

# How to Compute Equilibrium Prices in 1891

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ABSTRACT. Irving Fisher's Ph.D. thesis, submitted to Yale University in 1891, contains a fully articulated general equilibrium model presented with the broad scope and formal mathematical clarity associated with Walras and his successors. In addition, Fisher presents a remarkable hydraulic apparatus for calculating equilibrium prices and the resulting distribution of society's endowments among the agents in the economy. In this paper we provide an analytical description of Fisher's apparatus, and report the results of simulating the mechanical/hydraulic "machine," illustrating the ability of the apparatus to "compute" equilibrium prices and also to find multiple equilibria.

## I

### Introduction

IRVING FISHER'S Ph.D. thesis, submitted to Yale University in 1891, is remarkable in at least two distinct ways. The thesis contains a fully articulated general equilibrium model presented with the broad scope and formal mathematical clarity that we associate with Walras and his successors. But what is even more astonishing is the presentation, in the thesis, of Fisher's hydraulic apparatus for calculating equilibrium prices and the resulting distribution of society's endowments among the agents in the economy.

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Fisher's development of the general equilibrium model was done without any knowledge of the simultaneous achievement of Walras. In the introduction to his thesis, Fisher states that

[t]he equations in Chapter IV, Sec. 10, were found by me two years ago, when I had read no mathematical economist except Jevons. They were an appropriate extension of Jevons' determination of the exchange of *two* commodities between *two* trading bodies to the exchange of *any number* of commodities between *any number* of traders. . . . These equations are essentially those of Walras.

Even though Fisher's construction of what we now call the Walrasian model of equilibrium was a fully original achievement, he did have contemporaries: the central ideas of equilibrium theory were independently discovered at several locations in the final decade of the last century. But the second theme of Fisher's thesis was entirely novel in conception and execution. No other economist of his time suggested the possibility of exploring the implications of equilibrium analysis by constructing specific numerical models, with a moderately large number of commodities and consumers, and finding those prices that would simultaneously equate supply and demand in all markets. The profession would have to wait until rudimentary computers were available in the 1930s before Leontief turned his hand to a simplified version of this computation.

Fisher's hydraulic machine is complex and ingenious. It correctly solves for equilibrium prices in a model of exchange in which each consumer has additive, monotonic, and concave utility functions, and a specified money income; the market supplies of each good are exogenously given. Both additivity and fixed money incomes make the model of exchange to which his mechanism is applied less than completely general. But we know of no argument for the existence of equilibrium prices in this restricted model that does not require the full use of Brouwer's fixed point theorem. Of course fixed point theorems were not available to Fisher and in that section of his thesis in which first-order conditions are presented for a general model of exchange, Fisher argues for consistency by counting equations and unknowns.

It is hard to discover the source of Fisher's interest in computation. He was a student of J. Willard Gibbs, and perhaps the theme of

concrete models in mechanics was carried over to economics. But it is also possible that his hydraulic apparatus is simply an instance of an American passion for complex machinery that gets things done. Fisher, himself, had a passion for innovation. In the course of a long career, he invented an elaborate tent for the treatment of tuberculosis (described in the *Journal of the American Medical Association*, 1903), developed a mechanical diet indicator that permitted easy calculation of the daily consumption of fats, carbohydrates, and proteins, copywrote (1943) an icosahedral world globe with triangular facets that when unfolded was allegedly an improvement on the Mercator projection, and patented an “*index visible filing system*” (1913) sold to Kardex Rand, later Remington Rand, in 1925 for \$660,000.

## II

**Fisher's Cisterns**

LET US BEGIN by describing the special model of exchange that is solved by the Fisher machine. There are, say,  $m$  consumers, indexed by  $i = 1, \dots, m$  and  $n$  goods, indexed by  $j = 1, \dots, n$ . Consumer  $i$  has the utility function

$$u_i(x_1, \dots, x_n) = \sum_{j=1, n} u_{ij}(x_j),$$

with each  $u_{ij}$  increasing and concave. Society's endowment of good  $j$  is  $E_j$ , and consumer  $i$ 's income is  $Y_i$ .

At prices  $p = (p_1, \dots, p_n) \geq 0$ , consumer  $i$  is assumed to maximize utility subject to his budget constraint

$$\begin{aligned} &\max u_i(x_1, \dots, x_n) \text{ subject to} \\ &\sum_{j=1, n} p_j x_j \leq Y_i. \end{aligned}$$

If the marginal utility of income for consumer  $i$  is  $\lambda_i$ , the demands,  $x_{ij}$ , will satisfy the first order conditions:

$$u'_{ij}(x_{ij}) \leq \lambda_i p_j \quad (= \text{if } x_{ij} > 0).$$

A competitive equilibrium is given by a price vector  $p$  so that the market demands obtained by the summation of individual demands

are equal, commodity by commodity, to the market supply. In other words, we are asked to find  $p = (p_1, \dots, p_n) \geq 0$ ,  $\lambda = (\lambda_1, \dots, \lambda_m) \geq 0$  and a matrix of commodities  $[x_{ij}]$ , such that

$$u'_{ij}(x_{ij}) \leq \lambda_i p_j, \text{ (if } x_{ij} > 0)$$

$$\sum_{i=1, m} x_{ij} \leq E_j \text{ (if } p_j > 0) \text{ and}$$

$$\sum_{j=1, n} p_j x_{ij} \leq Y_i \text{ (if } \lambda_i > 0).$$

These are Irving Fisher's equations.

To solve these equations on an electronic computer we would input the endowment vector  $E$ , the income vector  $Y$ , and provide a description of the  $mn$  marginal utilities  $u'_{ij}(x_{ij})$  as functions of their arguments. Fisher invented a clever way to represent a typical marginal utility by an irregularly shaped cistern that is meant to receive varying amounts of water during the operation of the machine.

Imagine a cistern with a uniform thickness of unity, and with a shape shown in Figure 1.

The origin is placed at the northwest corner of the cistern, which has a vertical height of  $d$  units. For each  $0 \leq \xi \leq d$ , the width of the cistern is given by the function  $f(\xi)$ . If the cistern is filled with water up to  $\xi$  units below the top of the cistern, the volume of water it contains will be

$$\int_{\xi}^d f(t) dt.$$

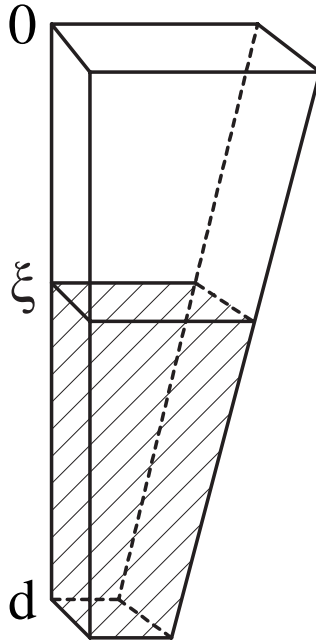
Fisher designs the cistern so that the depth  $\xi$  is equal to the marginal utility, for that consumer and that good, of the volume of water it contains, i.e.,

$$\xi = u' \left( \int_{\xi}^d f(t) dt \right)$$

as an identity for all  $\xi$ . With this construction, the maximal depth  $d$  is equal to  $u'(0)$ , and one needs to select utility functions with a finite marginal utility at zero in order to have cisterns of finite depth. Another feature of this utility function is that the consumer is satiated

Figure 1

A Cistern



when the cistern is filled to its top and consumption is equal to  $\int_0^d f(t)dt$ .

The connection between the utility function  $u(x)$  and the shape of the cistern  $f(\xi)$  is unusual, and the reader may find it useful to calculate  $f$  for various well-known utility functions. For example, if  $u(x) = -e^{-x}$ , the shape function is  $f(\xi) = 1/\xi$ , and the depth,  $d$ , is unity.

As a second example, if  $u(x) = dx - \frac{1}{2b}x^2$ , the shape function is  $f(\xi) = b$  and the box is rectangular. And finally if  $f(\xi) = b + at$ , with  $b + ad > 0$ , the utility function is given by

$$u(x) = \frac{1}{12a^2} \left( (b + 2ad)^3 - 6bax - ((b + 2ad)^2 - 4ax)^{3/2} \right).$$

In order to see a way in which these cisterns can be used to solve an optimization problem, consider a single consumer with utility function

$$u(x) = \sum u_j(x_j).$$

Construct a cistern for each utility function  $u_j(x_j)$  and let their tops be rigidly connected by a rod constrained to be horizontal, but free to move vertically, so that the cisterns will float in a tub of water without changing their relative positions. Insert a fixed amount of water,  $Y$ , into the right-most cistern, and let the interiors of the cisterns be connected by rubber tubes so that the water flows easily from one to another.

When the apparatus reaches an equilibrium, the input of water,  $Y$ , will be allocated among the  $n$  cisterns, with cistern  $j$  containing an amount  $x_j$ . The level of water will be the same in each cistern, say  $\lambda$  units below the top, so that

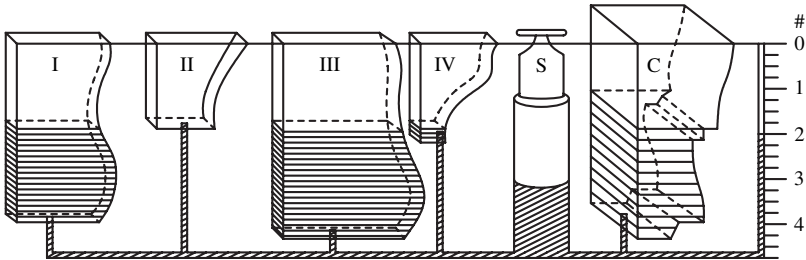
$$\sum x_j = Y \text{ and}$$

$$u'_j(x_j) = \lambda, \text{ for each } j.$$

We see that this assemblage of the Fisher cisterns solves the problem

$$\begin{aligned} \max \sum u_j(x_j) \text{ subject to} \\ \sum x_j = Y, \end{aligned}$$

Figure 2



simply by using the fact that freely flowing water will seek a common level in the series of interconnected cisterns, thereby equating the marginal utility for each good to a common price of unity.

We have evaluated the demand functions of this consumer at income  $Y$ , and at prices  $p = (1, \dots, 1)$ . In order to deal with general prices, and several consumers, we shall have to assemble the cisterns in a considerably more complex fashion.

### III

#### **A Preliminary Machine: Pareto Optimal Allocations**

LET US CONSIDER a construction that is not quite the one used by Fisher, but that allows us to explore models of exchange with  $m$  consumers. Figure 3 depicts the case of two consumers (I, II) and two commodities ( $A, B$ ). Construct a separate rod for each of the  $m$  consumers. On the rod for the  $i$ th consumer, we place  $n$  cisterns, one for each commodity, and for each such good we connect all of the  $m$  cisterns referring to that good by a tube in which water will flow. The west sides of the cisterns for each commodity are aligned parallel to a float  $F$  lying on the surface of the water. The left end of each rod is attached to this float,  $F$ . The  $i$ th rod will make an angle  $\alpha_i$  with the water surface.

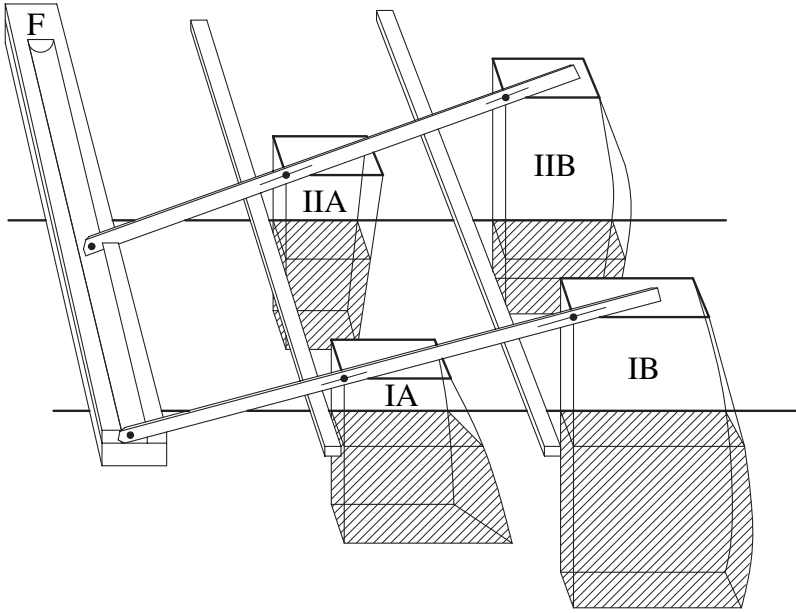
In this construction the float  $F$  is not permitted to move and the angles  $\alpha_i$  are fixed in advance. What does move are the cisterns along each rod; but in the following precisely constrained way.

#### *Constraining the Cisterns*

All of the cisterns relating to good  $j$  have their northwest corners at the *same horizontal distance*, say  $p_j$ , from their attachment to float  $F$ .

The distances from  $F$  may shift during the functioning of the machine but all distances relating to the same commodity are constrained to be the same. These distances are determined by inserting, into the tube for good  $j$ , an amount of water equal to the market supply  $E_j$  of that good. The water levels for the  $m$  cisterns for good  $j$  reach a common level, and together the cisterns move along the rods so that this common level is equal to the water level in the

Figure 3



surrounding tub (let us not be concerned that a line of cisterns may very well have to pass through an adjacent line).

Let the volume of water in cistern  $ij$  be  $x_{ij}$ , and the horizontal distance of the cisterns for commodity  $j$  from the float  $F$  be  $p_j$ . We then have

$$\sum_i x_{ij} = E_j$$

$$u'_{ij}(x_{ij}) = p_j \tan(\alpha_i),$$

so that  $x_{ij}$  is the allocation of the market supplies  $E_j$  that maximizes the social welfare function

$$\sum_i \lambda_i \sum_j u_{ij}(x_{ij}),$$

when  $\lambda_i$  is given by  $1/\tan(\alpha_i)$ , and is therefore on the Pareto frontier. And conversely, concavity of the individual utility functions implies



that any Pareto optimal allocation is obtained by some selection of  $\lambda_i$  and therefore by some selection of these angles  $\alpha_i$ . Of course, Fisher wrote before Pareto and this concept does not appear in his thesis.

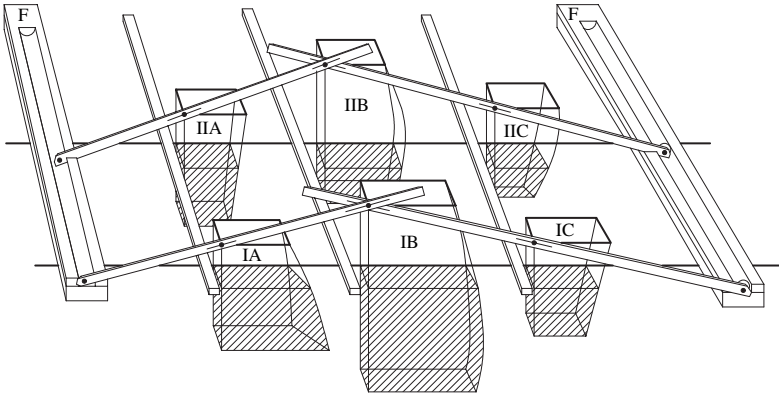
Typically this allocation will not satisfy the individual budget constraints  $\sum p_j x_{ij} = Y_i$  and the angles of the rods must be systematically varied so as to find those angles for which the value of consumption equals income for each consumer. But at this point in our exposition, it is far from clear how to modify the apparatus so as to compare consumption and income. Nothing in this machine permits us to calculate the cost of the commodity bundle demanded by a particular consumer. At no place is a quantity multiplied by a price. We shall return to this issue after describing Fisher's somewhat different arrangement of these cisterns.

*Fisher's Construction (Preliminary)*

There are some substantial differences between the instrument described in the previous section and Fisher's use of these cisterns, rods, and floats to accomplish the same end. In Fisher's arrangement, the horizontal locations of the northwest corners of the cisterns are fixed; the only possible movement of these corners is strictly vertical. This would present a serious constraint for the mechanism described above, since it does not provide enough flexibility to determine the relative prices,  $p_j$ , if the number of goods is greater than two. As we shall see, Fisher's modification of the mechanism permits him to extend the number of goods to three, but no further extension is possible for him. The number of consumers is arbitrary, though the actual model constructed by Fisher has three consumers as well.

Let us introduce a coordinate system to describe the fixed horizontal positions of the three cisterns for each consumer along their common line. The cistern for good 1 will be at the point on its line with coordinate  $-1$ , good 2 at the point with coordinate 0, and good 3 at 1. In Fisher's machine the northwest corners of the cisterns are not placed on a single rigid rod as in the previous section; instead they are attached to a connected pair of rods that pivot above the middle cistern, as shown in Figure 4. The left rod for each consumer begins at the point on the line with coordinate  $-1 - a$ , passes through

Figure 4



the corner of the cistern for good 1 at  $-1$ , and then through the corner of the middle cistern for good 2 at  $0$ . The right rod begins at the point on the line with coordinate  $1 + c$ , passes through the corner of the cistern for good 3 at  $1$ , and then joins the left rod at the corner of the cistern for good 2.

The beginnings of these pairs of rods are secured on two floats, one to the left at coordinate  $-1 - a$  and the other to the right at coordinate  $1 + c$ . The floats are permitted to move laterally, so that the values of  $a$  and  $c$  are arbitrary. Let us also assume for the moment that the northwest corners of the cisterns describing the same good, for the various consumers, are all connected by a rigid bar parallel to the surface of the water, so that all of these corners are forced to be at the same height. Movement of the floats implies sliding pivots.

As before, the cisterns for a common good are connected by tubing through which water will move freely. A quantity of water equal to the supply of that good is entered into the corresponding tub by setting the levels of certain plungers at the back of the tub. The water levels for the cisterns for each commodity will find a common level, and subsequently the floats to the right and left will move so that all of the cistern levels, for each consumer and for each good, will equilibrate at the water level of the tub. If the quantity of water in cistern

$ij$  is  $x_{ij}$  then  $\sum_i x_{ij} = E_j$  for each good. Moreover, the surface of the water in each cistern referring to good  $j$  is at the same depth below its northwest corner, so that for each  $j$  the marginal utilities  $u'_{ij}(x_{ij})$  are identical for all consumers  $i$ .

If we set  $p_j = u'_{ij}(x_{ij})$  we see that the apparatus has found an allocation  $x_{ij}$  and prices  $p$  such that

$$\sum_i x_{ij} = E_j$$

$$u'_{ij}(x_{ij}) = p_j.$$

As with the previous construction, the hydraulic instrument has produced a particular Pareto optimal allocation; in this case the allocation with the marginal utility of income equal to unity for all consumers. Other Pareto optimal allocations can be found by multiplying each utility function by a nonnegative constant, or by allowing the bars connecting the northwest corners of the cisterns to make an angle with the surface of the water different from zero.

But it is still not clear how to find an efficient allocation in which the value of consumption is equal to income for all consumers. A reasonable scholar might perfectly well have given up at this point, but Fisher persisted with an ingenious and intricate modification of the apparatus, designed to solve this problem.

It would be difficult to improve on Fisher's description of the final mechanism. The next section is taken directly from his thesis. He denotes the consumers by *I*, *II*, *III* and commodities by *A*, *B*, *C*. His statements about marginal utilities and their relationship to prices are only true at equilibrium.

*Fisher's Construction (The Schematic Figure 5 and Text Taken from Section 4 of His Thesis)*

The water in these cisterns must be subjected to two sets of conditions, first: the sum of all of the contents of *IA*, *IIA*, *IIIA*, etc., shall be a given amount (vis: the whole of the commodity *A* consumed during the given period) with a like given sum for the *B* row, *C* row, etc., secondly: the sum of *IA*, *IB*, *IC*, etc., each multiplied by a coefficient (the price of *A*, of *B*, of *C*, etc.), shall be given (vis: the whole income of *I* during the period) with a like given sum for the *II* row, *III* row, etc.

To realize these two sets of conditions each cistern is divided into two by a vertical partition of wood. The front compartments are all of unit thickness one inch (say). All front compartments belonging to the same front-and-back row are mutually connected by tubes (in the tank but not in connection with the water of the tank) thus fulfilling the first set of conditions.

The thickness of the back compartment is adjustable but is (as will soon appear) constrained to be always equal to the price, thus if the price of *A* is \$1, of *B* \$3 and *C* \$1.20, the thickness of all of the cisterns in the *A* row will be 1, in the *B* row 3 and in the *C* row 1.2 (inches).

Since the thickness of the front compartment is unity, the contents of each back compartment equals the contents of the front multiplied by the number of inches of thickness of the back cistern, that is the back compartment contains a volume of water equal the amount of the commodity multiplied by its price. It contains therefore the *money value* of the commodity. The double cistern represents the double light in which each commodity is commonly regarded—so many pounds, yards, etc. and so many dollars worth.

All back compartments of the same right and left rows are mutually connected by tubes—that is the sum of their contents is given—thus fulfilling the second set of conditions.

The back compartments can change their thickness, as the walls at the right, left and bottom are of flexible leather; the back plane is kept parallel to the wood partition by two double “parallel rules” not diagrammed.

There remains to be described the system of levers. The purpose of these levers is to keep the continuous ratio of marginal utilities, the same for all individuals and equal to the ratio of prices.

First there is a system of oblique levers [our Figure 5] connected by sliding pivots with the tops of the cisterns and having their lower extremities hinged to wooden floats *F*, the hinges being at the level of the water of the tank. The floats are free only to shift latterly. It is evident from . . . similar triangles in [our Figure 5] that the ordinates of the 2 cisterns *IA* and *IB* are proportional to the distances of the *A* and *B* rods . . . from the hinge in the left float *F*. Likewise in the row behind, the ordinates are proportional to each other and in general all of the ordinates of the front row are proportional to those of the row next behind and so on. Remembering that each ordinate is a marginal utility we have:

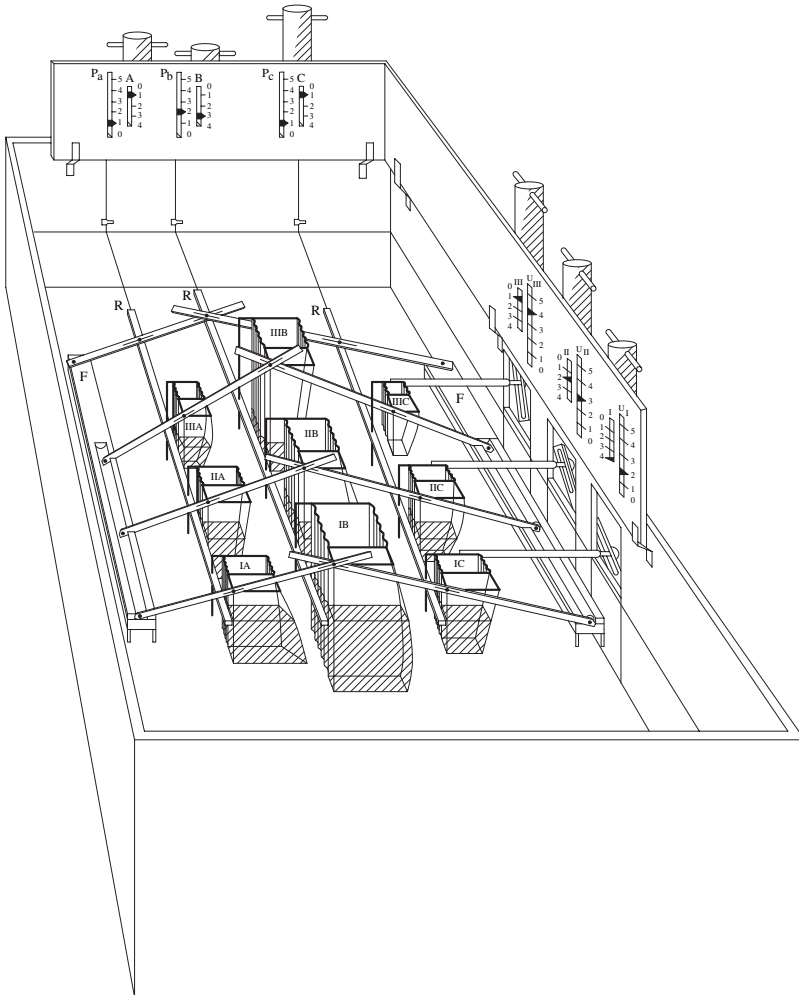
$$\frac{dU}{dA_1} : \frac{dU}{dB_1} = \dots = \frac{dU}{dA_2} : \frac{dU}{dB_2} = \dots = \frac{dU}{dA_3} : \frac{dU}{dB_3} = \dots = \dots$$

which is the required condition that the marginal utilities must be proportional.

Secondly there are the horizontal levers (*F34*, etc., Figure 10 [our Figure 6]) lying on the surface of the water in the tank. These relate to prices. The sliding pivots 3, 4 are connected to the rods *RRR*, which in

Figure 5

Fisher's apparatus.

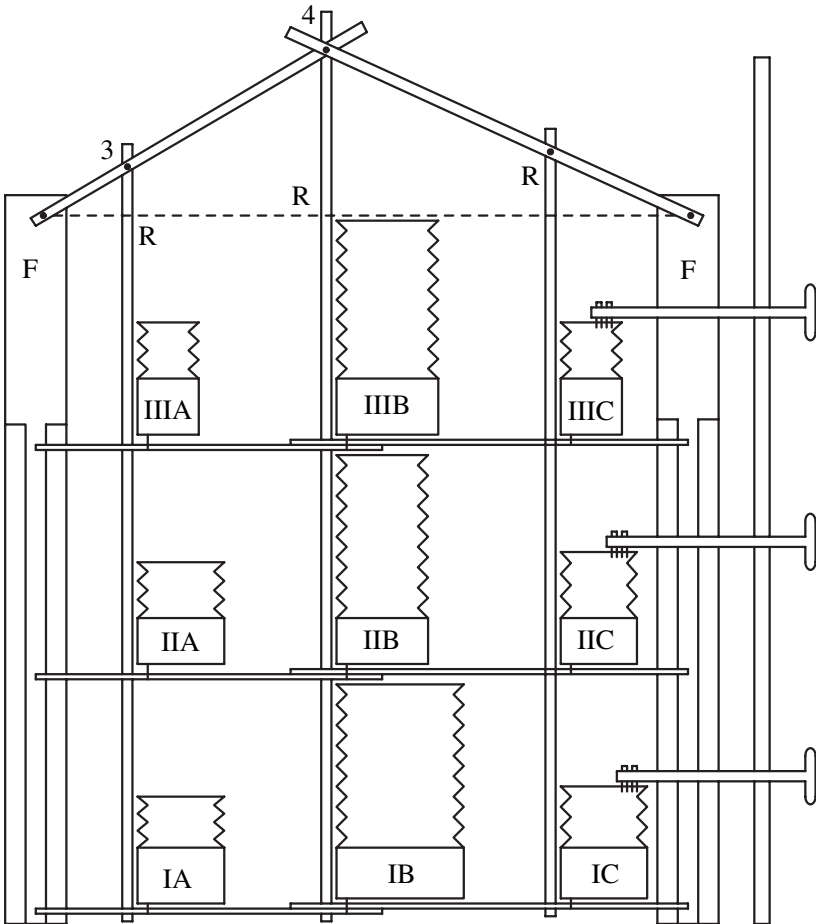


turn are connected by vertical pins with the rear walls of the cisterns. A motion of one of those rods causes all of the back compartments to expand or shrink in unison. The pivots 3, 4, etc. are so situated on these rods that if the levers  $F34$ , etc. should assume a right-and-left position along the dotted line  $FF$ , the back compartment of every cistern would

be completely closed. Hence  $R_3$  equals the thickness of each back compartment in the  $A$  row,  $R_4$  the corresponding thickness in the  $B$  row and so on.

By the similar triangles  $FR_3$  and  $F_3R_4$  in Figure 10 [our Figure 6], it is clear that the lines  $R_3$  and  $R_4$ , and consequently the rear thickness in the  $A$  and  $B$  rows are proportional to the distances of the  $A$  and  $B$  rods  $R$  and  $R$  from the float  $F$ . But we have just seen that the ordinates of  $IA$  and  $IB$  are proportional to these same distances. Hence the thickness of the back

Figure 6



compartments of the cisterns are proportional to the ordinates of those cisterns, that is to marginal utilities. Hence we are free to call the thickness of each back compartment, the money price of the commodity to which that cistern relates.

It is to be observed that the cisterns are free to move only *vertically*, the rods and rear cistern walls only *forward* and *backward*, the wooden floats can shift sidewise *right and left* while the levers assume such positions as the mechanism compels.

This ends our direct quotation from Fisher's thesis, and his description of his machine. In the next section, we describe our computer simulation of his model.

#### *The Digital Version of the Fisher Machine*

Did Fisher's machine work? We knew it must have, at least after a fashion, because Fisher reports that for 25 years he used his first model in teaching. But we were curious to see it work, and could not help wondering whether the floats would come to rest against their stops rather than in a "market" equilibrium. After some romantic thoughts about replicating the machine, tub, water, and all, we settled on simulating the machine in Matlab. The physical features of the Fisher apparatus fall into two groups. The properties of the first group—the quantity and expenditure cisterns and the system of rods controlling the elevations of the cisterns and the price dimension of the expenditure cisterns—determine the equilibrium prices and allocation, and when the system is out of equilibrium they determine the forces and pressures that move the rods, cisterns, floats, and bellows themselves. The second group of features—the tubes connecting the quantity and expenditure cisterns, the mass and surface area of the floats, the viscosity of water—govern the speed of adjustment of the system when it is out of equilibrium.

Our mathematical description of the first group of features is straightforward and faithful to the hydraulic model, except for ignoring the actual model's "imperfections"—the nonzero mass of the rods and the thickness and mass of the walls of the cisterns. For any position of the consumer and price rods, we use Archimedes principle and the weight of water in each of the cisterns to calculate the vertical forces they exert; these forces are resolved into a rotational

force on each consumer's rods, and a force inward or outward on the two floats. The inward or outward pressures on the price bellows of each consumer's expenditure boxes are calculated from the difference in the level of water inside and out of the expenditure cisterns. The resulting forces are transmitted to the price rods, and are also resolved into rotational forces and inward or outward forces on the floats.

The construction of the Fisher model guarantees that in equilibrium or out, the quantity of water summed over each individual's expenditure cisterns always equals that individual's money income, and the quantity of water, summed over an individual's quantity cisterns, always equals the social endowment for each commodity. Tubes allow these fixed quantities to flow among the expenditure cisterns for each individual, and among commodity cisterns across individuals for each commodity. This preservation of quantities is easy to mimic in the Matlab version. As explained in the previous section by Fisher, if the level of the water in each of the quantity and expenditure cisterns is the same as the level of water in the tub, each consumer will be maximizing utility subject to his or her budget constraint and the prices and allocations will correspond to a competitive equilibrium. This condition obviously implies that for each cistern, the Archimedes forces are exactly balanced by the weight of water in the cistern; in both the hydraulic and Matlab model, if the system is at rest in a competitive equilibrium, there are no net forces to disturb it. It is an intriguing question, whose answer we do not know, whether there are any rest points in which the level of water in the cisterns is different from the level in the tub.

In contrast to its accurate description of the features of the hydraulic model determining equilibrium, the Matlab model is an approximation of several features governing the dynamics of adjustment. Rather than the continuous, real-time, adjustments in the hydraulic machine, adjustments in the Matlab model are discrete, iteration by iteration. Three speeds of adjustment are crucial to the dynamics—the speed with which water flows among cisterns, the speed with which the cisterns move up or down in the tub and the price bellows expand or contract, and the speed with which the pivot floats move in or out. The diameter, length, and surface characteristics of the tubes



used by Fisher to connect the cisterns are not known to us, nor are the mass, displacement, or surface area of the floats. If the dimensions of the tubes were known, textbook fluid mechanics could be used to calculate the flow as a function of pressure differentials, viscosity, surface frictions, etc.; those same textbooks suggest that as a practical matter, there is no substitute for estimating the flow experimentally. Similarly, while it might be theoretically possible to calculate how far in a moment of time a given force would move a cistern—up or down—in the tub, as a function of its mass, shape, and surface areas, it would have been a daunting task. In our simulations, what is important are the relative speeds of adjustment. Our intuition suggested that the vertical movements of the cisterns, the inward or outward movements of the floats, and the adjustment of the price bellows were likely to be much quicker than the adjustments of the fluid levels through the tubes. The results of a casual experiment, using a one-liter coke bottle as a proxy for a cistern and a one-foot length of 5/8" hose for a tube, were consistent with that intuition; a bottle of water two-thirds full came to a displacement equilibrium in a tub much faster (by a factor of something like 10) than when the same bottle was emptied into the tub by the hose. This led us to choose the parameters in the model so that the adjustments of the consumer rods, bellows, and floats are much more rapid than the equalization of water levels in the commodity or expenditure cisterns.

Given these parameter choices, price and consumer rods adjust to almost completely eliminate the net forces on them, while significant differences in the level of water in the various cisterns remain. Starting from an arbitrary initial condition, there is typically a rapid adjustment of rods and floats, followed by a much longer period during which the water flows among the cisterns and the entire set of variables move slowly together. In our actual simulations it does not appear that the stability of the adjustment mechanism is much affected by the precise values of these parameters, so long as the step size, corresponding to an iteration, is kept small. As will be seen, the adjustment paths display this qualitative behavior; from an initial disequilibrium of both prices and quantities, prices adjust rapidly to levels that balance the forces on the rods, followed by much slower

adjustment as the water flows between cisterns. The Matlab model also ignores two physical phenomena that could, in principle, be important to the dynamics of adjustment—the increase in drag that results from increases in velocity, and inertia. Inertia could, even with continuous time, result in overshooting, with damped oscillations of the system, around equilibrium values. The equations used in the MatLab program are found in the Appendix.

#### *Fisher's Use of the Machine*

Fisher regarded his model as “the physical analogue of the ideal economic market” (Mathematical investigations, p. 44), with the virtue that “[t]he elements which contribute to the determination of prices are represented each with its appropriate role and open to the scrutiny of the eye . . .” providing “a clear and analytical picture of the interdependence of the many elements in the causation of prices.” Fisher also saw the machine as way of demonstrating comparative statics results “to employ the mechanism as an instrument of investigation and by it, study some complicated variations which could scarcely be successfully followed without its aid.” We do not know how transparent the model was to Fisher’s students, but it is easy to imagine the excitement they may have felt in watching the model at work, accompanied by enthusiastic commentary by Fisher.

In principle the Fisher model will find the competitive equilibrium for any three-commodity, three-person exchange economy with additively separable preferences, with the restriction that individuals’ endowments are some fraction of the aggregate commodity endowment bundle—a restriction implicit in the assumption that individuals are endowed with fixed money incomes rather than arbitrary bundles of commodities. To find a particular equilibrium merely requires specifying the shapes of the  $3 \times 3$  cisterns, and prescribing the aggregate supplies and individual endowments. As previously explained, for the Fisher machine to work, the preferences and quantity endowments must be assigned to rows so that in equilibrium the commodity in the middle row has the highest marginal utility. Given preferences and initial quantities, comparative static results can be obtained simply by altering the aggregate supplies or individual

endowments as desired and observing the new equilibrium prices and allocation.

Although we do not know what experiments Fisher actually ran with his machine, he does describe eight comparative static exercises. Some of these illustrate basic features of the system, for example, that proportional increases in money incomes result in an equal proportional increase in each price, with no change in the allocation of goods. Another simple exercise discussed by Fisher examines whether proportional increases in the endowment of goods necessarily result in proportional decreases in prices, as was apparently, and incorrectly, believed by Mill. Some exercises illustrate less intuitive properties of exchange economies: increasing one individual's income may make some other individual better off and also the possibility of "immiserating growth," i.e., increasing an individual's endowment of a good may actually lower his or her welfare.

Although Fisher discusses the way in which changing an individual's preferences alters the equilibrium, it was expensive to change the cisterns in his model and it seems unlikely they were changed after some initial experimentation. The cisterns in Fisher's original 1893 model included a fairly rich variety of shapes; the cisterns in his 1925 model look like they may all have rectangular faces, corresponding to quadratic utility functions. While Fisher's construction made it difficult to vary preferences, his set up made it easy to change quantities; by simply moving a plunger for a commodity or individual's income he could force more water into the associated column or row of cisterns.

#### *Dynamics of the Machine*

We used our Matlab representation of the hydraulic machine to simulate the path of adjustment for a variety of preferences, endowments, and initial conditions. We do not have a basis for comparison with Fisher's actual model; Fisher does not analyze or describe the dynamic behavior of his machine.

It is easy to imagine that, as Fisher slowly depressed a quantity plunger to change the equilibrium, the flows of water between cisterns and the movements of lever arms were rapid enough to keep

the system in the neighborhood of the shifting equilibrium; it is likely that in Fisher's experiments the initial conditions were never very far from equilibrium. In contrast, in our simulations we typically started the system with arbitrary allocations of the quantities to the cisterns (dividing the quantities equally across individuals and expenditures by individuals equally across commodities) and arbitrary prices. As a consequence our simulations show rapid movements of prices early in the adjustment process, with substantial discrepancies between the quantity of a commodity  $j$  being allocated to individual  $i$ ,  $q[i, j]$  and the quantity of that good that could be purchased by the allocation of expenditures on commodity  $j$  by individual  $i$ ,  $qe[i, j]$ , given current prices. These characteristics are evident in three examples given in Figure 5, which display paths of prices, and these discrepancies for the first commodity ( $q[i, 1] - qe[i, 1]$ ) during the adjustment process.

It is difficult to imagine an economic interpretation of Fisher's adjustment process. Throughout the adjustment process individuals' commodity allocations sum to the social endowment, and each individual's money income is exactly exhausted in that individual's expenditure cisterns. However, allocations are not on the contract curve (ratios of marginal utilities are not the same across individuals), individuals are not maximizing utility (the ratios of the marginal utilities of the quantities being consumed by an individual are not the same as the ratios of prices), and the value at current prices of a good allocated to an individual is typically not equal to the nominal income allocated to purchase that good by the individual (the water level in the front and back cisterns for a given commodity need not be equal).

#### *An Example of Multiple Equilibria*

The restrictive assumptions on preferences and endowments embodied in the Fisher machine are not sufficient to guarantee uniqueness. Indeed it is possible to have multiple equilibria even if the additively separable preferences are restricted to be quadratic. How would the machine behave if in fact there is more than one equilibrium? A combination of preferences and endowments that gives rise to multiple equilibria is specified in the following example.

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**Example**

Each cistern is a rectangle.

Consumer 1 has income of 5 and cisterns of size:

$$\boxed{46 \times 3.6} \quad \boxed{3 \times 12} \quad \boxed{13 \times 26}$$

Consumer 2 has income of 2.5 and cisterns:

$$\boxed{6.5 \times 26} \quad \boxed{1.5 \times 12} \quad \boxed{23 \times 3.6}$$

Consumer 3 has income of 2.5 and cisterns:

$$\boxed{6.5 \times 26} \quad \boxed{1.5 \times 12} \quad \boxed{23 \times 3.6}$$

The supplies of the three goods are:

$$\boxed{400, 49.5, 400}$$


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This example has several special features that helped make it relatively easy to construct, for example, two of the individuals are identical. It was not difficult to find the three equilibria of this example by starting the simulations in appropriate places. The paths of adjustment for the three different initial conditions shown in Figure 7 display the characteristics expected of the Matlab model. Consistent with the relatively rapid adjustment of the lever arms and floats compared to fluid flows, prices adjust rapidly at the start and are almost coincident with the  $y$  axis, whereas elimination of discrepancies in the levels of water in the cisterns (not shown), and discrepancies between  $q[\cdot]$  and  $qe[\cdot]$  are much more gradual. It is interesting to note that for equilibrium 2 the process of adjustment seems to have been almost completed after 2,500 iterations, but, reminiscent of Lorentz's examples from meteorology, after another 1,000 iterations the system moves off and converges to equilibrium 1. This adjustment path is

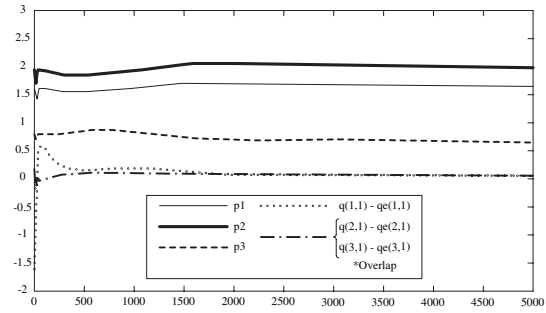
Figure 7

Dynamics of adjustment.

Equilibrium 1

$$p^* = [0.6199 \ 1.9600 \ 1.6370]$$

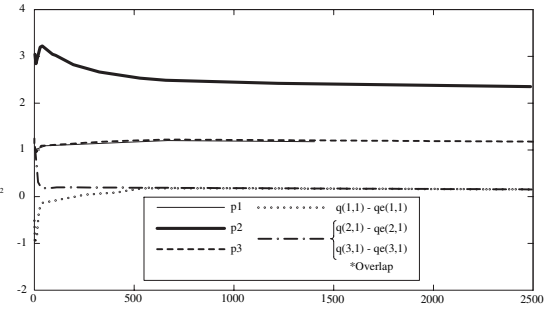
$$(q - qe)^* = \begin{bmatrix} 0.03E_1 & 0.44E_2 & -0.04E_3 \\ 0.005E_1 & 0.22E_2 & -0.07E_3 \\ 0.005E_1 & 0.22E_2 & -0.07E_3 \end{bmatrix} \times 10^{-2}$$



Equilibrium 2

$$p^* = [1.1054 \ 2.4060 \ 1.0964]$$

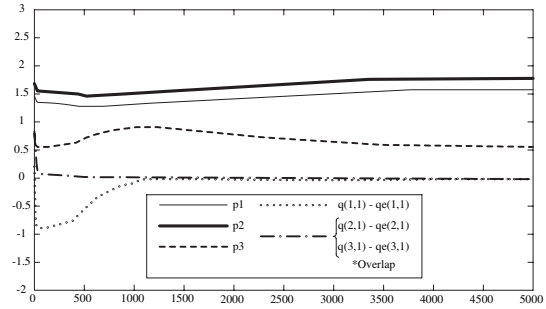
$$(q - qe)^* = \begin{bmatrix} -0.14E_1 & -0.10E_2 & -0.05E_3 \\ -0.005E_1 & -0.44E_2 & -0.04E_3 \\ -0.005E_1 & -0.04E_2 & -0.04E_3 \end{bmatrix} \times 10^{-2}$$



Equilibrium 3

$$p^* = [1.6560 \ 1.9310 \ 0.6082]$$

$$(q - qe)^* = \begin{bmatrix} -0.07E_1 & 0.13E_2 & -0.00E_3 \\ -0.00E_1 & 0.08E_2 & 0.02E_3 \\ -0.00E_1 & 0.08E_2 & 0.02E_3 \end{bmatrix} \times 10^{-2}$$



\*At last iteration

shown in Figure 8. Since we do not have an economic interpretation of the dynamics of adjustment in the hydraulic model, we cannot make use of the usual economic propositions about the equilibria, for example, the role of the assumption of gross substitutability in guaranteeing uniqueness of the competitive equilibrium and stability of the Walrasian price adjustment mechanism.

### Appendix

The equations used in the digital version of the Fisher machine are given in Figure 9. The state of the system is given by the values of 24 state variables—the nine commodity and nine expenditure allocations ( $x_{ij}$ ,  $y_{ij}$ ), two float locations ( $a$ ,  $c$ ), three consumer rod angles

Figure 8

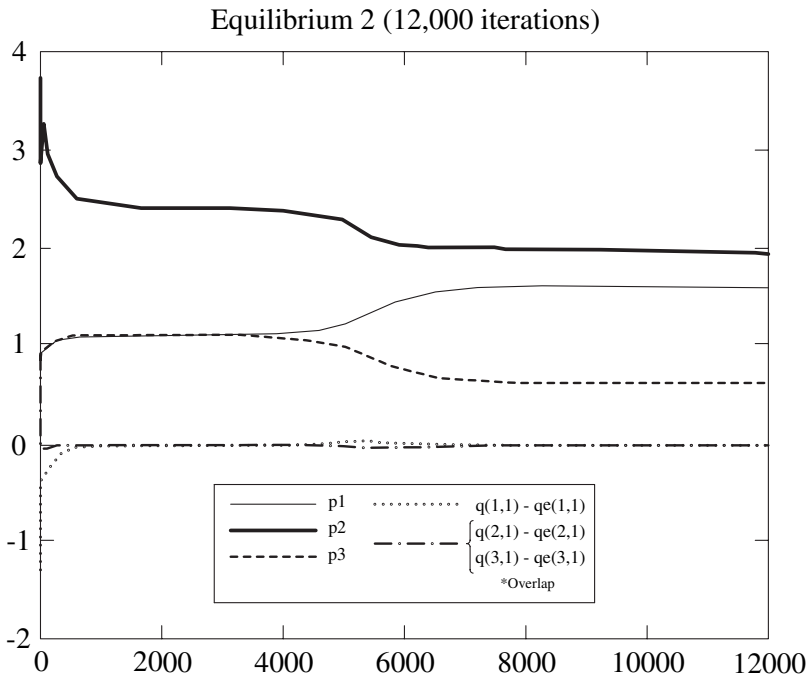


Figure 9

## Fisher Machine

		Equa- tions	Variables (New)
(1)	Market clearing	$\sum x_{ij} = E_j$	$j = 1, 3$ 3
(2)	Expenditure constraints	$\sum y_{ij} = Y_i$	$i = 1, 3$ 3
(3)	Heights ( $h_{ij}$ ) of cisterns above surface level where $\alpha_i =$ angle of consumer $i$ 's float A rod to surface	$h_{11} = l a_1 \tan(\alpha_i); h_{12} = l a_2 \tan(\alpha_i);$ $h_{13} = l a_3 \tan(\alpha_i)$	$i = 1, 3$ 9
(3')	$l a_i, l c_i =$ lever arm of cisterns relative to floats A, C, where floats A and C at locations $-(1 + a)$ and $(1 + c)$	$l a_1 = a, l a_2 = (1 + a),$ $l a_3 = (1 + a)c/(1 + c)$ $l c_1 = c, l c_2 = (1 + c),$ $l c_3 = (1 + c)a/(1 + a)$	3    3
(3'')	Angle ( $\gamma_i$ ) of consumer $i$ 's float C rod	$\gamma_i = \tan^{-1}((1 + a)\tan(\alpha_i)/(1 + c))$	3    3
(4)	Prices ( $p_j$ ) as function of price rod's angle $\beta$ to perpendicular to axis of float A	$p_1 = l a_1 \tan(\beta) = l c_3 \tan(\delta)$ $p_2 = l a_2 \tan(\beta) = l c_2 \tan(\delta)$ $p_3 = l a_3 \tan(\beta) = l c_1 \tan(\delta)$	3    4
(4')	Angle ( $\delta$ ) of float C price rod	$\delta = \tan^{-1}((1 + a)\tan(\beta)/(1 + c))$	1    1



- (5) Marginal utilities of commodity allocation  $(D_{ij} - \xi_{ij}) = u'_{ij}(x_{ij})$  9 9
- (6) Marginal utilities of expenditure allocation where  $\xi_{ij}$   
 $\psi_{ij}$  = depth of water in commodity, expenditure cistern;  $(D_{ij} - \psi_{ij}) = u'_{ij}(y_{ij}/p_i)$  9 9
- (7)  $D_{ij}$  = maximum depth of cistern  
 Change in angles of consumers' rods (a function of weight of undispaced water in cistern) where  $s_{ij}$  = speed of adjustment  

$$\Delta\alpha_i = S_{ij}/((1 + \tan(\alpha_i)^2)\Sigma_j \ell a_j^2) \times \Sigma_j \ell a_j [x_{ij} - u_{ij}^{-1}(b_{ij}) + y_{ij} - p_j u_{ij}^{-1}(b_{ij})]$$
 i = 1, 3 3 -
- (8) change in price rod angle (function of pressures on price bellows) where  $s_{\beta}$  = speed of adjustment  

$$\Delta\beta = S_{\beta}/((1 + \tan(\beta)^2)\Sigma_j \ell a_j^2) \times \Sigma_j \ell a_j p_j r_{ij}(\psi_{ij}, b_{ij})$$
 1 -
- (8')  $p_{r_{ij}}(\psi_{ij}, b_{ij})$  = pressure on price bellows ij where  $f(t)$  = width of cistern at depth  $t$   

$$p_{r_{ij}}(\psi_{ij}, b_{ij}) = \int_0^{\psi_{ij} - (D_{ij} - b_{ij})} f(t) dt$$

Figure 9 Continued

		Equa- tions	Variables (New)
(9)	Change in float locations (change in relative prices)	$\Delta a = -S_{\beta}/(1 + a)\Sigma_i \{\tan(\alpha_i)(x_{i1} - u_{i1}^{-1}(b_{i1}))$ $+ y_{i1} - p_i u_{i1}^{-1}(b_{i1})\}$ $- \tan(\beta) P_{\pi}(\psi_{\pi}, b_{\pi})\}$ $\Delta c = -S_{\delta}/(1 + c)\Sigma_i \{\tan(\gamma_i)(x_{i3} - u_{i3}^{-1}(b_{i3}))$ $+ y_{i3} - p_{i3} u_{i3}^{-1}(b_{i3})\}$ $- \tan(\delta) p_{i3}(\psi_{i3}, b_{i3})\}$	2
(10)	Quantity adjustments ( $S_x$ = flow as function of pressure differential)	$\Delta x_{1j} = S_x(\xi_{2j} - \xi_{1j}) + S_x(\xi_{3j} - \xi_{1j})$ $\Delta x_{2j} = S_x(\xi_{1j} - \xi_{2j}) + S_x(\xi_{3j} - \xi_{2j})$ $\Delta x_{3j} = S_x(\xi_{1j} - \xi_{3j}) + S_x(\xi_{2j} - \xi_{3j})$	$j = 1, 3$ 6
(11)	Expenditure adjustments	$\Delta y_{1j} = S_x(\psi_{2j} - \psi_{1j}) + S_x(\psi_{3j} - \psi_{1j})$ $\Delta y_{2j} = S_x(\psi_{1j} - \psi_{2j}) + S_x(\psi_{3j} - \psi_{2j})$ $\Delta y_{3j} = S_x(\psi_{1j} - \psi_{3j}) + S_x(\psi_{2j} - \psi_{3j})$	$j = 1, 3$ 6

$(a_i)$ , and a single price rod angle  $(b)$ . Together these 24 variables determine 40 other variables used in the solution of the model. The sets of equations in (1) through (6) give the constraints and relationships among these variables. The condition that the commodity and expenditure allocations add up to the exogenous commodities and incomes  $(E_j, Y_i)$  are given by six equations in (1) and (2). Not counting these equations there are 18 independent equations and variables. The equations in (3) give the heights of the cisterns above the surface of the water  $(h_{ij})$  as determined by the lever arms  $l_{aj}, l_{cj}$  implied by the float locations and consumer rod angles  $(a_i)$ . Equations in (4) give prices as a function of the same lever arms and the price rod angle  $(b)$ . Equations (5) and (6) give the depth of water in the commodity and expenditure cisterns  $(x_{ij}, Y_{ij})$  implied by the commodity and expenditure allocations.

In equilibrium the heights for the cisterns above the surface of the water in the tub  $(h_{ij})$ , the marginal utilities implied by the commodity allocations  $(u'_{ij}(x_{ij}))$  and the marginal utilities implied by the expenditure allocations  $(u'_{ij}(y_{ij}/p_j))$  are all equal and proportional to the prices.

Equations (7) through (11) give the adjustment of the 24 state variables iteration by iteration as a function of the forces on the rods and floats when the system is out of equilibrium. (For economy of presentation and computation the adjustments of  $a$  and  $c$  assume that the moments of the consumer rods are zero, equivalently that the  $(a_i)$  and  $(b)$  are changing so slowly that the motion of the cisterns through the water induces negligible resistance.)