)\,d\xi\,d\xi_1 + \alpha\int f_{n-1}(y - \xi)\varphi(\xi)\,d\xi\,,$$

subject to the constraint $u \leqq y \leqq x_2$.

We define $S_n(x_2)$ as the minimizing value of the independent variable in

(12) $$G_n(y) = cy + \alpha^2 \iint L(y - \xi - \xi_1)\varphi(\xi)\varphi(\xi_1)\,d\xi\,d\xi_1 + \alpha\int f_{n-1}(y - \xi)\varphi(\xi)\,d\xi\,,$$

subject to the condition $y \leqq x_2$, and let $s_n(x_2)$ be the point at which $G_n(s_n(x_2)) = K + G_n(S_n(x_2))$; see figure 1. It is easy to verify that the function $G_n(y)$ is itself K-convex. This implies that there is a unique choice of $s_n(x_2)$, and that the optimal policy for the above equation, given the restriction $y \leqq x_2$, is

$$y = \begin{cases} S_n(x_2) & \text{if} \quad x_1 + w_1 \leqq s_n(x_2)\,, \\ x_1 + w_2 & \text{if} \quad x_1 + w_1 > s_n(x_2)\,. \end{cases}$$

(The argument is identical with that given in [3].)

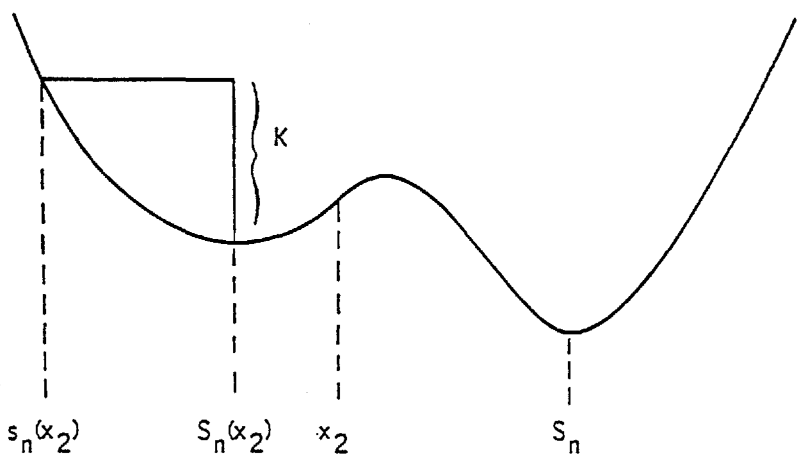

FIG. 1.

Now we shall consider two cases that depend on the relationship between $x_1 + w_1$ and $s_n(x_2)$.

Case 1. $x_1 + w_1 > s_n(x_2)$. In this case the minimizing choice of y is $x_1 + w_1$, and (11) reads

$$\begin{aligned} C_n(x_1, w_1, x_2) \leqq L(x_1) &+ \alpha\int L(x_1 + w_1 - \xi)\varphi(\xi)\,d\xi \\ &+ \min_{z \geq 0}\left\{c(z) + \bar{L}(x_2) + \alpha\int \bar{g}_{n-1}(x_2 + z - \xi)\varphi(\xi)\,d\xi\right\} \\ &+ G_n(x_1 + w_1) - c_1(x_1 + w_1) + f_n(x_1 + w_1) - f_n(x_1 + w_1)\,. \end{aligned}$$

The induction will be complete for this case if we can show that

(13) $$G_n(x_1 + w_1) - c_1(x_1 + w_1) - f_n(x_1 + w_1) \leqq \bar{\Lambda}_n(x_2) = G_n(x_2) - G_n(S_n)\,.$$

But

$$f_n(x_1 + w_1) = \begin{cases} K - c_1(x_1 + w_1) + G_n(S_n) & \text{if} \quad K + G_n(S_n) \leqq G_n(x_1 + w_1)\,, \\ -c_1(x_1 + w_1) + G_n(x_1 + w_1) & \text{if} \quad K + G_n(S_n) \geqq G_n(x_1 + w_1)\,. \end{cases}$$

In the latter case, the left-hand side of (13) is 0 and the right-hand side is nonnegative. In the former, (13) is equivalent to

$$G_n(x_1 + w_1) \leqq G_n(x_2) + K,$$

which is certainly true if $G_n(x_1 + w_1) \leqq G_n(S_n(x_2)) + K$. This latter inequality is correct, since we have assumed that $x_1 + w_1 > s_n(x_2)$.

Case 2. $x_1 + w_1 < s_n(x_2)$. In this case the minimizing choice of y is $S_n(x_2)$, and therefore (11) reads

$$\begin{aligned} C_n(x_1, w_1, x_2) \leqq L(x_1) + \alpha\int L(x_1 + w_1 - \xi)\varphi(\xi)\,d\xi \\ + \min_{z\geq 0}\left\{c(z) + \tilde{L}(x_2) + \alpha\int \bar{g}_{n-1}(x_2 + z - \xi)\varphi(\xi)\,d\xi\right\} \\ + K - c(x_1 + w_1) + G_n(S_n(x_2)) + f_n(x_1 + w_1) - f_n(x_1 + w_1), \end{aligned}$$

and our theorem will be correct if we can show that

$$K - c(x_1 + w_1) + G_n(S_n(x_2)) - f_n(x_1 + w_1) \leqq G_n(x_2) - G_n(S_n). \tag{14}$$

A simple argument based on K-convexity shows that $s_n(x_2) \leqq s_n$, and since $x_1 + w_1 < s_n(x_2)$, it follows that $x_1 + w_1 < s_n$; hence

$$f_n(x_1 + w_1) = K - c(x_1 + w_1) + G_n(S_n),$$

and the inequality (14) becomes $G_n(S_n(x_2)) \leqq G_n(x_2)$, which is correct by definition. This finishes the proof of the following theorem.

THEOREM 1. $C_n(x_1, w_1, x_2) \leqq C_n(x_1, w_1) + \bar{g}_n(x_2)$.

The policy described above has some use other than as an intermediary step in the proof of Theorem 1. As we shall see in the next section, if it is actually followed, the cost that will be incurred will be *between* $C_n(x_1, w_1, x_2)$ and $C_n(x_1, w_1) + \bar{g}_n(x_2)$. If this gap is small, the policy will have considerable merit. We need only remark at this stage that the computation of this policy is quite simple, and is based only on the calculation of functions of a single variable.

Let us now verify the lower bound in (8), again by induction. The first several steps in the proof will be identical with those in the proof of the upper bound, except that the inequalities will be reversed and $\bar{g}_{n-1}$ will be replaced by $\underline{g}_{n-1}$. The same quantities $S_n(x_2)$ and $s_n(x_2)$ will serve in this proof.

Again if $x_2 > S_n$, the proof is immediate. Let $x_2 \leqq S_n$, and consider two cases.

Case 1. $x_1 + w_1 > s_n(x_2)$. The analog of (11) is then

$$\begin{aligned} C_n(x_1, w_1, x_2) \geqq L(x_1) + \alpha\int L(x_1 + w_1 - \xi)\varphi(\xi)\,d\xi \\ + \min_{z\geq 0}\left\{c(z) + \tilde{L}(x_2) + \alpha\int \underline{g}_{n-1}(x_2 + z - \xi)\varphi(\xi)\,d\xi\right\} \\ + G_n(x_1 + w_1) - c_1(x_1 + w_1) + f_n(x_1 + w_1) - f_n(x_1 + w_1), \end{aligned}$$

and the result will be demonstrated, in this case, if we can show that

$$G_n(x_1 + w_1) - c_1(x_1 + w_1) - f_n(x_1 + w_1)$$
$$\geqq \Delta_n(x_2) = \max\{0, -K + G_n(x_2) - G_n(S_n)\} \qquad (x_2 \leqq S_n) .$$

Again we use the two possible representations for f_n. If $x_1 + w_1 > s_n$, then

$$f_n(x_1 + w_1) = -c_1(x_1 + w_1) + G_n(x_1 + w_1) ,$$

and our inequality becomes $0 \geqq \Delta_n(x_2)$. But if $x_1 + w_1 > s_n$, it is certainly true that $x_2 > s_n$, and therefore $\Delta_n(x_2) = 0$. On the other hand, if $x_1 + w_1 < s_n$, then

$$f_n(x_1 + w_1) = K - c_1(x_1 + w_1) + G_n(S_n) ,$$

and our inequality is

$$G_n(x_1 + w_1) - K - G_n(S_n) \geqq \max\{0, -K + G_n(x_2) - G_n(S_n)\} .$$

If $x_2 > s_n$, the right-hand side is zero, but the left-hand side is positive, since $x_1 + w_1 < s_n$. On the other hand, if $x_2 < s_n$, we are required to show that $G_n(x_1 + w_1) \geqq G_n(x_2)$. It is, however, a simple consequence of K-convexity that $G_n(x)$ is monotonically decreasing for $x < s_n$. If this were not so, the situation would result in which the function drops more than K units below its tangent line at a point to the right, contradicting K-convexity; see figure 2. Since $x_1 + w_1 \leqq x_2$, it follows that $G_n(x_1 + w_1) \geqq G_n(x_2)$ for $x_2 < s_n$, and this disposes of Case 1.

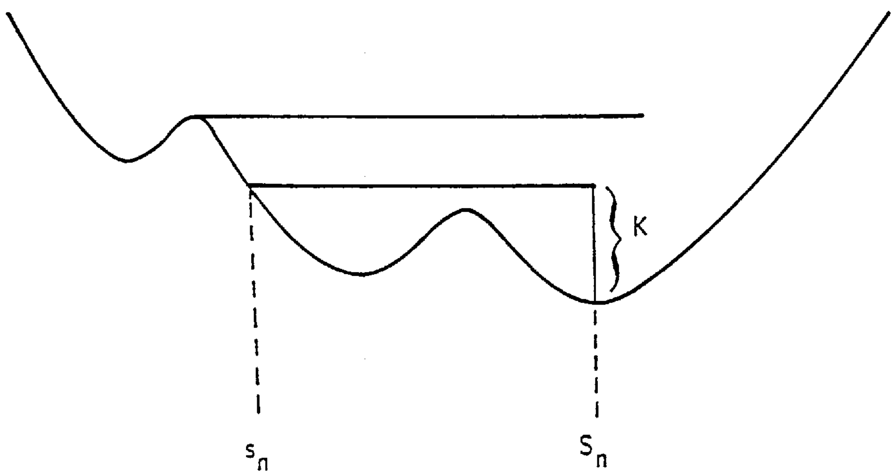

FIG. 2.

Case 2. $x_1 + w_1 < s_n(x_2)$. As before, the analog of (11) becomes

$$C_n(x_1, w_1, x_2) \geqq L(x_1) + \alpha\int L(x_1 + w_1 - \xi)\varphi(\xi)\,d\xi$$
$$+ \min_{z \geq 0}\left\{c(z) + \tilde{L}(x_2) + \alpha\int \underline{g}_{n-1}(x_2 + z - \xi)\varphi(\xi)\,d\xi\right\}$$
$$+ K - c(x_1 + w_1) + G_n(S_n(x_2)) + f_n(x_1 + w_1) - f_n(x_1 + w_1) ,$$

and the result will be demonstrated if we can show that

$$K - c(x_1 + w_1) + G_n(S_n(x_2)) - f_n(x_1 + w_1) \geqq \max\{0, -K + G_n(x_2) - G_n(S_n)\}.$$

Since $x_1 + w_1 < s_n(x_2) \leqq s_n$, it follows that $f_n(x_1 + w_1) = K - c_1(x_1 + w_1) + G_n(S_n)$; hence we are required to show that

$$G_n(S_n(x_2)) - G_n(S_n) \geqq \max\{0, -K + G_n(x_2) - G_n(S_n)\}.$$

Since the left-hand side is always positive, we may restrict our attention to values of x_2 that make the right-hand side positive, i.e., to $x_2 < s_n$, and we must demonstrate that $K + G_n(S_n(x_2)) \geqq G_n(x_2)$. But since G_n is monotonically decreasing in the region $x_2 < s_n$, it follows that $S_n(x_2) = x_2$, so that the inequality is immediate. This completes the argument and demonstrates the validity of the lower bound.

THEOREM 2. $C_n(x_1, w_1, x_2) \geqq C_n(x_1, w_1) + \underline{g}_n(x_2)$.

3. The Approximate Policy

The calculations of the preceding section suggest a pair of policies that are capable of being implemented in the simple multi-echelon situation that we are analyzing. We first compute for each x_2 the policy parameters $S_n(x_2)$ and $s_n(x_2)$. (If $x_2 \geqq S_n$, these numbers are taken to be S_n and s_n.) The policy, as far as the lower installation is concerned, is first of all to examine the entire system stock x_2. If $x_1 + w_1 > s_n(x_2)$, no order is to be placed. If $x_1 + w_1 < s_n(x_2)$, an order of size $S_n(x_2) - x_1 - w_1$ is requested from the upper installation.

As far as system purchases of stock are concerned, either sequence of functional equations (those involving $\underline{g}_n$ or $\bar{g}_n$) will furnish us with a policy in the sense that it will contain an explicit purchase rule z as a function of x_2. Either of these policies may be combined with the rule for the lower installation to provide a system policy. The problem is to evaluate the merits of either of these policies.

We shall have nothing to say about the merits of the policy associated with $\underline{g}_n(x_2)$. We shall show, however, that if the policy associated with $\bar{g}_n(x_2)$ is actually implemented, the expected cost will not be larger than $C_n(x_1, w_1) + \bar{g}_n(x_2)$.

To this end, let us define $\bar{C}_n(x_1, w_1, x_2)$ as the expected cost (not *minimum* expected cost) associated with the use of the latter policy in an n-period problem, which is begun in state (x_1, w_1, x_2). The $\bar{C}_n$'s will satisfy a functional equation similar to (10), but without the minimum operator, since we are not finding an optimal policy. Let us denote the system policy based on the functions $\bar{g}_n$ by z^*; this policy will, of course, be a function of x_2. Let us also denote the ordering policy at the lower installation by y^*. This is the policy defined by the pair of critical numbers $S_n(x_2)$ and $s_n(x_2)$. With this notation, it is easy to see that the equation satisfied by $\bar{C}_n$ will be

$$\begin{aligned}(15)\qquad \bar{C}_n(x_1, w_1, x_2) = \bar{L}(x_2) + c(z^*) + L(x_1) \\ + K\cdot\delta(y^* - x_1 - w_1) + c_1(y^* - x_1 - w_1) \\ + \alpha\int \bar{C}_{n-1}(x_1 + w_1 - \xi, y^* - x_1 - w_1, x_2 + z^* - \xi)\varphi(\xi)\,d\xi .\end{aligned}$$

Our objective is to show inductively that

$$\bar{C}_n(x_1, w_1, x_2) \leqq C_n(x_1, w_1) + \bar{g}_n(x_2) . \tag{16}$$

If we assume (16) to be correct for $n-1$, we obtain from (15)

$$\begin{aligned}\bar{C}_n(x_1, w_1, x_2) \leqq \bar{L}(x_2) + c(z^*) + L(x_1) + K\cdot\delta(y^* - x_1 - w_1) \\ + c_1(y^* - x_1 - w_1) + \alpha\int \bar{g}_{n-1}(x_2 + z^* - \xi)\varphi(\xi)\,d\xi \\ + \alpha\int C_{n-1}(x_1 + w_1 - \xi, y^* - x_1 - w_1)\varphi(\xi)\,d\xi .\end{aligned} \tag{17}$$

Since

$$\bar{g}_n(x_2) = \bar{L}(x_2) + \bar{\Lambda}_n(x_2) + c(z^*) + \alpha\int \bar{g}_{n-1}(x_2 + z^* - \xi)\varphi(\xi)\,d\xi$$

and

$$\begin{aligned}C_{n-1}(x_1 + w_1 - \xi, y^* - x_1 - w_1) = L(x_1 + w_1 - \xi) \\ + \alpha\int L(y^* - \xi - \xi_1)\varphi(\xi_1)\,d\xi_1 + f_{n-1}(y^* - \xi) ,\end{aligned}$$

(17) may be rewritten as

$$\begin{aligned}\bar{C}_n(x_1, w_1, x_2) &\leqq \bar{g}_n(x_2) - \bar{\Lambda}_n(x_2) + L(x_1) + \alpha\int L(x_1 + w_1 - \xi)\varphi(\xi)\,d\xi \\ &\quad + K\cdot\delta(y^* - x_1 - w_1) + c_1(y^* - x_1 - w_1) + \alpha\int f_{n-1}(y^* - \xi)\varphi(\xi)\,d\xi \\ &\quad + \alpha^2\iint L(y^* - \xi - \xi_1)\varphi(\xi)\varphi(\xi_1)\,d\xi\,d\xi_1 \\ &= \bar{g}_n(x_2) + C_n(x_1, w_1) - f_n(x_1 + w_1) - \bar{\Lambda}_n(x_2) \\ &\quad + K\cdot\delta(y^* - x_1 - w_1) - c_1(x_1 + w_1) + G_n(y^*) .\end{aligned}$$

Our theorem will therefore be proved if we can show that

$$-f_n(x_1 + w_1) - \bar{\Lambda}_n(x_2) + K\cdot\delta(y^* - x_1 - w_1) - c_1(x_1 + w_1) + G_n(y^*) \leqq 0 . \tag{18}$$

It is convenient to consider two cases:

Case 1. $x_1 + w_1 < s_n(x_2)$. In this case $y^* = S_n(x_2)$, and (18) becomes

$$-f_n(x_1 + w_1) - \bar{\Lambda}_n(x_2) + K - c_1(x_1 + w_1) + G_n(S_n(x_2)) \leqq 0 .$$

But if $x_1 + w_1 < s_n(x_2)$, then certainly $x_1 + w_1 < s_n$, and

$$f_n(x_1 + w_1) = K - c_1(x_1 + w_1) + G_n(S_n) .$$

If we also use the fact that $\bar{\Lambda}_n(x_2) = G_n(x_2) - G_n(S_n)$ unless $x_2 > S_n$, in which case $\bar{\Lambda}_n(x_2) = 0$, we see that the required inequality is

$$G_n(S_n(x_2)) \leqq G_n(x_2) \qquad (x_2 < S_n) ,$$

and is trivially satisfied for $x_2 \geqq S_n$. This disposes of Case 1.

Case 2. $x_1 + w_1 > s_n(x_2)$. In this case $y^* = x_1 + w_1$, and (18) becomes

$$-f_n(x_1 + w_1) - \bar{\Lambda}_n(x_2) - c_1(x_1 + w_1) + G_n(x_1 + w_1) \leqq 0 .$$

If $x_1 + w_1 > s_n$, then $f_n(x_1 + w_1) = c_1(x_1 + w_1) + G_n(x_1 + w_1)$, and since $\bar{\Delta}_n(x_2) \geqq 0$, the inequality is correct. On the other hand, if $x_1 + w_1 < s_n$, it follows that $f_n(x_1 + w_1) = K - c_1(x_1 + w_1) + G_n(S_n)$, and we need to show that

$$(19) \qquad -K - G_n(S_n) - \bar{\Delta}_n(x_2) + G_n(x_1 + w_1) \leqq 0 .$$

If $x_2 > S_n$, then $\bar{\Delta}_n(x_2) = 0$, but then $x_1 + w_1 > s_n$, so that $G_n(x_1 + w_1) \leqq K + G_n(S_n)$ and (19) holds. If $x_2 \leqq S_n$, then $\bar{\Delta}_n(x_2) = G_n(x_2) - G_n(S_n)$, and we need to show that

$$(20) \qquad K + G_n(x_2) \geqq G_n(x_1 + w_1) .$$

Since $G_n(x_2) \geqq G_n(S_n(x_2))$, relation (20) will obtain if

$$(21) \qquad K + G_n(S_n(x_2)) \geqq G_n(x_1 + w_1) ;$$

but (21) is an immediate consequence of the K-convexity of G_n and of the relations $G_n(s_n(x_2)) = K + G_n(S_n(x_2))$ and $x_1 + w_1 > s_n(x_2)$.

Thus we have proved the following theorem.

THEOREM 3. *The expected cost of using the policy associated with $\bar{g}_n$ will never be larger than*

$$C_n(x_1, w_1) + \bar{g}_n(x_2) .$$

It is possible to obtain a slight improvement over the bounds that we have given by using a somewhat different definition of $\bar{\Delta}_n$ and $\underline{\Delta}_n$, while maintaining Theorems 1, 2, and 3. The modified definitions are

$$\bar{\Delta}_n(x_2) = \begin{cases} G_n(S_n(x_2)) - G_n(S_n) & x_2 < S_n , \\ 0 & x_2 \geqq S_n , \end{cases}$$

and

$$\underline{\Delta}_n(x_2) = \begin{cases} -K + G_n(S_n(x_2)) - G_n(S_n) & x_2 < s_n , \\ 0 & x_2 > s_n . \end{cases}$$

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