GAMES WITH PARTIAL INFORMATION

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§1. INTRODUCTION

In this paper we shall discuss a particular class of games with partial information. The characteristic feature of the information pattern in these games is that each player is informed of his opponent's moves a fixed amount of time after they are made. More specifically, the players each make a sequence of choices, $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, respectively, from fixed finite sets $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$, in the order $a_1, b_1, a_2, b_2, \ldots$. The condition on the information pattern is that Player 1 (2) in selecting $a_n(b_n)$ is informed of his opponent's moves up to and including $b_{n-k}(a_{n-\ell})$, as well as his own previous moves. It is necessary that k be positive and ℓ nonnegative. The payoff is defined to be some function of the two sequences of choices. A general theorem is proved in [9] which implies that for games of this type continuity of the payoff is a sufficient condition for the existence of a value and optimal strategies for both players.

The number $\lambda = k + \ell - 1$ is defined to be the time lag of the game. The case of perfect information is given by $\lambda = 0$. This case has received a considerable amount of attention [2, 4], and the purpose of our paper is to generalize some of the properties of games with perfect information to games with positive time lags. In order to illustrate the properties that we wish to generalize, let us assume for the moment that the payoff function is continuous. Let us define $V^+(a_1, \ldots, b_n)$ to be the value of the perfect-information game in which the first n moves of both players have been fixed to be $a_1, b_1, \ldots, b_n$, the payoff being the same as the payoff in the original case. The subgame property of games with perfect information is expressed by the fact that the game which terminates after b_n, and whose payoff function is given by $V^+(a_1, \ldots, b_n)$, has the same value as the original game, and that the optimal strategies in the terminated game may be directly related to the optimal strategies of

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the original game [2].

The point about optimal strategies may perhaps be seen more clearly if we briefly describe the functional equations associated with these subgames. These equations will be treated in more detail in the body of the paper. An example of the functional equations for perfect information games is

$$V^+(a_1, \ldots, b_n) = \max_{a_{n+1}} \min_{b_{n+1}} V^+(a_1, \ldots, b_n, a_{n+1}, b_{n+1}) \quad ,$$

and their relationship to optimal strategies is expressed by the fact that if Player 1, when informed of the specific choices of $a_1, \ldots, b_n$, plays the choice of a_{n+1} which maximizes

$$\min_{b_{n+1}} V^+(a_1, \ldots, b_n, a_{n+1}, b_{n+1}) \quad ,$$

then this strategy constitutes an optimal strategy. The optimal strategies for Player 2 are derived from a corresponding set of functional equations which have the form

$$V^-(a_1, \ldots, b_n, a_{n+1}) = \min_{b_{n+1}} \max_{a_{n+2}} V^-(a_1, \ldots, b_{n+1}, a_{n+2}) \quad .$$

The case $k = 1$, $\ell = 1$, and $\lambda = 1$ is a so-called "simultaneous game." In this case the subgame property may be expressed by the functional equations

$$V(a_1, \ldots, b_n) = \max_{p(a_{n+1})} \min_{q(b_{n+1})} \sum_{\substack{a_{n+1} \\ b_{n+1}}} p(a_{n+1}) q(b_{n+1}) V(a_1, \ldots, b_{n+1})$$

$$= \min \max \cdots \quad ,$$

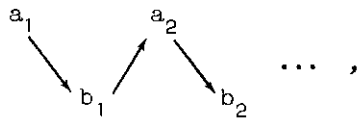
where the p's and q's are probability distributions, and $V(a_1, \ldots, b_n)$ is defined to be the value of the subgame in which the first n moves of both players have been fixed, and the game proceeds as a simultaneous game toward the same payoff. If Player 1, when informed of the specific choices of $a_1, \ldots, b_n$, plays a_{n+1} with a probability distribution $p(a_{n+1})$ which maximizes

$$\min_{b_{n+1}} \sum_{a_{n+1}} p(a_{n+1}) V(a_1, \ldots, b_{n+1}) \quad ,$$

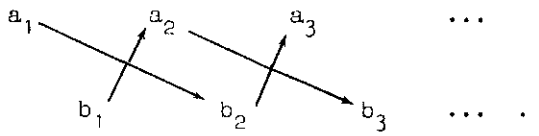
then this collection of distributions, called a behavior strategy, constitutes an optimal strategy. A similar remark is valid for Player 2.

As soon as we begin to discuss the case in which the time lag is greater than one, the subgame properties no longer exist. The basic reason for this failure is that if we fix the initial moves of both players and only inform the players of the moves of their opponents which they are entitled to know, then the information available to each player will be different at all times from that available to his opponent. We will never arrive at a situation which looks like the beginning of a new game, and subgames will therefore not exist.

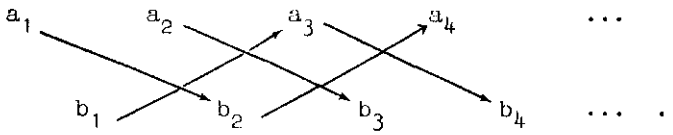
In order to clarify this remark, let us introduce a set of diagrams describing the different types of information patterns. The meaning of the diagrams will be clear from the examples. The diagram for the case of perfect information will be



whereas the diagram for the game $k = 1$, $\ell = 1$, and $\lambda = 1$ is



The diagram for $k = 2$, $\ell = 1$, and $\lambda = 2$ is given by



The subgames in the first diagram occur after any initial sequence of moves; in the second diagram they occur after any initial sequence which terminates with a move of Player 2. It is easy to see that these represent places in which both players have a common fund of information, and the last diagram points out the fact that in the game with time lag 2, there is no place in which both players have the same fund of information.

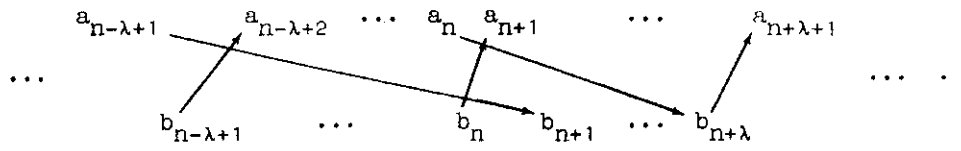
As we shall see, it is possible to introduce a collection of games associated with a game whose time lag is greater than one, which play somewhat the same role as the subgames described for the cases $\lambda = 0$, 1, and

which give rise to more complex functional equations than the ones mentioned above. It will also be true that every time-lag game will have associated with it two functional equations from which the optimal strategies of either player may be deduced. We should point out that these functional equations have been discussed by Isaacs [5, 6], Karlin [5, 7], and Dubins [3] for a particular game, with time lag 2.

§2. THE GENERALIZED SUBGAMES ($k = 1$, $\ell = \lambda > 0$)

We shall fix a specific value of $\lambda > 0$, and consider the case $k = 1$, $\ell = \lambda$. It is clear that any other information pattern with the same time lag can be transformed into the above case by a renumbering of the moves of one of the players, and the addition of several vacuous moves at the beginning of the game. We shall find it convenient, however, to discuss different combinations of (k, ℓ) with the same time lag separately (Section 4), and to introduce a particular set of functional relations for each combination. What this means, of course, is that any particular game will have several types of subgames and several sets of functional relations. In particular, the subgames and functional equations that we discuss in this section $(k = 1, \ell = \lambda)$ will apply with the appropriate renumbering to an arbitrary game with time lag λ.

The diagram for this case is given by

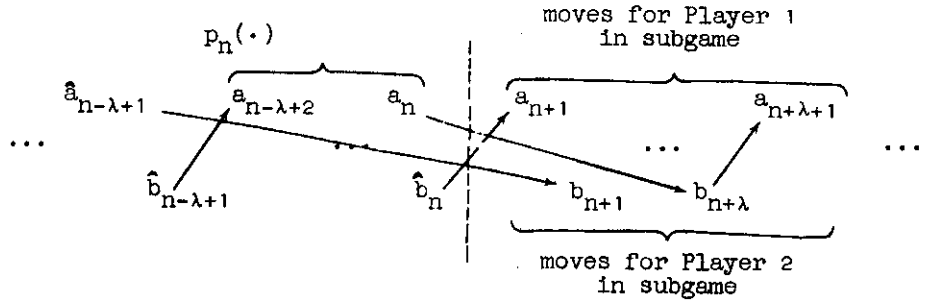


The generalized subgame that we are going to introduce will be described by a collection of parameters, which will summarize the information available to both players at the beginning of the subgame. This information consists of two parts:

1. The complete set of information that would be available to Player 2 after he makes his n-th move in the original game. This collection of information which we denote by I_n consists of a specification of the first n moves of Player 2 and the first $n - \lambda + 1$ moves of Player 1. As the above diagram shows, this information would also be available to Player 1 at this time.

2. A joint probability distribution on the moves $a_{n-\lambda+2}, \cdots, a_n$ which we represent by $p_n(\cdot)$.

The diagram for this subgame is as follows:



The notation $\hat{b}_n$, etc., is used to indicate that these are fixed choices and are involved in the specifications of the subgame. The subgame proceeds as follows: The moves $a_{n-\lambda+2}, \cdots, a_n$ are randomized from $p_n(\cdot)$, and told to Player 1, but not to Player 2. Player 1 then makes a choice of a_{n+1}, followed by a choice of b_{n+1} by Player 2. The choice of $a_{n-\lambda+2}$ which occurred as a result of the randomization is announced to Player 2. The choice of b_{n+1} is told to both players after it is made; but, according to the information requirements, the choice of a_{n+1} is kept secret from Player 2 until he is ready to make move $b_{n+\lambda+1}$. We then have a choice of a_{n+2} and b_{n+2}, respectively, and $a_{n-\lambda+3}$ is then announced to Player 2. This sequence of moves proceeds until all of the chance moves have been announced, and then continues using the information pattern of the original game. The payoff is defined to be the same payoff as for the original game. When we have occasion to refer to this subgame, it will be denoted by $G_n = G_n(I_n; p_n(\cdot))$. Clearly G_o is the original game.

The techniques of [9] may be used to show that the game G_n will have a value and optimal strategies if the payoff function is continuous, and in this case it is easy to see that the value will be continuously dependent on the joint probability distribution specifying the game. The next section of this paper will be devoted to a derivation of the functional equations associated with these subgames, and we shall assume in this derivation that the payoff function is continuous. Later on we shall discuss the relevance of the functional equation in other cases.

§3. THE FUNCTIONAL RELATIONS ($k = 1$, $\ell = \lambda$)

Let the value of G_n be denoted by $V(I_n; p_n(\cdot))$. Let us define a specific strategy for Player 1 in this game in the following way. Let him make his first move a_{n+1} according to the probability distribution $x(a_{n+1}|a_{n-\lambda+2}, \cdots, a_n)$. (We indicate the dependence of these moves upon the result of the randomization in the obvious way.) After Player 2 makes the move $\hat{b}_{n+1}$, and if the randomized value of $a_{n-\lambda+2}$ is denoted by $\hat{a}_{n-\lambda+2}$, then Player 1 has complete knowledge of $I_{n+1} = (\hat{a}_1, \cdots, \hat{a}_{n-\lambda+2}, \hat{b}_1, \cdots, \hat{b}_{n+1})$, plus, of course, the other randomized values of a. Let

him then continue his strategy by playing an optimal strategy in the game $G_{n+1}(I_{n+1};\ p_{n+1}(\cdot|\hat{a}_{n-\lambda+2}))$, where $p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})$ is meant to be the joint distribution on $a_{n-\lambda+3}, \cdots, a_{n+1}$ which is formed by combining $p_n(\cdot)$ with $x(a_{n+1}|a_{n-\lambda+2}, \cdots, a_n)$ and conditioning this joint distribution by $a_{n-\lambda+2} = \hat{a}_{n-\lambda+2}$. Let us see what Player 1 can obtain by using a strategy of this form, if he tells Player 2, at the beginning of G_n, that this is the strategy he will be using. In this case the common fund of information after both players have made their initial moves in G_n, and after $\hat{a}_{n-\lambda+2}$ is told to Player 2, is precisely I_{n+1} and $p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})$, and this is the common fund with probability $p(\hat{a}_{n-\lambda+2})$ derived from $p_n(\cdot)$. Player 1, of course, also knows the other results of the randomization. The way that we have chosen Player 1's strategy shows that he will get at least

$$V(I_{n+1};\ p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad .$$

Player 1 cannot determine the result of the randomization for $a_{n-\lambda+2}$, so that at the beginning of G_n he can guarantee himself only an expected value of

$$\sum p(\hat{a}_{n-\lambda+2})V(I_{n+1};\ p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad .$$

Again, Player 1 cannot dictate the choice of $\hat{b}_{n+1}$, so that he can only be sure of

$$\operatorname*{Min}_{\hat{b}_{n+1}} \sum p(\hat{a}_{n-\lambda+2})V(I_{n+1};\ p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad ;$$

and finally if he picks $x(a_{n+1}|a_{n-\lambda+2}, \cdots, a_n)$ judiciously, we can conclude that

$$V(I_n;\ p_n(\cdot))$$

$$\geq \operatorname*{Max}_{x(a_{n+1}|a_{n-\lambda+2},\cdots,a_n)} \operatorname*{Min}_{\hat{b}_{n+1}} \sum p(\hat{a}_{n-\lambda+2})V(I_{n+1};\ p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})). \quad .$$

The next step is to replace this inequality by an equality, and this is accomplished by the following reasoning. Let $x^*(a_{n+1}|a_{n-\lambda+2}, \cdots, a_n)$ be the initial component of an optimal behavior strategy for Player 1 in G_n. Since the strategy is optimal, it can be told to Player 2 without degrading Player 1's expected return. Let Player 2 choose $\hat{b}_{n+1}$ so as to minimize

$$\sum p(\hat{a}_{n-\lambda+2})V(I_{n+1};\ p^*_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad ,$$

where $p^*_{n+1}(\cdot|\hat{a}_{n+\lambda-2})$ is compounded from $p_n(\cdot)$ and $x^*(a_{n+1}|a_{n-\lambda+2}, \cdots, a_n)$ in the obvious way. Then with probability $p(\hat{a}_{n-\lambda+2})$ the common fund of information available to both players is I_{n+1}. Now if Player 2 continues his strategy by playing an optimal strategy in $G_{n+1}(I_{n+1}; p^*_{n+1}(\cdot|\hat{a}_{n-\lambda+2}))$, it is clear that he will prevent Player 1 from getting an expectation greater than

$$\sum p(\hat{a}_{n-\lambda+2})V(I_{n+1}; p^*_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad ,$$

which from the way that b_{n+1} was chosen is equal to

$$\operatorname*{Min}_{\hat{b}_{n+1}} \sum p(\hat{a}_{n-\lambda+2})V(I_{n+1}; p^*_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad .$$

Since Player 1 was assumed to be playing optimally, this last quantity must be no less than $V(I_n; p_n(\cdot))$, and we obtain

$$V(I_n; p_n(\cdot)) \leq \operatorname*{Max}_{x(a_{n+1}|\cdots a_n)} \operatorname*{Min}_{\hat{b}_{n+1}} \sum p(\hat{a}_{n-\lambda+2})V(I_{n+1}; p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad .$$

Combining this with the previous inequality, we obtain the desired functional relationship.

THEOREM 1. Let G_o be a game with time lag λ (written in the form $k = 1$, $\ell = \lambda$), which has a continuous payoff. Let $V(I_n; p_n(\cdot))$ be the value of the subgame in which both players' information about the past is $I_n = (\hat{a}_1, \cdots, \hat{a}_{n-\lambda+1}; \hat{b}_1, \cdots, \hat{b}_n)$ and in which Player 1's previous $\lambda - 1$ moves are governed by the joint probability distribution $p_n(\cdot) = p(a_{n-\lambda+2}, \cdots, a_n)$. Then

$$V(I_n; p_n(\cdot))$$

$$= \operatorname*{Max}_{x(a_{n+1}|a_{n-\lambda+2},\cdots,a_n)} \operatorname*{Min}_{\hat{b}_{n+1}} \sum p(\hat{a}_{n-\lambda+2})V(I_{n+1}; p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad ,$$

where

$$p(\hat{a}_{n-\lambda+2}) = \sum_{a_{n-\lambda+3},\cdots,a_n} p(\hat{a}_{n-\lambda+2}, \cdots, a_n)$$

and

$$p_{n+1}(\cdot|\hat{a}_{n-\lambda+2}) = \frac{p(\hat{a}_{n-\lambda+2},\ldots,a_n)x(a_{n+1}|\hat{a}_{n-\lambda+2},\ldots,a_n)}{p(\hat{a}_{n-\lambda+2})} \quad .$$

§4. OPTIMAL STRATEGIES FOR PLAYER 1

In this section, we shall show that a class of optimal strategies for Player 1 in the game with time lag λ can be derived from the functional equations that we have established in the preceding section. As before, we assume that the game is represented in the form $k = 1$, $\ell = \lambda > 0$. Optimal strategies for the case $\lambda = 0$ are obtained by the process outlined in the Introduction.

The first of the functional equations relates the value of G_0 (the actual value of the game itself) to the value of $G_1(I_1;\ p_1(\cdot))$. For $\lambda \geq 2$ the first equation is

$$V = \max_{x(a_1)} \min_{\hat{b}_1} V(I_1;\ p_1(\cdot))$$

with $I_1 = \hat{b}_1$ and $p_1(\cdot) = x(a_1)$. Let us define the components of a behavior strategy for Player 1 in the following recursive fashion. Let $x(a_1)$ be chosen so as to maximize

$$\min_{\hat{b}_1} V(I_1;\ p_1(\cdot)) \quad .$$

Call such a maximizing distribution $x^*(a_1)$. In general, if $x^*(a_1)$, $x^*(a_2|a_1;\ b_1)$, ..., $x^*(a_n|a_1,\ \ldots,\ a_{n-1};\ b_1,\ \ldots,\ b_{n-1})$ are known, then for each $I_n = (\hat{a}_1,\ \ldots,\ \hat{a}_{n-\lambda+1},\ \hat{b}_1,\ \ldots,\ \hat{b}_n)$ we form the joint probability distribution $p_n^*(\cdot)$ given by the following product of $\lambda - 1$ factors:

$$x^*(a_n|\hat{a}_1,\ \ldots,\ \hat{a}_{n-\lambda+1},\ \ldots,\ a_{n-1};\ \hat{b}_1,\ \ldots,\ \hat{b}_{n-1})$$

$$\ldots\ x^*(a_{n-\lambda+2}|\hat{a}_1,\ \ldots,\ \hat{a}_{n-\lambda+1};\ \hat{b}_1,\ \ldots,\ \hat{b}_{n-\lambda+1}) \quad .$$

Then $x^*(a_{n+1}|\hat{a}_1,\ \ldots,\ \hat{a}_{n-\lambda+1},\ \ldots,\ a_n;\ \hat{b}_1,\ \ldots,\ \hat{b}_n)$ is chosen equal to any $x(a_{n+1}|a_{n-\lambda+2},\ \ldots,\ a_n)$ that maximizes

$$\min_{\hat{b}_{n+1}} \sum p^*(\hat{a}_{n-\lambda+2})V(I_{n+1};\ p^*_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad .$$

As I_n takes on all conceivable values, we obtain the complete behavior

strategy component $x^*(a_{n+1}|a_1, \ldots, a_n; b_1, \ldots, b_n)$.

We want to show that this method for selecting the components of a behavior strategy for Player 1 leads to an optimal strategy. Suppose that such a sequence has been chosen. Let us define the sequence of functions $V^*(I_n)$ to be equal to $V(I_n; p_n^*(\cdot))$. They have the property that

I. $$\min_{b_{n+1}} \sum x^*(a_{n-\lambda+2}|a_1, \ldots, a_{n-\lambda+1}; b_1, \ldots, b_{n-\lambda+1})V^*(I_{n+1}) = V^*(I_n) \ ,$$

II. $$\min_{b_1} V^*(I_1) = V \ ,$$

and

III. $$\lim_{n \longrightarrow \infty} V^*(I_n) = M(a, b)$$

uniformly, where M is the payoff function.

Property I is a direct consequence of the definitions. Property II follows from the application of the initial functional equation. Property III is a direct consequence of the continuity of the payoff function, which implies that the values of the subgames approach the payoff function for large fixed initial segments.

Now let us suppose that the strategy $\{x^*\}$ is played against an arbitrary mixed strategy for Player 2, which is represented in behavior strategy form by the sequence of conditional distributions $y(b_{n+1}|a_1, \ldots, a_{n-\lambda+1}; b_1, \ldots, b_n)$. These two strategies give rise to a measure on the space of all sequences of a's and b's with the property that

$$\begin{aligned} \text{prob}\ (\hat{a}_1, \ldots, \hat{a}_n, \hat{b}_1, \ldots, \hat{b}_n) &= x^*(\hat{a}_1) \\ &\cdots x^*(\hat{a}_n|\hat{a}_1, \ldots, \hat{a}_{n-1}; \hat{b}_1, \ldots, \hat{b}_{n-1})y(\hat{b}_1) \\ &\cdots y(\hat{b}_n|\hat{a}_1, \ldots, \hat{a}_{n-\lambda}; \hat{b}_1, \ldots, \hat{b}_{n-1}) \ , \end{aligned}$$

and the functions $V(I_n)$ become a sequence of random variables. Then

$$E(V^*(I_{n+1})|I_n) = E(V^*(I_{n+1})|\hat{a}_1, \ldots, \hat{a}_{n-\lambda+1}, \hat{b}_1, \ldots, \hat{b}_n)$$

$$= \sum_{\substack{\hat{a}_{n-\lambda+2}, \ldots, \hat{a}_{n+1} \\ \hat{b}_{n+1}}} \frac{\text{prob}(\hat{a}_1, \ldots, \hat{a}_{n+1}, \hat{b}_1, \ldots, \hat{b}_{n+1})}{\text{prob}(\hat{a}_1, \ldots, \hat{a}_{n-\lambda+1}, \hat{b}_1, \ldots, \hat{b}_n)} \ V^*(I_{n+1})$$

which in turn is equal to

$$\sum_{\substack{\hat{a}_{n-\lambda+2} \\ \hat{b}_{n+1}}} x^*(\hat{a}_{n-\lambda+2}|\hat{a}_1, \ldots, \hat{a}_{n-\lambda+1}; \hat{b}_1, \ldots, \hat{b}_{n-\lambda+1})$$

$$y(\hat{b}_{n+1}|\hat{a}_1, \ldots, \hat{a}_{n-\lambda+1}; \hat{b}_1, \ldots, \hat{b}_n)V^*(I_{n+1}) \quad ,$$

because I_{n+1} does not depend on $\hat{a}_{n-\lambda+3}, \ldots, \hat{a}_{n+1}$. Property I tells us that this last expression is not less than $V^*(I_n)$, and we obtain

$$E(V^*(I_{n+1})|I_n) \geq V^*(I_n) \quad .$$

If we integrate out the conditioning variables and apply Property II, we obtain

$$E(V^*(I_{n+1})) \geq V \quad ;$$

and applying Property III yields

$$E(M) \geq V,$$

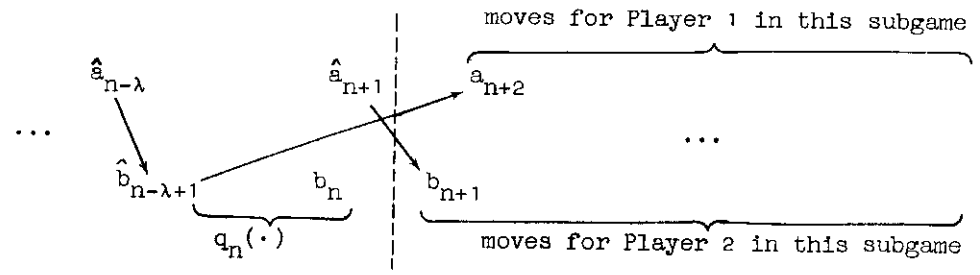
which tells us that our strategy is optimal.

There may be some question at this point as to which optimal strategies of Player 1 are obtained from the functional equation by the procedure outlined above. It is quite easy to give examples in which not all of Player 1's optimal strategies are obtained in this way. It is true, but we shall not prove it at this point, that the class of strategies obtained from the functional equation will include the class of "best" strategies for Player 1 [8, p. 84] (if we disregard those portions of a strategy that refer to situations of measure zero).

THEOREM 2. If the components of a behavior strategy for Player 1 are chosen recursively in the way outlined above, this strategy is optimal.

§5. OPTIMAL STRATEGIES FOR PLAYER 2

To obtain optimal strategies for Player 2, the game must be represented in the form $k = \lambda + 1$, $\ell = 0$. The diagram for the n-th subgame in this case is



The game is specified by $I_n = (\hat{a}_1, \ldots, \hat{a}_{n+1}, \hat{b}_1, \ldots, \hat{b}_{n-\lambda+1})$, and $q_n(\cdot) = \text{prob}\,(b_{n-\lambda+2}, \ldots, b_n)$. We notice that the first move in this subgame is made by Player 2. If we denote its value by $V(I_n; q_n(\cdot))$, the functional equation is

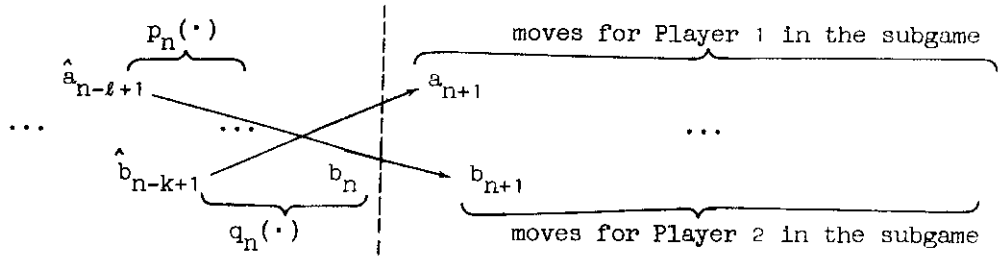
$$V(I_n; q_n(\cdot)) = \min_{y(b_{n+1}|b_{n-\lambda+2},\ldots,b_n)} \max_{\hat{a}_{n+2}} \sum q(\hat{b}_{n-\lambda+2}) V(I_{n+1}; q_{n+1}(\cdot|\hat{b}_{n-\lambda+2})) \quad ,$$

and by using the same techniques as in Section 4, it is possible to compute an optimal strategy for Player 2 in a recursive fashion from this functional equation.

§6. THE GENERALIZED SUBGAMES (OTHER VALUES OF k AND ℓ)

In this section we consider the representation of our game with time lag λ for general values of k and ℓ ($k > 1$, $\ell > 0$, $\lambda = k + \ell - 1$). For each value of (k, ℓ) there will be two classes of subgames, depending on which player moves first. These will be generalizations of either the games discussed in Section 3 or those discussed in Section 5. In what follows we shall restrict our attention to the former.

The diagram for the n-th subgame is given by



This subgame is described, first of all, by the fund of information known to both players after Player 2 has made his n-th move. In this

case it will be a specification of Player 1's first $n - \ell + 1$ moves, and Player 2's first $n - k + 1$ moves, say $(\hat{a}_1, \ldots, \hat{a}_{n-\ell+1}, \hat{b}_1, \ldots, \hat{b}_{n-k+1}) = I_n$. We also have given an arbitrary pair of joint probability distributions $p_n(\cdot) = \text{prob}(a_{n-\ell+2}, \ldots, a_n)$ and $q_n(\cdot) = \text{prob}(b_{n-k+2}, \ldots, b_n)$. The game will be denoted by $G_n(I_n; p_n(\cdot), q_n(\cdot))$, and it proceeds as follows: The moves $a_{n-\ell+2}, \ldots, a_n$ are randomized from $p_n(\cdot)$ and told to Player 1 but not to Player 2. Simultaneously, the moves $b_{n-k+2}, \ldots, b_n$ are randomized from $q_n(\cdot)$ and told to Player 2 but not to Player 1. They then proceed as they would in the original game, with the same payoff $M(a, b)$. We still assume that this payoff is continuous, so that the general theorem of [9] applies and the game has a value, which we denote by $V(I_n; p_n(\cdot), q_n(\cdot))$. As before, there exists a sequence of functional relations. They take the form

$$V(I_n; p_n(\cdot), q_n(\cdot))$$

$$= \underset{x(a_{n+1}|a_{n-\ell+2},\ldots,a_n)}{\text{Max}} \; \underset{y(b_{n+1}|b_{n-k+2},\ldots,b_n)}{\text{Min}}$$

$$\sum p(\hat{a}_{n-\ell+2})q(\hat{b}_{n-k+2})V(I_{n+1}; p_{n+1}(\cdot|\hat{a}_{n-\ell+2}), q_{n+1}(\cdot|\hat{b}_{n-k+2}))$$

$$= \text{Min Max} \ldots \; .$$

The proof is quite similar to the proof given in Section 3 and we shall not repeat it here.

We would like to indicate the major difference between the functional equations in this case and the functional equations that were discussed in Sections 3 and 4. In those sections we showed how an optimal strategy for Player 1 could be computed recursively from the functional equation. The corresponding procedure for the present case would be the following: Suppose that $x^*(a_1) \ldots x^*(a_n|a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-k})$ have been computed. Then for any $I_n = (\hat{a}_1, \ldots, \hat{a}_{n-\ell+1}, \hat{b}_1, \ldots, \hat{b}_{n-k+1})$, we would consider a game $G_n(I_n; p_n(\cdot), q_n(\cdot))$ with $p_n(\cdot)$ defined by

$$p_n(\cdot) = x^*(a_{n-\ell+2}|\hat{a}_1, \ldots, \hat{a}_{n-\ell+1}, \hat{b}_1, \ldots, \hat{b}_{n-\lambda+1})$$

$$\ldots x^*(a_n|\hat{a}_1, \ldots, a_{n-1}, \hat{b}_1, \ldots, \hat{b}_{n-k}) \; ,$$

and for some $q_n(\cdot)$ which for the moment we leave undefined. Then $x(a_{n+1}|a_{n-\ell+2}, \ldots, a_n)$ would be chosen so as to maximize

$$y(b_{n+1}|b_{n-k+2},\ldots,b_n) \quad \operatorname{Min}_{\hat{b}_{n-k+2}} \sum p(\hat{a}_{n-\ell+2})q(\hat{b}_{n-k+2})$$

$$V(I_{n+1};\ p_{n+1}(\cdot|\hat{a}_{n-\ell+2}), q_{n+1}(\cdot|\hat{b}_{n-k+2})) \quad .$$

It is clear that this choice would depend on $q_n(\cdot)$, and we would therefore only be able to show that the strategy we have chosen is optimal against a particular choice of Player 2's strategy.

This means that in order to obtain an optimal strategy for Player 1 in an arbitrary (k, ℓ) representation $(\lambda > 0)$, we must transform this representation into $k = 1$, $\ell = \lambda$ by renumbering the moves of one of the players and the addition of several vacuous moves at the beginning of the game, and then apply the method of Theorem 2. To obtain optimal strategies for Player 2, we must transform into $k = \lambda + 1$, $\ell = 0$.

§7. REMARKS

In our discussion we have consistently assumed that the payoff function is continuous. This has permitted us to say that each subgame under discussion has both a value and optimal strategy for either player, and there is at least one point in the proof that we have given for the validity of the functional equation in which the existence of optimal strategies was specifically used. There are other conditions under which our subgames may be shown to have a value. For example, as is shown in [9], if the payoff function is upper (lower) semi-continuous, then each subgame has a value and optimal strategies exist for Player 1 (2). The question arises as to whether the functional equations relating the values of these subgames are still valid. It can be shown that a modification of the argument of Section 3 yields this same functional equation, with Max Min replaced by Max Inf (Sup Min). It is also true that optimal strategies for the player who has them in the semi-continuous case can also be generated by means of the functional equations. On the other hand, very little can be said about the other player's strategies from the functional equation. To illustrate this point, let us assume that the payoff is lower semi-continuous, so that the maximizing player does not necessarily have an optimal strategy. Let us recursively pick strategies for Player 1, by choosing an $x(a_{n+1}|a_{n-\lambda+2}, \ldots, a_n)$ which is $\epsilon/2^n$-effective in

$$\operatorname{Min}_{\hat{b}_{n+1}} \sum p(\hat{a}_{n-\lambda+2})V(I_{n+1};\ p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad ,$$

and as before, define $V^*(I_n) = V(I_n;\ p_n^*(\cdot))$. Then it will be true that

against any strategy for Player 2 we have $E(V^*(I_{n+1})|I_n) \geq V^*(I_n) - \epsilon/2^n$, and therefore $E(V^*(I_n)) \geq V - \epsilon$. But lower semi-continuity weakens Property III of Section 4 to:

$$\text{III}'. \quad \overline{\lim_{a,b \longrightarrow \hat{a},\hat{b}}} M(\hat{a}, \hat{b}) \geq \overline{\lim_{n \longrightarrow \infty}} V^*(I_n) \geq \underline{\lim_{n \longrightarrow \infty}} V^*(I_n) \geq M(\hat{a}, \hat{b}) \quad .$$

As a result, we can conclude that

$$E\left(\lim_{a,b \longrightarrow \hat{a},\hat{b}} M(a, b)\right) \geq V - \epsilon \quad ,$$

but not that $E(M(\hat{a}, \hat{b})) \geq V - \epsilon$.

Even without the conditions of semi-continuity on the payoff function, it is still meaningful to talk about the functional equations. Without any conditions on the payoff function, if we can find a solution of the functional equations

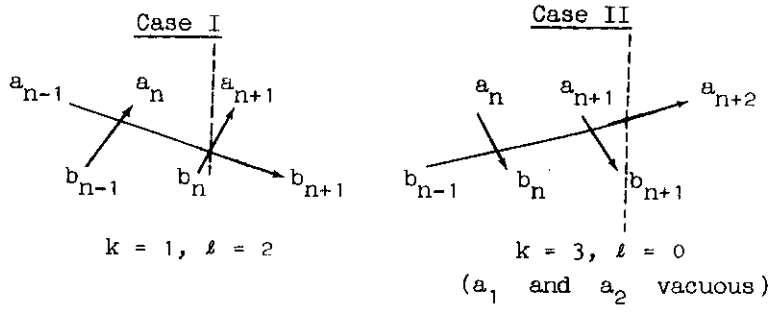
$$V(I_n; p_n(\cdot))$$

$$= \max_{x(a_{n+1}|a_{n-\lambda+2},\ldots,a_n)} \min_{\hat{b}_{n+1}} \sum p(\hat{a}_{n-\lambda+2}) V(I_{n+1}; p_{n+1}(\cdot|\hat{a}_{n-\lambda+2})) \quad ,$$

with the property that $V(I_n; p_n(\cdot)) \longrightarrow M(\hat{a}, \hat{b})$ (say boundedly), then the strategy for Player 1, which is generated recursively from these equations, will guarantee Player 1 at least $V(I_o)$ against any strategy of Player 2.

§8. AN EXAMPLE

It may be instructive to show how the above techniques can be applied to the celebrated "bomber-battleship" game to yield functional equations. (References [3], [5], [6], [7].) We shall do this in two ways, according to the two arrangements



The b_i will each be 0 or 1; the a_i will each be 0, 1, 2, or $\oplus$ (pass). The first $a_n \neq \oplus$ is interpreted as a prediction that

$$b_n + b_{n+1} = a_n \quad \text{(in Case I).} \qquad \text{or} \qquad b_{n-2} + b_{n-1} = a_n \quad \text{(in Case II)} .$$

The payoff is 1 to the a-player for a correct prediction, 0 for an incorrect prediction or no prediction. This payoff is lower semi-continuous, so optimal strategies are assured only for the b-player. (This formulation follows Blackwell [1].)

In Case I we observe that the generalized subgame $G_n(I_n; p_n(\cdot))$ is trivial if any $a_i \neq \oplus$ in I_n. On the other hand, $G_n(I_n; p_n(\cdot))$ and $G_m(I_m'; p_m(\cdot))$ are completely isomorphic provided that $p_n = p_m$, $b_n = b_m'$, and all a_i, a_i' are $\oplus$ (i.e., the earlier b_i, b_i' do not matter). Hence we may write

$$V_n(I_n; p_n) = f_\nu(x_0, x_1, x_2) \quad \text{if} \quad n \geq 1$$

where $\nu = b_n$ and $x_\alpha = p_n(\alpha)$. Moreover, symmetry tells us that $f_0(x_0, x_1, x_2) = f_1(x_2, x_1, x_0)$. Hence the functional equations of Theorem 1, $n \geq 1$, reduce to the single equation:

$$(1) \qquad f_0(x, y, z) = \operatorname*{Max}_{u,v,w} \operatorname{Min} \begin{cases} tf_0(u, v, w) + x \\ \\ tf_0(w, v, u) + y \end{cases}$$

where $t = 1 - x - y - z$, and u, v, w are restricted to be nonnegative with sum ≤ 1. The first equation ($n = 0$) becomes

$$V = V_0 = \operatorname*{Max}_{x,y,z} \operatorname{Min} \begin{cases} f_0(x, y, z) \\ \\ f_0(z, y, x) \end{cases}$$

for the value of the game. Equation (1) is the same as equation (34) in Isaacs' paper [6] and forms the basis of his analysis of ϵ-optimal strategies for the a-player. The unique "ideal" (locally optimal) strategy is given by $x_0 = x_1 = x_2 = 0$ — i.e., never predict — and is clearly not optimal.

In Case II, $G_n(I_n; q_n(\cdot))$ is again trivial if any $a_i \neq \oplus$ in I_n, while the other games are entirely independent of <u>all</u> b_i in I_n.

Thus, we have

$$V_n(I_n;\ q_n(\cdot)) = g(x) \quad \text{if} \quad n \geq 1 \ ,$$

where $x = q_n(0)$, $0 \leq x \leq 1$. Symmetry gives us $g(x) = g(1 - x)$. Hence, the functional equations of Section 5 reduce to

$$(2) \qquad g(x) = \operatorname*{Min}_{\substack{0\leq u\leq 1\\ 0\leq v\leq 1}} \operatorname{Max} \begin{cases} xu & (0) \\ x(1-u) + (1-x)v & (1) \\ (1-x)(1-v) & (2) \\ xg(u) + (1-x)g(v) & (\oplus) \end{cases}$$

with

$$V = V_0 = \operatorname*{Min}_x g(x) \ .$$

No Max appears because a_2 is an automatic pass in this case. This functional equation has been studied exhaustively. It develops that a minimizing x^* can be chosen for (3) with the property that, if we set $x = x^*$ in (2), the minimum is achieved at $u = x^*$, $v = 1 - x^*$. This makes an optimal strategy for the b-player extremely easy to describe: always choose $b_{n+1} = b_n$ with probability x^*, $b_{n+1} = 1 - b_n$ with probability $1 - x^*$. There is reason to believe that this phenomenon will not occur in other related cases, such as the $\lambda = 3$ version of the present example, but the theory remains obscure on this point.

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