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CHAPTER 13

THE LIMIT OF THE CORE OF AN ECONOMY

GERARD DEBREU and HERBERT SCARF[1]

1. Introduction. – 2. The core of an economy. – 3. Edgeworth on the limit of the contract curve. – 4. A general limit theorem on the core of an economy.

1. Introduction

In an economy made up of agents initially owning certain quantities of commodities and trading with each other, the final result of the exchange process is a new allocation of the total quantities of commodities available. The core of the economy is the set of final allocations that no coalition of agents can, and wants to, prevent. EDGEWORTH (1881) introduced this concept under the name of the "contract curve" and studied the limit of the core under the following conditions. Two agents are said to be of the same type if they have the same preferences and the same initial commodity-vector. Edgeworth proved that, under assumptions to be specified later, in an economy with two commodities and two types of agents, if the number of agents of each type increases indefinitely, then the core of the economy decreases, or stays constant, and tends to the set of competitive allocations. This fundamental result provided the first precise explanation of competitive behavior. Yet, it received little notice for many decades,[2] to a large extent because of the way in which Edgeworth presented his ideas. We shall, as part of this paper, give a detailed account of his contribution.

The concept of the core reappeared in a different form in the theory of games with transferable utility. In this context it was made explicit, received its name, and was studied in 1953 by D. B. Gillies and L. S. Shapley. The connection between Edgeworth's contract curve and the core of a game was

[1] We thank Tjalling Koopmans for his comments on an earlier version and the National Science Foundation for its support of our work.

[2] See, however, J. A. SCHUMPETER (1954), pp. 830–831, 984.

then perceived by M. SHUBIK (1959), who called attention to the contribution of *Mathematical Psychics*.

During the last decade several developments have taken place in the study of the core of an economy. The first has consisted in a generalization of Edgeworth's theorem from the case of two commodities and two types of agents to the case of arbitrary finite numbers of commodities and types of agents with convex preferences (SCARF (1962), DEBREU (1963), DEBREU and SCARF (1963)).[3] The behavior of the core of large economies with non-convex preferences was then investigated by L. S. SHAPLEY and M. SHUBIK (1966) in the context of transferable utility, and by K. J. ARROW and F. HAHN (Chapter 8) in the context of nontransferable utility. The limit theorem of DEBREU and SCARF (1963), which will be described in Section 4 of this paper, was extended by J. JASKOLD-GABSZEWICZ (1968) to a class of economies with an infinity of commodities.

A second development began with the proof by R. J. AUMANN (1964) that if the agents of an economy form a continuum, then the core equals the set of competitive allocations, a result for which a different derivation was given by K. VIND (1964). The introduction of a measure-theoretic point of view for the set of agents of an economy is of great importance for models attempting to explain competitive behavior. We do not discuss it further here only because we cannot do so without a mathematical apparatus far more elaborate than the one to which we wish to restrict ourselves in this expository article. We refer instead to the two recent surveys by R. J. AUMANN (1972) and W. HILDENBRAND (1971).

A third direction has been the development of algorithms for calculating a point in the core. In SCARF (1967) a theorem is given, which provides sufficient conditions for the core of an n person game to be nonempty. The theorem applies, in particular, to an exchange economy with convex preferences and the method of proof permits the approximate calculation of a point in the core. The theorem may also be used to demonstrate the non-emptiness of the α-core (R. J. AUMANN (1961)) for a game in normal form with convex strategy spaces and quasi-concave utilities. These algorithms are contained in a more general class of numerical techniques which can be used to approximate a fixed point of a continuous mapping of a simplex into itself, and to determine approximate price equilibria in a general Walrasian model (SCARF (1969), HANSEN and SCARF (1969)).

[3] An alternative proof of the main theorem of the last paper was offered by K. VIND(1965).

2. The Core of an Economy

Let $\mathscr{E}$ be an economy with l commodities and m agents. Each one of the agents, say the i-th one, initially owns a commodity-vector ω_i, an element of the l-dimensional commodity space R^l. We assume that the commodity-vectors to be assigned to the i-th agent are always in Ω, the nonnegative orthant of R^l, and that he has preferences among the elements of Ω. This preference relation is denoted by $\precsim_i$ and $x \precsim_i y$ is read "the commodity-vector y is at least as desired by the i-th agent as the commodity-vector x." The relation $\precsim_i$ has the properties:

(1) for every x in Ω, $x \precsim_i x$ (reflexivity);
for every x, x', x'' in Ω, if $x \precsim_i x'$ and $x' \precsim_i x''$, then $x \precsim_i x''$ (transitivity);
for every x, x' in Ω, $x \precsim_i x'$ and/or $x' \precsim_i x$ (completeness).

We say that y is preferred to x, and we write $x \prec_i y$, when $x \precsim_i y$ and not $y \precsim_i x$. It will be assumed that the preferences of every agent satisfy the following conditions of insatiability (2) and strong-convexity (3).

(2) If x is in Ω, then there is x' in Ω such that $x \prec_i x'$.

(3) Let x_1, x_2 be distinct vectors in Ω and let α_1, α_2 be positive real numbers *adding up to* 1. *If* $x_1 \succsim_i x_2$, *then* $\alpha_1 x_1 + \alpha_2 x_2 \succ_i x_2$.

From this last assumption, we can derive Lemma 1 which will be used in the proof of Lemma 2.

LEMMA 1. *For* $r \geqq 2$, *let* $x_1, \ldots, x_r$ *be vectors in* Ω *that are not all equal and let* $\alpha_1, \ldots, \alpha_r$ *be positive real numbers adding up to* 1. *If* $x_q \succsim_i x_r$ *for every* $q = 1, \ldots, r$, *then* $\sum_{q=1}^{r} \alpha_q x_q \succ_i x_r$.

Proof. We re-index the first $r-1$ vectors in such a way that $x_q \succsim_i x_{r-1}$ for every $q = 1, \ldots, r-1$ and note that for $r = 2$, the proposition is assumption (3) itself. We give a proof by induction making the hypothesis that the assertion is true for $x_1, \ldots, x_{r-1}$.

Observe that

$$\text{(i)} \qquad \sum_{q=1}^{r} \alpha_q x_q = \alpha_r x_r + \left(\sum_{q=1}^{r-1} \alpha_q\right) \left[\frac{1}{\sum_{q=1}^{r-1} \alpha_q} \sum_{q=1}^{r-1} \alpha_q x_q\right]$$

and denote the vector between brackets by x'. Two cases may arise:

(a) $x_1, \ldots, x_{r-1}$ are all equal. Then x' is their common value and since all the $x_q (q = 1, \ldots, r)$ are not equal, one has $x_r \neq x'$. Moreover, $[x' = x_{r-1}$ and $x_{r-1} \precsim_i x_r]$ implies $[x' \succsim_i x_r]$.

(b) $x_1, \ldots, x_{r-1}$ are not all equal. Then, by the induction hypothesis $x' \succ_i x_{r-1}$. Since $x_{r-1} \succsim_i x_r$, one has $x' \succ_i x_r$.

Thus, in either case, $x' \neq x_r$ and $x' \succsim_i x_r$. It now suffices to apply assumption (3) to equality (i) to obtain $\sum_{q=1}^{r} \alpha_q x_q \succ_i x_r$. Q.E.D.

An allocation for the economy $\mathscr{E}$ specifies the commodity-vector assigned to each agent. Thus an allocation is an m-tuple $(x_1, \ldots, x_i, \ldots, x_m)$ of points of Ω, where x_i denotes the commodity-vector assigned to the i-th agent. Such an allocation is attainable if the total commodity-vector $\sum_{i=1}^{m} x_i$ assigned to the set of all agents equals the available total commodity-vector $\sum_{i=1}^{m} \omega_i$. Consider now an allocation $(x_1, \ldots, x_i, \ldots, x_m)$. A coalition S of agents *blocks* the allocation (x_i) if its members can redistribute their own initial commodity-vectors among themselves so that every one of them is at least as satisfied as he is with the allocation (x_i) and at least one of them is more satisfied. In symbols, the coalition S blocks the allocation $(x_1, \ldots, x_i, \ldots, x_m)$ if one can find for each i in S a commodity-vector x_i' in Ω such that

(ii) $$\sum_{i \in S} x_i' = \sum_{i \in S} \omega_i$$

and

(iii) for every $i \in S$, $x_i' \succsim_i x_i$, while for some $i \in S$, $x_i' \succ_i x_i$.

The *core* of the economy $\mathscr{E}$ is then formally defined as the set of attainable allocations that no coalition of agents blocks.

The second term of the comparison with which we are concerned is the set of competitive allocations of the economy $\mathscr{E}$. In a precise manner, the agents of $\mathscr{E}$ are said to behave competitively if (I) total demand equals total supply and (II) there is a price-vector p in R^l such that for every $i = 1, \ldots, m$, *the i-th agent chooses his demand x_i according to his preferences among the commodity-vectors in Ω whose value does not exceed that of his initial commodity-vector ω_i*. Denoting the value of the commodity-vector z relative to the price-vector p by $p \cdot z$, we can express the italicized part of condition (II) as

(iv) $p \cdot x_i \leqq p \cdot \omega_i$ and $[x \in \Omega$ and $p \cdot x \leqq p \cdot \omega_i]$ implies $[x \precsim_i x_i]$.

Thus an allocation $(x_1, \ldots, x_i, \ldots, x_m)$ is defined as competitive if it is attainable and there is a price-vector p such that (iv) is satisfied for every $i = 1, \ldots, m$.

The asymptotic equality of the core and of the set of competitive allocations is established by Theorems 1 and 2.

THEOREM 1. *Every competitive allocation of $\mathscr{E}$ is in the core of $\mathscr{E}$.*

Proof. Let $(x_1, \ldots, x_i, \ldots, x_m)$ be a competitive allocation of $\mathscr{E}$ and let p be the associated price-vector. We first remark that, according to (iv),

(v) $[x_i' \in \Omega$ and $x_i' \succ_i x_i]$ implies $[p \cdot x_i' > p \cdot \omega_i]$.

We also remark that

(vi) $[x_i' \in \Omega$ and $x_i' \succsim_i x_i]$ implies $[p \cdot x_i' \geqq p \cdot \omega_i]$.

The latter statement is correct because there is, by assumption (2), a commodity-vector x_i'' in Ω such that $x_i'' \succ_i x_i'$. Therefore, by assumption (3), for

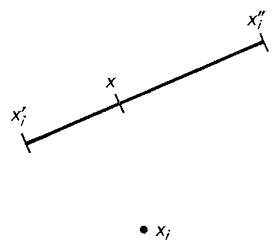

Fig. 1

every point x of the segment joining x_i' and x_i'' different from x_i', one has $x \succ_i x_i'$, hence, $x \succ_i x_i$. Consequently, by (v), $p \cdot x > p \cdot \omega_i$. Letting x tend to x_i', one obtains in the limit $p \cdot x_i' \geqq p \cdot \omega_i$.

Suppose now that coalition S blocks the given competitive allocation. For every i in S, there is a commodity-vector x_i' in Ω such that (ii) and (iii) hold. By (vi), for every i in S, one has $p \cdot x_i' \geqq p \cdot \omega_i$. By (v), for some i in S, one has $p \cdot x_i' > p \cdot \omega_i$. Therefore, $\sum_{i \in S} p \cdot x_i' > \sum_{i \in S} p \cdot \omega_i$, a contradiction of (ii). Q.E.D.

In order to be able to state the asymptotic converse of this theorem we introduce the economy $\mathscr{E}^r$ with m types of agents and r agents of each type. Each one of the r agents of the i-th type initially owns the commodity-vector ω_i and has the preferences $\precsim_i$, where $i = 1, \ldots, m$. An allocation for $\mathscr{E}^r$ is an mr-tuple $(x_{11}, \ldots, x_{iq}, \ldots, x_{mr})$ of commodity-vectors in Ω where x_{iq} is the commodity-vector assigned to the q-th agent of the i-th type. Therefore, the core is a subset of the space R^{lmr}. We will now prove that the core can be represented by a subset of the space R^{lm}. An essential simplification will result from the fact this space does not depend on r.

The proposition that Edgeworth expressed by saying that "all the field is collected at one point" (EDGEWORTH (1881), middle of page 35) is

LEMMA 2. *An allocation in the core of $\mathscr{E}^r$ assigns the same commodity-vector to all agents of the same type.*

Proof. Let $(x_{11}, \ldots, x_{iq}, \ldots, x_{mr})$ be an allocation in the core of $\mathscr{E}^r$ and, for each i, denote by x_i the worst of the commodity-vectors $(x_{i1}, \ldots, x_{iq}, \ldots, x_{ir})$ according to the preferences $\precsim_i$. If there are several worst commodity-vectors, indifferent to each other, x_i is arbitrarily chosen among them. Thus one has $x_{iq} \succsim_i x_i$ for every $q = 1, \ldots, r$.

Then let S denote the coalition of m agents formed by taking for each i one of the least privileged agents, i.e., one of the agents to whom x_i is assigned. Since the allocation (x_{iq}) is attainable, $\sum_{i=1}^{m} \sum_{q=1}^{r} x_{iq} = r\left(\sum_{i=1}^{m} \omega_i\right)$, from which one obtains $\sum_{i=1}^{m} \frac{1}{r}\left(\sum_{q=1}^{r} x_{iq}\right) = \sum_{i=1}^{m} \omega_i$. In other words, coalition S can distribute its total initial commodity-vector $\sum_{i=1}^{m} \omega_i$ so as to give to its member of the i-th type the commodity-vector $x_i' = \frac{1}{r}\left(\sum_{q=1}^{r} x_{iq}\right)$ for every $i = 1, \ldots, m$. Consider a certain value of i. If all the x_{iq} $(q = 1, \ldots, r)$ are identical, then clearly $x_i' = x_i$. If all the x_{iq} $(q = 1, \ldots, r)$ are not identical, then, by Lemma 1, $x_i' \succ_i x_i$. Therefore if for some $i = 1, \ldots, m$, all the x_{iq} $(q = 1, \ldots, m)$ are not identical, coalition S blocks the proposed allocation (x_{iq}) by giving x_i' to its member of the i-th type $(i = 1, \ldots, m)$. Q.E.D.

Thus an allocation in the core of $\mathscr{E}^r$ can be represented by an m-tuple $(x_1, \ldots, x_i, \ldots, x_m)$ of points of the space R^l and the core of $\mathscr{E}^r$ can be represented by a subset C^r of the space R^{lm}. Obviously, C^{r+1} is contained in C^r, for if an allocation belongs to the core of $\mathscr{E}^{r+1}$, it is blocked by no coalition of $\mathscr{E}^{r+1}$ and, in particular, by no coalition having at most r members of each type. Therefore, when r increases indefinitely, the sets C^r form a nested sequence having their intersection as a limit. The characterization of this limit is the problem to which we turn.

3. Edgeworth on the Limit of the Contract Curve

The solution offered by Edgeworth is contained in *Mathematical Psychics*, pp. 34–38. In presenting his proof of the asymptotic equality of the core of the economy $\mathscr{E}^r$ and of the set of its competitive allocations for the case of two commodities and two types of agents, we follow his notation and, therefore, abandon in this section the more convenient symbols that we introduced in Section 2.

Initially every agent of the first type owns a positive quantity a of the first commodity and nothing of the second, every agent of the second type owns nothing of the first commodity and a positive quantity b of the second. For the economy $\mathscr{E}$ consisting of one agent of each type, an allocation is represented by the point of the plane R^2 with coordinates (x, y) where

x is the quantity of the first commodity assigned to the agent of the second type;

y is the quantity of the second commodity assigned to the agent of the first type.

Therefore, the initial allocation is $(0, 0)$, while an attainable allocation satisfies $0 \leqq x \leqq a$ and $0 \leqq y \leqq b$. The set of points of R^2 satisfying these inequalities is a rectangle E, the "Edgeworth box." Commodity-vectors assigned to the agents will be denoted by brackets. For the agent of the first type it is $[a-x, y]$, for the agent of the second type it is $[x, b-y]$.

Edgeworth assumes that the preferences of the agents of the first type (respectively, of the second type) are represented by a utility function F (respectively, Φ) with continuous first and second derivatives and such that

$$\frac{\partial F}{\partial x} < 0, \quad \frac{\partial F}{\partial y} > 0, \quad \frac{\partial^2 F}{\partial x^2} < 0, \quad \frac{\partial^2 F}{\partial x \partial y} < 0, \quad \frac{\partial^2 F}{\partial y^2} < 0;$$

$$\frac{\partial \Phi}{\partial x} > 0, \quad \frac{\partial \Phi}{\partial y} < 0, \quad \frac{\partial^2 \Phi}{\partial x^2} < 0, \quad \frac{\partial^2 \Phi}{\partial x \partial y} < 0, \quad \frac{\partial^2 \Phi}{\partial y^2} < 0.$$

In Fig. 2, the indifference curves I_1, I_2 through the origin for each type of agent are drawn.

The above conditions imply that the preferences of the agents satisfy assumption (3). To see this, consider an indifference curve for the agents of the first type determined by

$$F(x, y) = k, \tag{i}$$

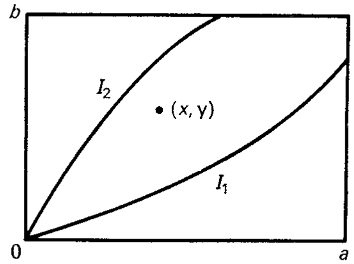

Fig. 2

where k belongs to the range of F. There is a function f such that (i) is equivalent to $y=f(x)$. By taking derivatives with respect to x in (i), one obtains

(ii) $$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\mathrm{d}f}{\mathrm{d}x}=0\,.$$

By performing the same operation in (ii), one obtains

(iii) $$\frac{\partial^2 F}{\partial x^2}+2\,\frac{\partial^2 F}{\partial x \partial y}\frac{\mathrm{d}f}{\mathrm{d}x}+\frac{\partial^2 F}{\partial y^2}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2+\frac{\partial F}{\partial y}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}=0\,.$$

It follows from (ii) that $\mathrm{d}f/\mathrm{d}x>0$. Therefore the first three terms in (iii) are negative. Since $\partial F/\partial y>0$, one has $\mathrm{d}^2 f/\mathrm{d}x^2>0$. The function f is strictly convex, which implies (3).

In a similar manner, one can prove that (3) is satisfied for the agents of the second type.

Edgeworth now introduces the economy $\mathscr{E}^r$ made up of r agents of each one of the two types. According to Lemma 2 the core of $\mathscr{E}^r$ can be represented by a subset C^r of the rectangle E. Consider an allocation $\alpha=(\xi,\eta)$ that belongs to C^r for every r. Let A_i be the set of allocations β in E that are at least as desired by an agent of the i-th type as α. In symbols,

$$A_i=\{\beta\in E \mid \beta \succsim_i \alpha\} \qquad i=1,2\,.$$

The central point of the proof (EDGEWORTH (1881), middle of page 38) consists of establishing that

(iv) the convex sets A_1, A_2 can be separated by a straight line L through 0 in Fig. 3 or in Fig. 4.

Since the allocation α is Pareto optimal for the economy $\mathscr{E}^1$ there is a straight line through α separating A_1 and A_2 by an argument too familiar to be repeated. Therefore, if α coincides with 0, assertion (iv) is established.

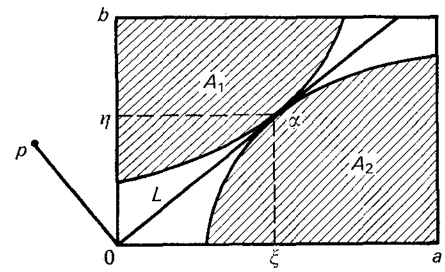

Fig. 3

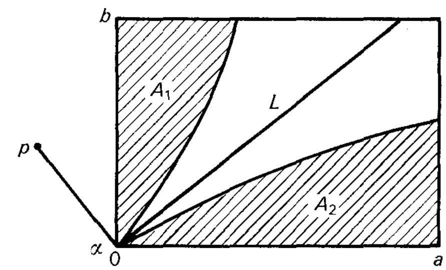

Fig. 4

We now prove that if α is distinct from 0, the straight line through 0 and α separates A_1 and A_2. Suppose that it is *not* supporting for one of them,

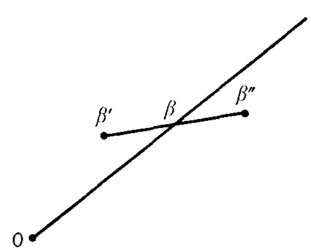

Fig. 5

say A_1. Then there are points β', β'' of A_1 on different sides of the line 0, α. One has $\beta' \succsim_1 \alpha$ and $\beta'' \succsim_1 \alpha$. Consequently, by assumption (3), the point β

where the segment β', β'' intersects 0, α satisfies $\beta \succ_1 \alpha$. Two cases have to be considered.

(*a*) β is between 0 and α.

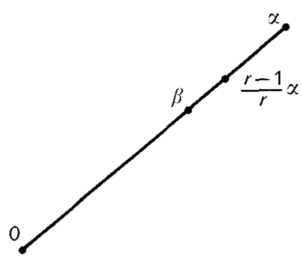

Fig. 6

For r large enough, the point $[(r-1)/r]\alpha$ is between β and α. Therefore, by assumption (3), $[(r-1)/r]\alpha \succ_1 \alpha$. Form the coalition S with r agents of the first type, $r-1$ agents of the second type and give to each agent of the first type the commodity-vector $\{a-[(r-1)/r]\,\xi, [(r-1)/r]\,\eta\}$, to each agent of the second type the commodity-vector $[\xi, b-\eta]$. The total commodity-vector assigned to coalition S is $[ra, (r-1)b]$ which is the total initial commodity-vector owned by coalition S. Moreover, every agent of the first type in S prefers $[(r-1)/r]\,\alpha$ to α, while every agent of the second type in S is at least as satisfied with α as with α. Consequently, coalition S blocks allocation α in $\mathscr{E}^r$, a contradiction of the hypothesis that α belongs to C^r for every r.

(*b*) β is not between 0 and α.

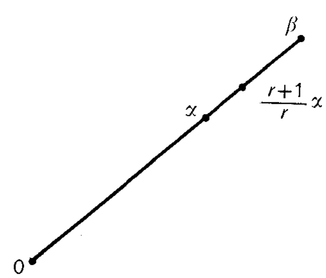

Fig. 7

In this case, repeat the reasoning of (*a*), replacing $[(r-1)/r]\,\alpha$ by $[(r+1)/r]\,\alpha$.

Thus the proof of assertion (iv) is complete.

We notice further that

(v) the straight line L is not supporting for E.

To establish this assertion, remark that the coalition made up of one agent of the first type does not block α. Therefore, $\alpha \succsim_1 0$. However, the utility $F(x, 0)$ is a decreasing function of x, hence, $0 \succ_1 (a, 0)$. Consequently, $\alpha \neq (a, 0)$. Similarly, $\alpha \neq (0, b)$.

Since F is decreasing in x and increasing in y, one has $(0, b) \succ_1 \alpha$. It follows from this relation and from the continuity of F that all the points of E in a sufficiently small neighborhood of $(0, b)$ are also preferred to α by an agent of the first type. As this neighborhood is contained in A_1, which is supported by L from below, the point $(0, b)$ is strictly above L. Similarly, $(a, 0)$ is strictly below L. Thus assertion (v) is established.

Choose now a price-vector p orthogonal to L as in Figs. 3 and 4. There remains to check that α and p form a competitive equilibrium. Specifically, we must prove that there is no point of E on or below L and preferred to α by the agents of the first type, and no point of E on or above L and preferred to α by the agents of the second type. Let us, therefore, suppose that β is a point of E on or below L for which $\beta \succ_1 \alpha$. Actually, β must be on L since β belongs to A_1, which is supported by L from below. Observe that, according to (v), there are points of E in the closed southeast quadrant with vertex β that are strictly below L. Select one of them γ close enough to β so that, by continuity of F, one has $\gamma \succ_1 \alpha$. Therefore, γ belongs to A_1 and is strictly below L, a contradiction.

In a similar manner, one shows that there can be no point β of E on or above L for which $\beta \succ_2 \alpha$, thereby completing the proof of Edgeworth's theorem.

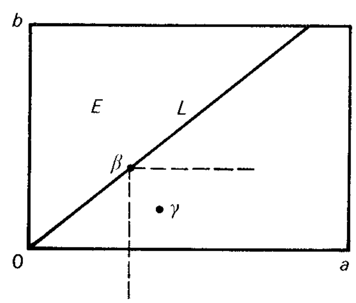

Fig. 8

4. A General Limit Theorem on the Core of an Economy

We now return to the notation of Section 2 and consider an economy $\mathscr{E}$ with l commodities and m agents. For every $i = 1, \ldots, m$, we make assumptions (1), (2), (3),

(4) the vector ω_i has all its components positive,

(5) for every x' in Ω, the set of x in Ω such that $x \precsim_i x'$ is closed.

(4) postulates that every agent initially owns a positive quantity of every commodity. (5) postulates that for any commodity-vector x' in Ω and any convergent sequence $\{x^q\}$ of commodity-vectors in Ω, if x^q is at most as desired by the i-th agent as x' for every q, then the limit of $\{x^q\}$ is at most as desired by the i-th agent as x'.

According to Lemma 2, the core of the economy $\mathscr{E}^r$ consisting of r replicas of the economy $\mathscr{E}$ can be represented by a subset C^r of R^{lm}. Let α be a competitive allocation of $\mathscr{E}$. The allocation α^r of $\mathscr{E}^r$ consisting of r replicas of α is competitive for $\mathscr{E}^r$. According to Theorem 1, α^r belongs to the core of $\mathscr{E}^r$. Therefore, α belongs to C^r. Denote by C^* the set of competitive allocations of $\mathscr{E}$. When r increases indefinitely, the C^r form a nested sequence and they all contain C^*.

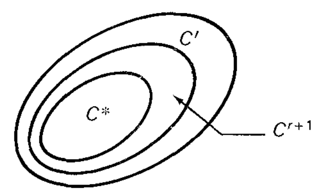

Fig. 9

The limit theorem of DEBREU and SCARF (1963), which we now state, asserts that under assumptions (1)–(5) the intersection of the sets $C^1, \ldots, C^r, \ldots$ representing the cores of the economies $\mathscr{E}^1, \ldots, \mathscr{E}^r, \ldots$ is the set C^* of the competitive allocations of the economy $\mathscr{E}$. In other words, C^* is the limit of C^r when r becomes indefinitely large.

THEOREM 2. *Every allocation of $\mathscr{E}$ that is in C^r for all r is competitive.*

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