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Chapter 21

NOTES ON THE CORE OF A PRODUCTIVE ECONOMY

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Preface

It gives me great pleasure to present the following paper on the core of an economy with non-convex production sets to my close friend and colleague on this occasion of his 65th birthday. The paper was written in 1963 and never published; the problem of incorporating increasing returns to scale in production in an analytical framework with the generality of the Walrasian model of equilibrium is one that I have been concerned with since that time.

In 1963, I was a Fellow at the Center for Advanced Study in the Behavioral Sciences at Stanford. Debreu and I had met several years previously and had already collaborated on our joint paper on the convergence of the core to the set of competitive equilibria. Our original treatment of the convergence theorem involved a model of exchange; as an afterthought we realized that the theorem was also valid when each coalition had access to the same convex cone as its production possibility set. In the classical case in which production exhibits constant returns to scale and convexity the two apparently distinct modes of analysis – one game theoretic and the other based on competitive assumptions – yielded virtually identical results.

In the presence of increasing returns to scale in production the competitive equilibrium will typically fail to exist. A firm engaged in maximizing profits will select ever higher scales of output if its choices influence neither the price of its output nor the unit cost of its factors of production. But a game theoretic analysis does not require the passive response to prices embodied in the competitive model. There are, in fact, very elementary economic models in

*I am extremely grateful to Donald Brown, Alvin Klevorick, Tatsuro Ichiishi and Martine Quinzii for conversations on the subject matter of this paper

which production involves increasing returns to scale, in which the core is non-empty and for which no competitive equilibria exist.

Consider an example consisting of two goods: a labor input l and a generalized output y. Labor is used to produce output according to the production function $y = f(l)$. Assume that the average productivity $f(l)/l$ is increasing in the labor input l, that each consumer's utility is a function of output alone, and that initial endowments consist solely of labor. Since leisure is not valued, the total initial endowment of labor is used to produce a level of output, which is then allocated among the various consumers. To be in the core such an allocation must have the property that higher levels of output cannot be provided to all of the members of any coalition, using their initial endowments alone. The allocation which gives each consumer an output level proportional to that consumer's initial labor supply will certainly be in the core if the proportionality constant is taken to be the average productivity at the point of total labor supply.

The validity of this example derives from the observation that large coalitions can exploit the economies of scale inherent in the production function $f(l)$ and therefore have proportionately more economic power than do smaller coalitions. The coalition of all consumers is, in this sense, stronger than any other coalition; since the core is not empty under constant returns to scale, it should certainly not be empty when increasing returns prevail.

When I first became aware of examples of this sort in the early 1960s, it seemed to me that this same intuition would apply whenever production exhibited increasing returns to scale – regardless of the generality of the model – and that a game theoretic analysis could be used to replace the unavailable competitive formulation. This is precisely the problem which is studied in the accompanying paper "Notes on the Core of a Productive Economy". The model of the paper involves an arbitrary number of commodities. Each consumer has preferences for non-negative commodity bundles and owns a vector of goods and potential services prior to production. The same production possibility set Y, satisfying several technical conditions, is assumed to be available to each coalition.

Perhaps the most significant conclusion of the paper is that the conjecture asserting the existence of a non-empty core, in the presence of increasing returns to scale, is false. If Y is not a convex cone, then it is possible to find a collection of consumers with conventional preferences and with specific vectors of initial endowments, for which the core is empty. In the counterexamples of the paper the preferences are actually described by the same utility function – which is concave and homogeneous of degree one – for each consumer; moreover the initial holdings for each consumer are strictly positive. The conclusion is a special case of Theorem 5, which appears at the end of Section

3; in spite of its special nature it requires the full force of the arguments of these three sections. The implication of the conclusion for the example given above with two goods is that there is an economic model using this production function for which the core is empty; in this model there will be initial holdings of output as well as labor, and the common utility function will value leisure as well as output.

An uncomfortable aspect of the models exhibiting an empty core when Y is not a convex cone is their reliance on preferences which give positive weight to all commodities. Clearly there are many commodities which are owned directly or indirectly, which are significant in production and which appear in no consumer's utility function – steel mills are an obvious example. The present paper distinguishes two types of commodities: consumer goods, which appear in consumers' utility functions, and producer goods or inputs into production, which do not. A new class of production possibility sets called *distributive sets* is then introduced. These sets, which generalize the ordinary concept of a convex cone, satisfy a number of minor conditions in addition to the following basic property: Let y be an arbitrary non-negative linear combination of the production plans $y^1, y^2, \ldots, y^n$, each of which is in Y. Then, for Y to be a distributive set, y must itself be in Y if it uses more of every producer good than does each of the component plans, y^j. Theorem 5 then states that if Y is not a distributive set, an economy may be found, with utility functions involving consumer goods only, for which the core is empty. Distributive sets are therefore a natural generalization of convex cones when certain commodities are excluded from consumer preferences.

In the special case of a single output, the production set Y is determined by a production function $y = f(x)$ which describes the maximum output obtainable from the vector of inputs x. Y will be a distributive set if and only if – again in addition to minor technical conditions – the production function has the following property: let $x = \sum a_j x^j$ with $a_j \geqq 0$ and let x be $\geqq x^j$ for each j. Then $f(x) \geq \sum a_j f(x^j)$. An equivalent formulation in terms of prices is that for every $x \geqq 0$, there exists a price vector $p \geqq 0$ such that $p \cdot x = f(x)$ and $p \cdot x' \geq f(x')$ for all $x' \leqq x$.

A positive converse to Theorem 5 is provided by Theorem 6, which states that the core will be non-empty if the common production set available to each coalition is a distributive set, and if a number of conventional technical assumptions hold. The argument is based on showing the existence of a particular price guided equilibrium – which I described by the fanciful term *social equilibrium* – and then arguing that a social equilibrium will be in the core. A social equilibrium is based on a price vector for all of the goods and services in the economy. Given the price vector, each consumer maximizes utility subject to his budget constraint, with income determined by the market evaluation of initial holdings. On the production side, a feasible production plan is selected which maximizes profit when compared to all other feasible production plans *which use fewer inputs of all producer goods*. The price vector, production and distribution plans form a social equilibrium if demand equals supply for all consumer and producer goods. The basic distinction between a social equilibrium and the usual concept of a competitive equilibrium is in the more limited comparison of profit making opportunities embodied in the newer concept.

Since 1963 a rather large number of works have appeared on a subject that has been termed "Coalition Production Economies": game theoretic models in which each coalition is assumed to possess its own production possibility set. Economies of scale are introduced into such models by allowing the production possibility set associated with the union of two coalitions to be distinctly larger than the sum of the associated sets themselves. This is in contrast to the treatment of the accompanying paper in which the productive possibilities open to a coalition are obtained by restricting the aggregate production possibility set to be consistent with the factor endowment of the coalition. I shall, however, call attention to the paper in this genre by Oddou (1976), which includes a discussion of distributive sets and social equilibria in models with a continuum of agents. In Oddou's model, there is an aggregate production possibility set, indexed by a vector of parameters; the production set associated with each coalition is obtained by specifying the collection of these vectors possessed by the coalition.

Sharkey (1979, 1982) discussed the core of an economy in which the common production set available to each coalition involves a single input, not appearing in any consumer's utility function. In Sharkey's model the productive side of the economy is summarized by the cost function $c(y)$, which represents the minimum amount of the input required to produce the vector of outputs y. Outputs are not initially owned by any consumer, prior to production. Sharkey then describes a series of properties of the cost function which are sufficient to imply that the corresponding game is "balanced" [Scarf (1967)] and therefore has a non-empty core.

The major assumptions made by Sharkey are that the cost function be quasi-concave and exhibit decreasing average cost along any ray. Since Sharkey's model is a special case of the more general problem discussed above, it is not surprising that these two conditions are sufficient for the associated production possibility set to be a distributive set. For a cost function $c(y)$ to lead to a distributive set the following property must be satisfied: Let $\sum a_j c(y^j) \geq c(y^j)$ for all j, with $a_j \geq 0$. Then $\sum a_j c(y^j) \geq c(\sum a_j y^j)$. An equivalent formulation is that every output vector y has associated with it a non-negative price vector p such that $c(y) = p \cdot y$ and $c(y') \geq p \cdot y'$ for all y' with $c(y') \leq c(y)$. [This should contrasted with the concept of a supportable cost function as used by Sharkey and Telser (1982); $c(y)$ is *supportable* if there exists a non-negative price vector p such that $c(y) = p \cdot y$ and $c(y') \geq p \cdot y'$ for all y' with $y' \leqq y$.]

Sharkey's model must have a social equilibrium as a particular outcome in the core: for such an equilibrium there will be prices for outputs and for the single input; at these prices the production plan will make a profit of zero and any other feasible production plan – using a smaller quantity of the input – will make a profit less than or equal to zero, when evaluated at these same prices. The concept of a social equilibrium is related to, though not identical with, various concepts arising in the theory of natural monopoly: *sustainable* equilibria [see, for example, Sharkey and Telser (1978), Sharkey (1982) or Baumol, Panzar and Willig (1982)] and *anonymously equitable* equilibria [Faulhaber and Levinson (1981) and Raa (1981)]. In a social equilibrium the alternative production plans which are tested for profitability are those with a smaller cost of production: in asking whether a particular plan is anonymously equitable, however, the alternative production plans are those which produce smaller quantities of all outputs.

How compelling are my counter-examples to the existence of a non-empty core when the production set is not a distributive set? Do they rest on an excessive requirement of generality which, when relaxed, permits us to exhibit an outcome in the core? For example, I was able to verify, several years after the accompanying paper was written, that the model with a single output and a single labor input does indeed have a non-empty core, with quite general utility functions for output and leisure, as long as the initial endowments consist of labor alone [Scarf (1973)]. My argument was based on showing that the corresponding n-person game was balanced; an alternative proof was subsequently provided by Mas-Colell (1980). Of course, if there are increasing returns to scale and leisure is valued by consumers, the production set is not a distributive set; nevertheless, counterexamples do not appear if the distribution of initial endowments is restricted in a suitable way.

A substantial generalization of this result was obtained by Quinzii (1982) who considered the case of a single output, not initially owned by any consumer, and an arbitrary number of inputs. Production is described by a function which is quasi-concave and exhibits increasing returns to scale; consumers have preferences for retained inputs as well as for the single output. Quinzii's assumptions on the production function are sufficient to verify that the corresponding production set is a distributive set – if there were no preferences for retained inputs. But such preferences are assumed in her model and an example with an empty core can be constructed as long as the number of inputs is greater than one [Scarf (1963)]. What Quinzii has shown is that the core will be non-empty if a particular relationship holds between a demand elasticity based on the individual utility functions and an elasticity derived from the production function. The relationship will be satisfied, for example, if consumer preferences are described by a Cobb–Douglas utility function and if the production function also has a Cobb–Douglas form. Ichiishi and Quinzii (1983) show that the same relationship is sufficient for the existence of a non-empty core in a more general model in which production involves a number of outputs as well as inputs. Their argument proceeds by exhibiting a particular price guided equilibrium, different from a social equilibrium, which may also be shown to be in the core.

To my way of thinking, the major difficulty with the game theoretic approach to the study of increasing returns to scale is that one is forced to work with production sets which do not permit the analysis of the most basic act of economic choice: the possibility of substituting between alternative production plans. Unfortunately distributive sets are not closed under addition – if Y and Y' are both distributive sets it is not necessarily correct that $Y + Y'$ is a distributive set. An economy, composed of a number of firms, each of whose technologies is described by a distributive set, may very well have an empty core if the technologies are used simultaneously.

Of course, one may forgo any attempt to explain decentralized economic activity and be content with an aggregate production set for which the core is available. But there is, in fact, a much more serious difficulty arising from this lack of substitutability. If the model is to extend over several periods of time some of the outputs at the end of a particular period will be used as inputs at the beginning of the next period. For investment activities to be captured by the analysis we must be capable of adding production possibility sets over time. Since this is not possible, no explanation is forthcoming for the stock of productive assets at the beginning of any particular time period. Inputs are merely parameters which index the production possibility sets available to each coalition at one instant of time, and this is not a rich, technologically based theory of economies with increasing returns to scale.

In publishing this paper so many years after its writing, I am offering a public argument for my reluctantly acquired feelings that a replacement for the Walrasian model, incorporating economies of large scale production, cannot be based on the concepts of cooperative game theory. In order to obtain such a theory we must have recourse either to considerations of imperfect competition and non-cooperative game theory [Hart (1982)], to non-strategic equilibrium concepts [Brown and Heal (1983, 1985)], or to the study of indivisibilities in production [Scarf (1981a, 1981b, 1984)].

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1. Introduction

In the last few years, several papers have described the close connection between the core of an economy and its competitive equilibria. In the present set of notes, I shall examine an extensive class of economies, in which the production possibility set is a member of a new class of sets, to which I have given the name "distributive" sets. These sets will generally display increasing returns to scale so that a competitive equilibrium cannot be expected. Economies in which production is described by a distributive set do, however, have a type of price guided equilibrium which differs from the competitive equilibrium in that profit maximization is over a range of production plans restricted by the availability of certain inputs. As we shall see, this modified type of equilibrium is in the core. This will provide us with a class of economies for which there is an analysis of economic allocations based on the core. We shall also see that such an analysis is not possible if the production possibility set is not a distributive set.

I will begin by reviewing the standard notation and definitions. The number of commodities will be n so that a typical commodity vector $x = (x_1, \ldots, x_n)$ will be represented by a point in n-dimensional space. There will be m consumers each of whom has preferences for non-negative commodity bundles. The preference preordering of the ith consumer will be denoted by $\succsim_i$, and throughout these notes I will make the following standard assumptions:

A.1. Monotonicity: If $x' \geqq x$, then $x' \succsim_i x$.
A.2. Convexity: If $x' \succ_i x$ and $0 < \alpha < 1$, then $\alpha x' + (1-\alpha)x \succ_i x$.
A.3. Continuity: For any given x the sets $\{x' | x' \succsim_i x\}$ and $\{x' | x \succsim_i x')$ are both closed.
A.4. Insatiability: For any x, there is an $x' \succ_i x$.

All of the commodities in the economy will be assumed to be privately owned, with ω^i representing the vector of commodities initially owned by the ith consumer.

In the study of the core it is necessary to describe the productive knowledge which is available to any specific group of participants in the economy. In these notes I shall take a simple, but perhaps unrealistic approach; I shall assume the existence of a production possibility set Y available to all groups. Y will be a set in n-dimensional space consisting of all possible production plans; the negative entries in a particular plan are the inputs into production and the positive entries are the outputs. The following general assumptions will be made throughout these notes:

B.1. Y is closed.
B.2. The vector all of whose components are zero, is in Y.
B.3. If $y \in Y$ and $y' \leqq y$, then $y' \in Y$.

A typical allocation will be described by the m commodity bundles $x^1, \ldots, x^m$ received by the various consumers. For the allocation to be feasible we must have

$$\sum_{i=1}^{m} x^i = \sum_{i=1}^{m} \omega^i + y \quad \text{with} \quad y \in Y.$$

In order to define the core of the economy it is necessary to consider the possibility of an allocation being blocked by a set of consumers S. The set S will block the allocation described above if it is possible to find commodity bundles $\tilde{x}_i$ (for $i \in S$), with

$$\sum_{i \in S} \tilde{x}^i = \sum_{i \in S} \omega^i + \tilde{y} \quad \text{for some} \quad \tilde{y} \in Y,$$

and such that $\tilde{x}^i \succsim_i x^i$ for all i in S, with strict preference for at least one member of S. The core of the economy will consist of the set of all feasible allocations which are not blocked by any set of consumers.

My primary purpose will be to investigate those conditions on Y which imply the existence of a non-empty core. Debreu and Scarf have discussed this problem in the case in which Y is a convex cone. In this case there will be a competitive equilibrium if, in addition to the assumptions made above about preferences and production, it is also assumed that $\omega^i > 0$, and that 0 is the only vector in Y with all components greater than or equal to zero. It is a simple matter to verify that a competitive equilibrium is in the core. The question of the existence of the core may therefore be answered in the affirmative in this case.

We shall see that if Y is not a convex cone, then it is possible to make a choice of m, the number of consumers, a choice of the preferences, subject to the regularity conditions given above, and a selection of strictly positive initial holdings ω^i, so that the resulting economy has an empty core. This seems to suggest that subject to regularity conditions, convexity and constant returns to scale in production are necessary in order to construct a theory of economic allocations based on the concept of the core.

This result is somewhat unrealistic if it is examined closely. As we shall see the types of preferences which are required for the construction of an economy with an empty core, are preferences in which all of the commodities in the economy enter. Clearly there are many commodities which are used in production, and which do not directly enter in the preferences of any consumer. If we restrict our attention to preferences which involve a subset of the commodities in the economy – the consumer commodities – then there will be a non-empty core for many production possibility sets other than convex cones.

Let me therefore consider the commodities in the economy as being divided into two categories. The first category, consisting of commodities $1, 2, \ldots, k$, will refer to the commodities which do not enter into the preferences of consumers. The remaining commodities, will be in the second category. It will be convenient to describe the commodities in the first category as producer commodities, and those in the second as consumer commodities. The reader should take note of the fact that consumer commodities may enter as factors of production; labor, for example, will be designated as a consumer commodity if leisure is involved in the preferences of consumers.

In the light of these remarks, I will make the following additional assumption on preferences.

A.5. All consumers will be indifferent to the two commodity bundles x and x' if $x_j = x'_j$ for $j = k+1, \ldots, n$.

Producer commodities may enter into production either as inputs or as outputs. The *net* result of production is to use all of the initially owned producer commodities as inputs. This remark will, of course, be correct even in a dynamic model in which new producer commodities emerge as the result of production, or in which current producer commodities are transformed into new producer commodities by the act of being used in production. I will, therefore, restrict my attention to those production plans in which the producer commodities appear as inputs into production.

B.4. If $y \in Y$, then $y_i \leq 0$ for $i = 1, 2, \ldots, k$.

2. *Distributive sets*

Now I shall turn to the definition of a class of sets which will be of considerable use in analyzing the existence of outcome in the core.

It will be useful first of all to introduce the cone $\Omega = \{(y_1, \ldots, y_n) |$ all $y_i \geq 0\}$ and a cone Λ in which only the first k components are restricted to be non-negative, $\Lambda = \{(y_1, \ldots, y_n) | y_1 \geq 0, \ldots, y_k \geq 0\}$. This notation is helpful in describing properties of the production possibility sets; for example, B.4 is equivalent to $Y \subset -\Lambda$.

Definition. Let Y be a set in n-dimensional space with $-\Omega \subset Y \subset -\Lambda$. We say that Y is a distributive set if for any finite number of points $y^i \in Y$, and any non-negative α_i, the point $y = \sum \alpha_i y^i$ is also in Y, *if y satisfies the* *conditions* $y^{\prime} - y \in \Lambda$. In other words, a non-negative weighted sum will be in Y, if it uses more of the producer commodities than any of the original plans.

The reader will notice that properties B.2, B.4 and a version of free disposal have been incorporated directly in the definition of a distributive set. This has been done deliberately since distributive sets without these properties have a vastly different character.

The definition is rather complex and I will spend the next several pages discussing properties of distributive sets and giving some examples. First of all, it should be noticed that if all of the commodities in the economy are consumer commodities, then Λ is n-space itself and a distributive set is a convex cone with vertex at the origin.

To see that distributive sets need not be cones, consider the following example, which involves two commodities, the first of which is a producer commodity and the second a consumer commodity. Let f be a non-negative, continuous, increasing function of a single variable, defined for all non-negative values of its argument. Consider the set

$$Y = \{(y_1, y_2) | y_1 \leq 0;\ y_2 \leq f(-y_1)\},$$

which corresponds to the case in which the consumer good is produced from the producer good according to the production function f.

On the basis of Figure 1 the reader should have no difficulty in verifying that Y is a distributive set, if and only if f has non-decreasing returns to scale, i.e., $f(\lambda u) \geq \lambda f(u)$ for $\lambda \geq 1$.

In the general case of several commodities in each category distributive sets exhibit non-decreasing returns to scale. If $y \in Y$ and $\lambda \geq 1$, then the first k components of $y - \lambda y$ will be non-negative, and therefore λy will be in Y.

Even though distributive sets are generally not convex, the collection of points in a distributive set which have the first k coordinates fixed, does form a convex set. To see this let

$$Y_a = \{y \in Y | y_j = a_j \text{ for } j = 1, 2, \ldots, k\}.$$

Let y^1 and y^2 be in Y_a. Then if $y = \alpha y^1 + (1 - \alpha) y^2$, we have y equal to y^i in the first k coordinates and therefore $y^i - y \in \Lambda$. Using the definition of a distributive set this tells us that $y \in Y_a$.

It should be easy for the reader to verify, using the same argument, that if all of the inputs of producer commodities are set at zero, then the resulting set of production plans will actually be a cone.

It may be verified directly from the definition that distributive sets are

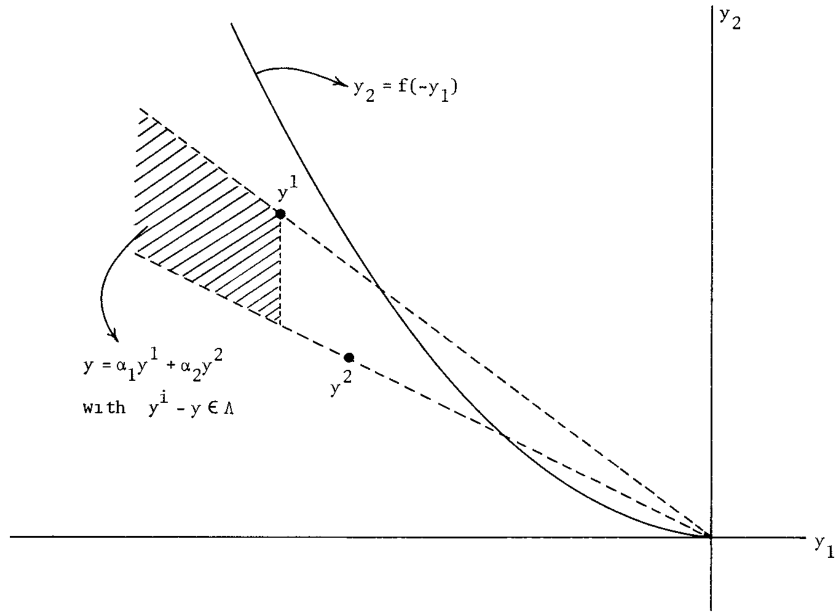

Figure 1

super-additive. If y^1 and y^2 are both in Y, then $y^1 + y^2$ is in Y. This result, in conjunction with $-\Omega \subset Y$ implies the free disposal property B.3.

In order to gain some additional insight into the nature of distributive sets, I shall turn to a version of the separating hyperplane theorem which is valid for sets of this sort.

Let ξ be a point which is not in Y. I will show that there is a non-zero price vector π such that $\pi \cdot \xi \geq 0$ and $\pi \cdot y \leq 0$ for all y in Y with $y_j \geq \xi_j$ for $j = 1, 2, \ldots, k$. This will yield a hyperplane which separates ξ from that part of the set Y using fewer productive commodities as inputs than does ξ.

If there are no points $y \in Y$ with $y - \xi \in \Lambda$, then of course any hyperplane will do. If there are such points then define

$$T = \left\{ \sum \alpha_i y^i \middle| \alpha_i \geq 0,\ y^i \in Y \text{ and } y^i - \xi \in \Lambda \right\}.$$

T is the smallest convex cone with vertex 0, containing the points $Y \cap [\Lambda + \xi]$. ξ cannot be in T, since this would imply $\xi \in Y$. We may therefore separate ξ

from T by a hyperplane. In other words there exists a non-zero vector π with $\pi \cdot \xi \geq 0$ and $\pi \cdot y \leq 0$ for $y \in Y \cap [\Lambda + \xi]$.

It will be useful to describe a stronger version of this result which is valid when Y is a closed set.

Theorem 1. Let Y be a closed distributive set and let ξ be a point which is not in Y. Then there is a non-negative price vector π such that $\pi \cdot \xi > 0$ and $\pi \cdot y \leq 0$ for all $y \in Y \cap [\Lambda + \xi]$. Moreover if any of the first k components of ξ are zero, then the associated price may be selected as zero.

The theorem is, of course, trivial if any of the first k components of ξ are strictly positive; it is therefore sufficient to consider the case $\xi_j \leq 0$ for $j = 1, 2, \ldots, k$.

We shall demonstrate the theorem first in the case where $\xi_j < 0$ for $j = 1, 2, \ldots, k$. Consider the cone T defined above. It is easy to see that this cone has the free disposal property, since it is additive and contains $(-\delta_1, \ldots, -\delta_n)$ for all sufficiently small non-negative δ's. This implies that T^*, the closure of T, does not contain ξ. For let $\xi = \lim_{v \to \infty} \xi^v$ with $\xi^v \in T$. Using the free disposal property of T we see that there is no loss in generality in assuming that $\xi^v \leqq \xi$, since if this were not correct we could lower the sequence of ξ^v's by small amounts and still retain the convergence. But we may write $\xi^v = \sum \alpha_i y^i$ with $y^i \in Y$ and $y^i - \xi \in \Lambda$, and this implies that $y^i - \xi^v \in \Lambda$. Using the definition of a distributive set we see that $\xi^v \in Y$, and since Y is closed this would imply $\xi \in Y$ contrary to our assumption.

We may, therefore, find a non-zero price vector π with $\pi \cdot \xi > 0$ and $\pi \cdot y \leq 0$ for all y in Y with $y - \xi \in \Lambda$. Since $y = (-\delta_1, \ldots, -\delta_n) \in Y$ and satisfies $y - \xi \in \Lambda$ for all sufficiently small non-negative δ, we see that $\pi \geq 0$.

On the other hand, if some of the producer commodities in ξ are at zero level, then we consider the following argument:

Assume that the notation has been selected so that $\xi_1 = \cdots = \xi_l = 0$ and that $\xi_{l+1} < 0, \ldots, \xi_k < 0$. Consider the set in $(n - l)$-dimensional space $\tilde{Y} = \{(y_{l+1}, \ldots, y_n) | (0, \ldots, 0, y_{l+1}, \ldots, y_n) \in Y\}$. It is a trivial matter to verify that $\tilde{Y}$ is a distributive set with commodities $l + 1, \ldots, k$ being the producer commodities, and that $(\xi_{l+1}, \ldots, \xi_n)$ is not in this set. The previous argument, applied to this set, gives a system of prices $\pi_{l+1}, \ldots, \pi_n \geq 0$ and not all zero, such that

$$\sum_{l+1}^{n} \pi_j \xi_j > 0 \quad \text{and} \quad \sum_{l+1}^{n} \pi_j y_j \leq 0,$$

for all $(y_{l+1}, \ldots, y_n) \in \tilde{Y}$ with $y_j \geq \xi_j$ for $j = l + 1, \ldots, k$. The price system

$(0,\ldots,0,\pi_{l+1},\ldots,\pi_n)$ may be applied in Y, and this demonstrates the theorem.

This theorem has an immediate application to efficient production plans. For the purposes of these notes I shall adopt a slightly different definition of efficiency than that customarily given. $y^* \in Y$ will be said to be efficient, if there is no $y \in Y$ with $y_j \geq y_j^*$ and at least one $y_j > y_j^*$ for $j = k+1,\ldots,n$. By a minor modification of the previous argument we see that an efficient production plan y^* has associated with it a non-negative price vector π, with $\pi \cdot y^* = 0$, and such that y^* maximizes profit when compared with all other plans in Y using smaller quantities of the producer commodities as inputs. Moreover if $y_j^* = 0$ for some producer commodity, then $\pi_j = 0$.

The reader may perhaps be troubled by this combination of profit maximization, zero profit and increasing returns to scale. Profit maximization, however, is over a restricted range, so that the prices associated with the producer commodities are not "marginal". In the example considered above, in which a single input is transformed into a single output by means of a production function f, the price system associated with an efficient point $(-u, f(u))$ will be $(f(u)/u, 1)$ and will not involve the marginal product.

A very useful class of functions is obtained if we generalize this example to consider production functions which transform k producer commodities into a single consumer commodity.

Definition. Let $f(u_1,\ldots,u_k)$ be defined for all non-negative values of its argument. We say that f is a distributive function if the set

$$Y = \{(y_1 \ldots y_k, y_{k+1}) | y_{k+1} \leq f(-y_1,\ldots,-y_k) \text{ and } y_i \leq 0 \text{ for } i = 1,\ldots,k\}$$

is a closed distributive set, with commodities $1,\ldots,k$ regarded as producer commodities.

The properties of distributive sets that have been described above may be carried over directly to obtain corresponding properties of distributive functions. For example, the fact that Y is closed becomes

B.1′. $f(\lim_{l\to\infty} u^l) \geq \overline{\lim}_{l\to\infty} f(u^l)$, which is of some technical use.

The fact that $0 \in Y$ becomes

B.2′. $f(0) \geq 0$,

and free disposal implies

B.3′. $f(u) \geq f(u')$ if $u \geqq u'$,

so that distributive functions are non-negative, monotonic non-decreasing, and upper semi-continuous.

These properties are, of course, relatively minor; the important property of distributive functions is obtained by translating Theorem 1. Let $u = (u_1, \ldots, u_k)$ be an arbitrary non-negative vector different from zero, so that $(-u_1, \ldots, -u_k, f(u))$ is an efficient point of the set Y. Theorem 1 then tells us that there is a system of non-negative prices $(\pi_1, \ldots, \pi_k, \pi_{k+1})$ not all zero, with

$$-\sum_1^k \pi_j u_j + \pi_{k+1} f(u) = 0 \quad \text{and} \quad -\sum_1^k \pi_j u_j' + \pi_{k+1} f(u') \leq 0,$$

for $0 \leqq u' \leqq u$. If $u_j = 0$, then π_j is also equal to zero. π_{k+1} must be different from zero, for otherwise we would have

$$\sum_1^k \pi_j u_j = 0,$$

with the $\pi_j \geq 0$ and with at least one strictly positive π_j associated with a strictly positive u_j. We may therefore normalize the prices and assume that $\pi_{k+1} = 1$.

I shall summarize this important property of distributive functions as follows.

Theorem 2. Let $u = (u_1, \ldots, u_k) \geq 0$. *Then there is a non-negative vector* $\pi = (\pi_1, \ldots, \pi_k)$ *different from zero, such that* $\pi \cdot u = f(u)$ *and* $\pi \cdot u' \geq f(u')$ *for* $0 \leqq u' \leqq u$. *Moreover if* $u_j = 0$, *then* π_j *may be selected as zero.*

This result should be compared with the theorem on the existence of a tangent plane for a function which is concave and homogeneous of degree one. For such functions the tangent plane at a point u lies above the function everywhere. For a distributive function, on the other hand, the analogue of the tangent plane is required to be above the function only for $0 \leqq u' \leqq u$.

The properties B.1′, B.2′, B.3′ and the property demonstrated in Theorem 2 actually characterize the class of distributive functions.

Theorem 3. *Let $f(u)$ be defined for all $u = (u_1, \ldots, u_k) \geqq 0$, and satisfy B.1′, B.2′, B.3′. If for every u, there exists a $\pi \geq 0$ with $\pi \cdot u = f(u)$ and $\pi \cdot u' \geq f(u')$ for $0 \leqq u' \leqq u$, then f is a distributive function.*

To demonstrate this theorem we must show that

$$Y = \{(y_1, \ldots, y_{k+1}) | y_{k+1} \leq f(-y_1, \ldots, -y_k) \text{ and } y_j \leq 0 \text{ for } j = 1, \ldots, k\}$$

is a distributive set. The first three conditions imply B.1, B.2, B.3. Let $y^i = (y_1^i, \ldots, y_k^i, y_{k+1}^i)$ be in Y so that $y_{k+1}^i \leq f(-y_1^i, \ldots, -y_k^i)$. Let $y = \sum \alpha_i y^i$ and let $y_j^i \geq y_j$ for $j = 1, \ldots, k$ and all i. We wish to show that $y \in Y$, or in other words that $y_{k+1} \leq f(-y_1, \ldots, -y_k)$. Let π be a price associated with the point $(-y_1, \ldots, -y_k)$. Then since $(-y_1^i, \ldots, -y_k^i) \leqq (-y_1, \ldots, -y_k)$, we see that

$$y_{k+1}^i \leq f(-y_1^i, \ldots, -y_k^i) \leq -\sum \pi_j y_j^i.$$

Therefore

$$y_{k+1} = \sum \alpha_i y_{k+1}^i \leq -\sum \pi_j \sum \alpha_i y_j^i = -\sum \pi_j y_j = f(-y_1, \ldots, -y_k).$$

Y is therefore a distributive set and f a distributive function.

Some examples of distributive functions may be useful.

$$f(u) = \prod_{j=1}^{k} u_j^{\alpha_j}$$

is a distributive function if

$$\sum_1^k \alpha_j \geq 1.$$

Any vector $\pi \geq 0$ with $\pi_j \leq (\alpha_j / u_j) f(u)$ and $\pi \cdot u = f(u)$ will serve as a system of prices at the point u. This example is useful in indicating that generally there is more than one system of prices appropriate at a given point. If $\sum \alpha_j > 1$, this production function exhibits increasing returns to scale; all distributive functions have non-decreasing returns to scale in the sense that $f(\lambda u) \geq \lambda f(u)$ for $\lambda \geq 1$.

The isoquants associated with this production function are convex. This need not be true for distributive functions, as the example

$$f(u) = \sum_{1}^{k} u_j^2$$

indicates. For this function $\pi = (u_1, \ldots, u_k)$ serves as a system of prices at the point u. Distributive functions have many additional properties that I shall develop in subsequent notes. For example the sum, product and minimum of distributive functions are also distributive. Distributive functions are also quite useful in constructing distributive sets with more than one consumer commodity.

3. Distributive sets and the core

In the present section, I will begin the study of the relationship between distributive sets and the core.

In the use of distributive sets as production possibility sets, one additional assumption will be made throughout the remainder of these notes.

B.5. Let $\omega = (\omega_1, \ldots, \omega_n) \geq 0$. Then the set of points $y \in Y$ with $y \geqq -\omega$ is a bounded set.

This assumption is a very natural one and says merely that if we begin with a fixed supply of initial commodities and transform these commodities according to the production possibility set Y, then the resulting commodity bundles available for consumption form a bounded set.

We shall begin with the study of a simple type of economy in which there are k producer commodities and a single consumer commodity. There are m consumers in the economy, with the initial holdings of the ith consumer designated by ω^i. Since there is a single consumer commodity the preferences of a typical consumer are described by his preference for a larger quantity of this commodity over a smaller quantity.

Let the production possibility set be Y, which is assumed to have properties B.1–B.5. For the moment it is not assumed that Y is a distributive set. The question is what additional conditions have to be placed on Y so as to guarantee the existence of a non-empty core for this economy.

Define

$$f(u_1, \ldots, u_k) = \max \{ y_{k+1} | (-u_1, \ldots, -u_k, y_{k+1}) \in Y \}.$$

The maximum exists and is finite because of B.1 and B.5, and f satisfies B.1′, B.2′, B.3′. If Y were a distributive set then f would of course be a distributive function.

Now let $x^1, \ldots, x^m$ be an assignment of non-negative commodity bundles representing a point in the core for this economy. We have

$$\sum_{i=1}^{m} x^i - \sum_{i=1}^{m} \omega^i = y \in Y,$$

if the allocation is to be feasible. Using free disposal and the fact that $x^i \geqq 0$, we see that

$$\left(-\sum \omega_1^i, \ldots, -\sum \omega_k^i, \sum \left(x_{k+1}^i - \omega_{k+1}^i\right)\right) \in Y,$$

and therefore

$$\sum_{i=1}^{m} \left(x_{k+1}^i - \omega_{k+1}^i\right) \leq f\left(\sum_1^m \omega_1^i, \ldots, \sum_1^m \omega_k^i\right).$$

I claim, however, that for any set S, including the set of all consumers, we must have

$$\sum_S \left(x_{k+1}^i - \omega_{k+1}^i\right) \geq f\left(\sum_S \omega_1^i, \ldots, \sum_S \omega_k^i\right),$$

for otherwise the set S would block the allocation $x^1, \ldots, x^m$. To see this let S be a set with

$$\sum_S \left(x_{k+1}^i - \omega_{k+1}^i\right) < f\left(\sum_S \omega_1^i, \ldots, \sum_S \omega_k^i\right).$$

Then using the definition of f, we see that there is a vector

$$y = \left(-\sum_S \omega_1^i, \ldots, -\sum_S \omega_k^i, y_{k+1}\right) \in Y,$$

with

$$y_{k+1} > \sum_S \left(x_{k+1}^i - \omega_{k+1}^i\right) \quad \text{or} \quad y_{k+1} + \sum_S \omega_{k+1}^i > \sum_S x_{k+1}^i.$$

The vector

$$y + \sum_S \omega^i = \left(0, \ldots, 0, y_{k+1} + \sum_S \omega^i_{k+1}\right)$$

may therefore be distributed among the members of S in such a way as to block the original allocation.

Define $u^i = (\omega^i_1, \ldots, \omega^i_k)$ and $\alpha^i = (x^i_{k+1} - \omega^i_{k+1})$. In order for there to be a non-empty core for our economy we must therefore be able to find numbers $\alpha^1, \ldots, \alpha^m$ with

$$\sum_1^m \alpha^i = f\left(\sum_1^m u^i\right) \quad \text{and} \quad \sum_S \alpha^i \geq f\left(\sum_S u^i\right) \quad \text{for all sets } S.$$

If we are to have a production possibility set which guarantees the existence of a non-empty core for all distributions of initial holdings, and any number of consumers in the economy, then there must be such a collection of α's for every collection of non-negative vectors $\{u^i\}$.

If Y is a distributive set and f, therefore, a distributive function, then such a point in the core may always be found. Let π be a non-negative price vector associated with the point

$$u = \sum_1^m u^i,$$

and define $\alpha^i = \pi \cdot u^i$. Then

$$\sum_1^m \alpha^i = \sum_1^m \pi \cdot u^i = \pi \cdot u = f(u).$$

If S is a subset of the set of consumers, then

$$u' = \sum_S u^i \leqq u,$$

and we have

$$\sum_S \alpha^i = \sum_S \pi \cdot u^i = \pi \cdot u' \geq f(u').$$

As the following theorem indicates, the converse of this result is also correct, in the sense that f must be a distributive function if there is always to be a non-empty core.

Theorem 4. Let $f(u)$ be defined for all non-negative u, and satisfy B.1′, B.2′, B.3′. Assume that for every set of strictly positive vectors $u^1, \ldots, u^m$, there is a set of numbers $\alpha^1, \ldots, \alpha^m$ with

$$\sum_1^m \alpha^i = f\left(\sum_1^m u^i\right) \quad \text{and} \quad \sum_S \alpha^i \geq f\left(\sum_S u^i\right).$$

Then f is a distributive function.

Let u be an arbitrary positive vector in k-space, and let $\{u^i\}$ be a set of positive vectors with

$$\sum_{i=1}^m u^i = u.$$

Let q be an arbitrary integer (eventually tending to $+\infty$) and consider the economy with $m \cdot 2^q$ participants, indexed by the pair (i, j) with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, 2^q$. The consumer with the label (i, j) will have an initial holding $u^i/2^q$, so that the total initial holdings add up to u.

According to the assumptions of the theorem there will be a set of numbers α^{ij} with

$$\sum_i \sum_j \alpha^{ij} = f(u) \quad \text{and} \quad \sum_S \alpha^{ij} \geq f\left(\sum_S (u^i/2^q)\right).$$

It will be convenient to regard the consumers as being indexed in such a way that $\alpha^{i1} \leq \alpha^{i2} \leq \cdots \leq \alpha^{i2^q}$.

Let us define

$$\delta^i = \sum_{j=1}^{2^q} \alpha^{ij}.$$

Then

$$\sum_{i=1}^m \delta^i = f(u).$$

We also have

$$\delta^i \geq 2^q \sum_{j=1}^{k_i} \left(\alpha^{ij}/k_i\right),$$

for any integer $k_i \leq 2^q$. If we then take for S the set of consumers labeled (i, j) with $j = 1, 2, \ldots, k_i$, we see that

$$\sum_{i=1}^{m} \frac{k_i}{2^q}\delta^i \geq \sum_{i=1}^{m} \sum_{j=1}^{k_i} \alpha^{ij} \geq f\left(\frac{k_1 u^1}{2^q} + \cdots + \frac{k_m u^m}{2^q}\right).$$

For each q, the set of δ's satisfying the conditions

$$\sum_{i=1}^{m} \delta^i = f\left(\sum_{i=1}^{m} u^i\right) \quad \text{and} \quad \sum_{i=1}^{m} \frac{k_i}{2^q}\delta^i \geq f\left(\sum_{i=1}^{m} \frac{k_i}{2^q} u^i\right),$$

$$\text{for all integral} \quad k^i \leq 2^q,$$

is therefore a non-empty closed set. The set of such δ's for $q + 1$ is obviously contained in the set for q. This implies that there is at least one δ which satisfies the above inequalities for all q.

We wish to show, that for this δ

$$\sum_1^m \theta_i \delta^i \geq f\left(\sum_1^m \theta_i u^i\right) \quad \text{for} \quad 0 \leq \theta_i \leq 1.$$

We do this by letting k_i be the smallest integer greater than or equal to $\theta_i \cdot 2^q$, and then passing to the limit.

We are now ready to determine the system of prices associated with an arbitrary point ξ. Let $\xi = (\xi_1, \ldots, \xi_l, 0, \ldots, 0)$ with $\xi_i > 0$ for $i = 1, \ldots, l$. Let ε be a small positive number and define

$$u^1 = (\xi_1 + \varepsilon, \varepsilon, \ldots, \varepsilon, \varepsilon, \ldots, \varepsilon),$$

$$\vdots$$

$$u^l = (\varepsilon, \varepsilon, \ldots, \xi_l + \varepsilon, \varepsilon, \ldots, \varepsilon).$$

For each ε there will be a set of δ's which satisfy the relations

$$\sum_{i=1}^{l} \delta^i = f(\xi_1 + l\varepsilon, \xi_2 + l\varepsilon, \ldots, \xi_l + l\varepsilon, l\varepsilon, \ldots, l\varepsilon),$$

and

$$\sum_{i=1}^{l} \theta_i \delta^i \geq f(\theta_1 \xi_1, \theta_2 \xi_2, \dots, \theta_l \xi_l, 0, \dots, 0),$$

for all $0 \leq \theta_i \leq 1$. As ε tends to zero we may select a limit point of these δ's, and using the fact that $f(\lim) \geq \overline{\lim} f$, we obtain

$$\sum_{i=1}^{l} \delta^i = f(\xi_1, \dots, \xi_l, 0, \dots, 0),$$

and

$$\sum_{i=1}^{l} \theta_i \delta^i \geq f(\theta_1 \xi_1, \dots, \theta_l \xi_l, 0, \dots, 0).$$

The vector

$$\pi = \left(\delta^1/\xi_1, \dots, \delta^l/\xi_l, 0, \dots, 0\right)$$

will therefore serve as a system of prices for f at ξ, and this demonstrates the theorem.

The results thus far obtained may be summarized as follows: Let Y be a production possibility set satisfying the regularity conditions B.1–B.5. Assume that there is a single consumer commodity. Then a necessary and sufficient condition that there be a non-empty core for an arbitrary number of consumers and an arbitrary, strictly positive distribution of initial holdings, is that Y be a distributive set. Our purpose is now to generalize this result to an arbitrary number of consumer commodities.

Let us return, therefore, to the general case in which there are n commodities, the first k of which are producer commodities. Let Y be the production possibility set, assumed to satisfy conditions B.1–B.5. I wish to show that if Y is not a distributive set, then there is a collection of consumers, with preferences satisfying A.1–A.5, and some strictly positive distribution of initial holdings, such that the resulting economy has an empty core.

In the construction of these examples it will be sufficient to assume that all consumers have identical preferences given by a utility function $g(x_{k+1}, \dots, x_n)$ defined for all non-negative values of its argument. The preferences are then described by saying that $(x_1, \dots, x_n) \succsim (x'_1, \dots, x'_n)$ if and only if $g(x_{k+1}, \dots, x_n) \geq g(x'_{k+1}, \dots, x'_n)$. The reader will have no difficulty in verifying that the common preferences will have properties A.1–A.5 if it is assumed that g is monotonically non-decreasing, concave, continuous, homogeneous of degree one, and strictly positive if all of its arguments are strictly positive.

The question of whether such an economy has a non-empty core depends upon the function $f(u_1, \ldots, u_n)$ defined for all non-negative vectors u, as follows:

$$f(u) = \max_{\substack{x-u \in Y \\ x \geqq 0}} g(x).$$

Let there be n consumers, with the initial holdings of the ith consumer designated by $\omega^i > 0$. If the economy has a non-empty core then – as the following argument indicates – there will be a set of numbers α^i with

$$\sum_{i=1}^{m} \alpha^i = f\left(\sum_{i=1}^{m} \omega^i\right) \quad \text{and} \quad \sum_S \alpha^i \geq f\left(\sum_S \omega^i\right) \quad \text{for all sets } S.$$

To see this let $x^i = (x_1^i, \ldots, x_n^i)$, $i = 1, \ldots, m$, be an assignment of commodity bundles which is in the core, and define $\alpha^i = g(x_{k+1}^i, \ldots, x_n^i)$. The numbers α^i will be strictly positive, since $g(x_{k+1}^i, \ldots, x_n^i) \geq g(\omega_{k+1}^i, \ldots, \omega_n^i) > 0$. Suppose that, for some set S, we have

$$\sum_S \alpha^i < f\left(\sum_S \omega^i\right).$$

Using the definition of the function f, this means that we can find a commodity bundle $x \geqq 0$, with

$$x - \sum_S \omega^i \in Y \quad \text{and} \quad g(x_{k+1}, \ldots, x_n) > \sum_S g\left(x_{k+1}^i, \ldots, x_n^i\right).$$

Define, for $i \in S$,

$$\tilde{x}^i = \frac{g(x^i)}{\sum_{j \in S} g(x^j)} x.$$

Then the assignment of commodity bundles $\tilde{x}^i$ to the members of S is feasible for this set of consumers, and moreover

$$g(\tilde{x}^i) = \frac{g(x^i)}{\sum_{j \in S} g(x^j)} g(x) > g(x^i),$$

which contradicts the assumption that $\{x^i\}$ is in the core. To see that

$$\sum_1^m \alpha^i = f\left(\sum_1^m \omega^i\right),$$

we need only show that

$$\sum_1^m g(x^i) \leq f\left(\sum_1^m \omega^i\right),$$

and this is correct since

$$\sum_1^m g(x^i) \leq g\left(\sum_1^m x^i\right) \quad \text{and} \quad \sum_i^m x^i - \sum_1^m \omega^i \in Y.$$

We now know how to proceed. Given a fixed utility function g, if there is to be a non-empty core for all distributions of initial holdings (subject to $\omega^i > 0$), then on the basis of Theorem 4, the function $f(u_1,\ldots,u_n)$ must be a distributive function, when all of its arguments are considered as producer commodities. (The remaining assumptions of Theorem 4 are easy to verify.) The next step is to demonstrate that if this is to be true for all utility functions g then Y, itself, must be a distributive set.

The procedure is to show, first of all, that an efficient point y^* in Y has a non-negative price system π associated with it such that $\pi \cdot y^* = 0$ and $\pi \cdot y \leq 0$ for all y in Y with $y_j \geq y_j^*$ for $j = 1,\ldots,k$. As we shall see this implies that Y is a distributive set.

Let y^*, then, be an efficient point in Y. For the purposes of this proof, efficiency will be used in the strict sense that there is no $y \in Y$ with $y \geqq y^*$ and $y_j > y_j^*$ for all $j = k+1,\ldots,n$. Let $u^* = (-y_1^*,\ldots,-y_k^*, u_{k+1}^*,\ldots,u_n^*) \geq 0$ with $u_j^* > \max(0, -y_j^*)$ for $j = k+1,\ldots,n$. I will show, first of all, that for each fixed u^*, there is a non-negative price system π, such that $\pi \cdot y^* = 0$ and $\pi \cdot y \leq 0$ if $y \in Y$, $y_1 \geq y_1^*,\ldots, y_k \geq y_k^*$; and if, in addition, $(y_j^* - u_j^*)/2 \leq y_j \leq (y_j^* + u_j^*)/2$ for $j = k+1,\ldots,n$. As $u_j^* \to +\infty$ for $j = k+1,\ldots,n$, the price vectors with these properties form a non-increasing sequence of closed sets and we may therefore obtain a non-negative vector π with $\pi \cdot y^* = 0$ and $\pi \cdot y \geq 0$ for *all* y in Y with $y_j \geq y_j^*$ for $j = 1,\ldots,k$.

To obtain such a price vector, define the function

$$g(x_{k+1},\ldots,x_n) = \min\left(\frac{x_{k+1}}{u_{k+1}^* + y_{k+1}^*},\ldots,\frac{x_n}{u_n^* + y_n^*}\right).$$

As a utility function g has all the properties previously described. Then, as before, define

$$f(u) = \max_{\substack{x-u\in Y\\ x\geqq 0}} g(x),$$

so that f is a distributive function, when all of its n arguments are considered as producer commodities. Consider the point $u = u^*$. Since $(0,\ldots,0, u_{k+1}^* + y_{k+1}^*,\ldots, u_n^* + y_n^*) - u^* = y^* \in Y$, we see that $f(u^*) \geq g(u_{k+1}^* + y_{k+1}^*,\ldots, u_n^* + y_n^*) = 1$. On the other hand if $f(u^*) > 1$, we can find an $x \geqq 0$ such that $x - u^* \in Y$ and $x_j > u_j^* + y_j^*$ for $j = k+1,\ldots, n$. The vector $y = x - u^*$ would therefore be in Y with $y \geqq y^*$ and with $y_j > y_j^*$ for $j = k+1,\ldots, n$. This is impossible since y^* was assumed to be efficient in the strict sense given above. Therefore $f(u^*) = 1$.

Since f is a distributive function there is a non-negative price vector π different from zero, such that $\pi\cdot u^* = 1$ and $\pi\cdot u \geq f(u)$ for $0 \leqq u \leqq u^*$. Moreover if $y_j^* = 0$ for some $1 \leq j \leq k$, then $u_j^* = 0$, and the corresponding π_j may be taken to be zero. Let me show, first of all, that $\pi\cdot y^* = 0$. Since $u^* \geq 0$ with $u_j^* > 0$ for $j = k+1,\ldots, n$, we may find a positive α such that $u^* \geqq \alpha(u^* + y^*) \geqq 0$. Then

$$\alpha\pi(u^* + y^*) \geq f(\alpha(u^* + y^*)) \geq g(\alpha(u^* + y^*)) = \alpha.$$

Since $\pi\cdot u^* = 1$, this implies $\pi\cdot y^* \geq 0$. To obtain the reverse inequality, define $u = (1-\alpha)u^* + \alpha(-y^*)$. Since $u^* \geqq -y^*$, we have $u \leqq u^*$. Also since $u_j^* = -y_j^*$ for $j = 1,\ldots, k$, and $u_j^* > 0$ for $j > k$, we may find a sufficiently small positive α such that $u \geqq 0$, so that $\pi\cdot u \geq f(u)$. But $(1-\alpha)(u^* + y^*) - u = y^* \in Y$, so that $f(u) \geq g((1-\alpha)(u^* + y^*)) = (1-\alpha)$, and therefore $(1-\alpha)\pi\cdot u^* - \alpha\pi\cdot y^* \geq (1-\alpha)$ or $\pi\cdot y^* \leq 0$. These two inequalities taken together tell us that $\pi\cdot y^* = 0$.

The next step is to show that $\pi\cdot y \leq 0$ if $y \in Y$, $y_j \geq y_j^*$ for $j = 1,\ldots, k$, and if $(y_j^* - u_j^*)/2 \leq y_j \leq (y_j^* + u_j^*)/2$ for $j = k+1,\ldots, n$. Let y be such a point, and define $u = \frac{1}{2}(u^* + y^*) - y$, so that $0 \leqq u \leqq u^*$. Then $\pi\cdot u \geq f(u)$ and since $\frac{1}{2}(u^* + y^*) - u = y \in Y$, we see that $f(u) \geq \frac{1}{2}$. The inequality then reads $\frac{1}{2}\pi(u^* + y^*) - \pi\cdot y \geq \frac{1}{2}$, and since $\pi(u^* + y^*) = 1$, this implies $\pi\cdot y \leq 0$. As I indicated before, if $u_{k+1}^*,\ldots, u_n^* \to +\infty$, we obtain a non-negative price vector π, different from zero, with $\pi\cdot y^* = 0$ and $\pi\cdot y \leq 0$ for $y \in Y$ with $y_j \geq y_j^*$ for $j = 1,\ldots, k$. In addition $\pi_j = 0$ if $y_j^* = 0$ for $1 \leq j \leq k$.

The final argument which tells us that Y is a distributive set, proceeds as follows. Let $y^i \in Y$ and $y = \sum\alpha_i y^i$ with $\alpha_i \geq 0$ and $y^i - y \in \Lambda$ for all i. We want to conclude that $y \in Y$. Suppose that this is not so. Then, using the specific notion of efficiency described above, we may obtain a point $y^* = (y_1, \ldots, y_k, y_{k+1} - \varepsilon, \ldots, y_n - \varepsilon)$, for some positive ε, which is efficient in Y. Then the results of the preceding argument may be used to obtain a non-zero, non-negative vector π such that $\pi \cdot y^* = 0$ and $\pi \cdot y \leq 0$ for $y^i \in Y$ and $y_j^i \geq y_j^* = y_j$ for $j = 1, \ldots, k$. Moreover $\pi_j = 0$ if $y_j^* = 0$ for $1 \leq j \leq k$. We must certainly have $\pi_j > 0$ for at least one $j > k$, for if not then

$$\sum_1^k \pi_j y_j^* = 0,$$

which is impossible since at least one of these terms $\pi_j y_j^*$ must be < 0. This means that $\pi \cdot y > 0$, which is impossible since $y = \sum \alpha_i y^i$ and $\pi \cdot y^i \leq 0$. This concludes the proof that Y is a distributive set.

Let me summarize the results of this section in the following important theorem.

Theorem 5. Let Y be a production possibility set satisfying B.1–B.5. Assume that there is a non-empty core for every economy in which the consumers' preferences satisfy A.1–A.5, and in which the initial holdings of all consumers are strictly positive. Then Y is a distributive set.

4. The existence of a core for distributive sets

The results of the previous section indicate that if a theory of economic allocations is to be based on the concept of the core, then production must be described by a distributive set. In the present section, I will demonstrate that if Y is distributive, if A.1–A.5 and B.1–B.5 are satisfied, and if $\omega^i > 0$, then there will indeed be a non-empty core. The procedure is to select a specific allocation, to which I shall give the name "social equilibrium" as distinct from "competitive equilibrium", and to show that this allocation is in the core.

Consider an economy with m consumers, with initial holdings $\omega^i \geqq 0$ and with preferences satisfying A.1–A.5. Define $\omega = \sum_1^m \omega^i$. Let Y be the production possibility set. An allocation $x^1, \ldots, x^m, y$ with $x^i \geqq 0$, $y \in Y$ and $\sum_1^m (x^i - \omega^i) = y$, is said to be a social equilibrium if there is a price vector $\pi \geq 0$, such that

(1) x^i maximizes the preferences of the ith consumer subject to the budget constraint $\pi \cdot \bar{x}^i \leq \pi \cdot \omega^i$; and

(2) $\pi \cdot y = 0$ and $\pi \cdot \bar{y} \leq 0$ for all $\bar{y} \in Y$ with $\bar{y}_j \geq -\omega_j$ for $j = 1, \ldots, k$.

The definition of a social equilibrium is virtually identical with that of a competitive equilibrium. The primary difference is that in a social equilibrium, the production side of the economy considers only those plans which use no more of the producer commodities than are available. As we shall see, a social equilibrium will exist if production is described by a distributive set, whereas a competitive equilibrium will generally exist only if production is described by a convex cone. The concept of a social equilibrium therefore provides a theory of economic allocations in cases which were previously intractible.

A social equilibrium must be in the core, as the following argument indicates. For let x^i, y be a social equilibrium and π the associated system of prices. Since $\pi \cdot y = 0$, it follows that $\pi \cdot x^i = \pi \cdot \omega^i$. Let this allocation be blocked by S, so that there exists $\bar{x}^i$ for $i \in S$ with $\sum_S(\bar{x}^i - \omega^i) = \bar{y} \in Y$, and $\bar{x}^i \succsim_i x^i$ with strict preference for at least one member of S. Then using our assumptions on preferences,

$$\sum_S \pi \cdot \bar{x}^i > \sum_S \pi \cdot x^i = \sum_S \pi \cdot \omega^i \quad \text{and} \quad \pi \cdot \bar{y} > 0.$$

But

$$\bar{y} \in Y \quad \text{and} \quad \bar{y} \geqq -\sum_S \omega^i \geqq -\omega,$$

which contradicts the second condition for a social equilibrium.

Our final result is embodied in the following theorem.

Theorem 6. Consider an economy in which the preferences of the consumers satisfy A.1–A.5, and in which the initial holdings are strictly positive. Let Y be a closed distributive set satisfying B.5. Then there exists a social equilibrium and consequently a non-empty core.

As before let $\omega = \sum_1^m \omega^i$, and let T be the smallest closed convex cone with vertex at the origin and containing $Y \cap [\Lambda - \omega]$. We shall begin by examining the competitive equilibrium for an economy in which the production set is T rather than Y, but with the other aspects of the economy unchanged. In order to deduce that this new economy has a competitive equilibrium it is sufficient to verify the following properties:

(1) $T \supset (-\Omega)$. This is the free disposal assumption for T. Since T is a cone it is sufficient to demonstrate that $(-\delta_1, \ldots, -\delta_n) \in T$ for all small non-negative δ_i, and this is certainly true since $\omega > 0$, and Y has the free disposal property.

(2) $T \cap \Omega = \{0\}$. This states that there is no vector in T with all components $\geqq 0$ other than zero. Let ξ be a point in $T \cap \Omega$ which is different from zero. Since all of the vectors in T have their first k components ≤ 0, we must have $\xi = (0, \ldots, 0, \xi_{k+1}, \ldots, \xi_n)$ with $\xi_j \geq 0$ and at least one component > 0. Consider the point $(-\omega_1, \ldots, -\omega_k, \xi_{k+1}, \ldots, \xi_n)$. I claim that this latter point must be in Y, for if it is not, then we may apply Theorem 1, and obtain a price vector $\pi \geq 0$ with

$$-\sum_1^k \pi_i \omega_i + \sum_{k+1}^n \pi_i \xi_i > 0,$$

and

$$\pi \cdot y \leq 0 \quad \text{for all} \quad y \in Y,$$

with $y_j \geq -\omega_j$ for $j = 1, \ldots, k$. But this implies that $\pi \cdot y \leq 0$ for all $y \in T$ and therefore $\pi \cdot \xi \leq 0$. This contradicts the previous inequality. Therefore $(-\omega_1, \ldots, -\omega_k, \xi_{k+1}, \ldots, \xi_n) \in Y$. The same argument may be repeated with $\lambda\xi$ instead of ξ and we see that $(-\omega_1, \ldots, -\omega_k, \lambda\xi_{k+1}, \ldots, \lambda\xi_n) \in Y$ for all λ and this contradicts B.5.

There is therefore a competitive equilibrium if T is used as the production possibility set. In other words there will be an assignment of non-negative commodity bundles x^i, a vector $y \in T$ with $\sum x^i - \omega = y$ and a price vector $\pi \geq 0$ such that x^i maximizes the preferences of the ith consumer subject to $\pi \cdot x^i \leq \pi \cdot \omega^i$, and such that $\pi \cdot y = 0$, and $\pi \cdot \bar{y} \leq 0$ for all $\bar{y} \in T$. From the last condition we see that $\pi \cdot \bar{y} \leq 0$ for all $\bar{y} \in Y$ with $\bar{y}_j \geq -\omega_j$ for $j = 1, \ldots, k$. To demonstrate that this allocation is a social equilibrium for the original economy we need only demonstrate that $y \in Y$ itself.

First of all notice that $y_j \geq -\omega_j$, since $x^i \geqq 0$. I claim that if $y_j > -\omega_j$ for $1 \leq j \leq k$ then $\pi_j = 0$. To see this assume that $y_1 > -\omega_1$ and that $\pi_1 > 0$. Then x_1^i must be > 0 for at least one i. Then define $\bar{x}^i = (0, \ldots, 0, x_{k+1}^i + \delta, \ldots, x_n^i + \delta)$ for some strictly positive δ with

$$\delta \sum_{k+1}^n \pi_i \leq \pi_1 x_1^i.$$

We have $\pi \cdot \bar{x}^i \leq \pi \cdot \omega^i$ and, by using A.1–A.5, we see that $\bar{x}^i \succ_i x^i$ which is a contradiction. This means that if $y_j > -\omega_j$ for $1 \leq j \leq k$, we may lower those components of y_j, without disturbing the preferences, maintaining $\pi \cdot y = 0$, and the modified y will still be in T, since T has the free disposal property. Let us assume that this has been done and that $y_j = -\omega_j$ for $j = 1, \ldots, k$.

Now, suppose that $y \notin Y$. Using Theorem 1, we obtain a non-zero price vector ρ with $\rho \cdot y > 0$ and $\rho \cdot \bar{y} \leq 0$ for all $\bar{y} \in Y$ with $\bar{y}_j \geq -\omega_j$ for $j = 1, \ldots, k$. But this implies that $\rho \cdot \bar{y} \leq 0$ for all $\bar{y} \in T$. Since $y \in T$ this is a contradiction which demonstrates that $y \in Y$. This concludes the proof of Theorem 6.