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THE FROBENIUS PROBLEM AND MAXIMAL LATTICE FREE BODIES

HERBERT E. SCARF AND DAVID F. SHALLCROSS

Let $p = (p_1, \ldots, p_n)$ be a vector of positive integers whose greatest common divisor is unity. The Frobenius problem is to find the largest integer f^* which cannot be written as a nonnegative integral combination of the p_i. In this note we relate the Frobenius problem to the topic of maximal lattice free bodies and describe an algorithm for $n = 3$.

1. Introduction. Let $p = (p_1, \ldots, p_n)$ be a vector of positive integers whose greatest common divisor is unity. The Frobenius problem is to find the largest integer f^* which cannot be written as a nonnegative integral combination of the p_i. For $n = 2$, it is well known that $f^* = p_1 p_2 - p_1 - p_2$. For $n = 3$, there is an algorithm of Rödseth (1977) which finds the Frobenius number f^* in polynomial time. Recently, Kannan (1989) has produced an algorithm for the Frobenius problem which runs in polynomial time for all fixed n, but which is doubly exponential in n.

The question of whether a single linear equation $\Sigma p_i h_i = f$ is solvable in nonnegative integers is NP complete, and we cannot expect to resolve its solvability by means of an algorithm which is polynomial in the number of variables as well as the bit size of the data. For fixed n, Lenstra's algorithm (1983) will execute in polynomial time for any particular linear equation. The significance of the Frobenius problem is that it is concerned with a family of linear equations, $\Sigma p_i h_i = f$, as f varies over all positive integers, rather than with a single equation itself. For any instance of the Frobenius problem, the Frobenius number f^* will typically be sufficiently large so that its determination by an exhaustive search over all f less than some established upper bound will not yield a polynomial algorithm.

In this note we shall relate the Frobenius problem to a different area under recent investigation, that of maximal closed convex sets containing no interior lattice points. Given a matrix A, the body $\{x: Ax \leqslant b\}$ is a *maximal lattice free body* if it contains no lattice points in its interior and if any strictly larger body obtained by relaxing some of the inequalities does contain an interior lattice point. A polytope is a maximal lattice free body if it contains no lattice points in its interior and if each facet contains a lattice point in its relative interior. We demonstrate that if we can maximize a linear function over the set of b's yielding maximal lattice free bodies for a matrix with n rows and $n - 1$ columns, then we can solve the Frobenius problem with n variables. One consequence is an algorithm for the three-variable problem—somewhat similar to Rödseth's algorithm—which runs in linear time in the bit size of the integers p_i. We also relate the Frobenius number to the covering radius of a simplex in R^{n-1}, in a somewhat different fashion than that established by Kannan.

Lovász (1988) has conjectured that if n is fixed and A is integral, the set of b yielding maximal lattice free bodies is the union of the set of lattice points in a polynomial number of polyhedra—with a particular lattice for each polyhedron.

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Maximizing a linear function over the lattice points in each such polyhedron is a standard integer program which can be solved in polynomial time for a fixed number of variables. If the Lovász conjecture were correct, this would yield an alternative polynomial algorithm for the Frobenius problem.

2. The relationship to maximal lattice free bodies. Let A be a matrix of size n by $n - 1$, whose columns generate the $(n - 1)$-dimensional lattice of integers h satisfying $p \cdot h = 0$. In this case the bodies $\{x\colon Ax \leqslant b\}$ will be simplices which are nonempty if $p \cdot b \geqslant 0$. Our main result is

THEOREM 1. *$f^* = \max\{p \cdot b | b$ is integral and $\{x\colon Ax \leqslant b\}$ contains no lattice points$\}$.*

PROOF. To demonstrate Theorem 1, we observe that if b is an integer vector such that $\{x\colon Ax \leqslant b\}$ contains no lattice points, then $f = p \cdot b$ cannot be written as $p \cdot h$ with h nonnegative integers. For if this were possible then $0 = p \cdot (b - h)$ so that $b - h$ is in the $(n - 1)$-dimensional lattice generated by the columns of A. It follows that $b - h = A\xi$ for some integral ξ and therefore the set $\{x\colon Ax \leqslant b\}$ contains a lattice point.

Conversely, if b is an integral vector such that $\{x\colon Ax \leqslant b\}$ contains a lattice point ξ, then $f = p \cdot b = p \cdot (b - A\xi)$, with $b - A\xi$ a nonnegative integer vector. Since the p_i are relatively prime, every integer f can be written as $p \cdot b$ for some integral b. It follows from these observations that f^* is the largest value of $p \cdot b$ for those integral b such that $\{x\colon Ax \leqslant b\}$ is free of lattice points. □

Theorem 1 permits us to calculate the Frobenius number f^* from a description of the set of vectors b such that $K_b = \{x\colon Ax \leqslant b\}$ is a maximal lattice free body, according to our previous definition. We simply remark that, for integral b, the simplex $\{x\colon Ax \leqslant b\}$ contains no lattice points in its interior if, and only if, $\{x\colon Ax \leqslant b - e\}$ contains no lattice points at all, where e is the vector all of whose components are unity. It follows that

$$f^* = \max\{p \cdot b | Ax \leqslant b \text{ is a maximal lattice free body}\} - \Sigma p_i.$$

Aside from lattice translates of $\{x\colon Ax \leqslant b\}$, which do not change the value of $p \cdot b$, there are a finite number of maximal lattice free bodies associated with the matrix A.

Kannan shows that the calculation of the Frobenius number is equivalent to finding the covering radius of a particular $(n - 1)$-dimensional simplex. The *covering radius* of a body K in R^{n-1} is the smallest ρ such that the lattice translates of ρK cover R^{n-1}. Our discussion yields the following relation between the Frobenius number and the covering radius of $\{x\colon Ax \leqslant b\}$.

THEOREM 2. *Let the covering radius of $\{x\colon Ax \leqslant b\}$ be ρ_b, for any particular b with $p \cdot b > 0$. Then $f^* = (p \cdot b)\rho_b - \Sigma p_i$.*

PROOF. If K_{b^*} is that maximal lattice free simplex $\{x\colon Ax \leqslant b^*\}$ which maximizes $p \cdot b$, then its covering radius is unity. For if x is not covered by any lattice translate of K_{b^*}, then $K_{b^*} - x$ contains no lattice points and it can be expanded to a maximal lattice free body strictly larger than K_{b^*}. On the other hand, a slight contraction of K_{b^*} contains no lattice points, and, therefore, its lattice translates do not cover the origin. The covering radius of K_{b^*} is therefore equal to unity. For any other b with $p \cdot b > 0$, the simplex K_b is similar to K_{b^*}; it can be brought to K_{b^*} by a suitable translation and expansion by a factor $p \cdot b^*/p \cdot b$. It follows that the covering radius of K_b is $p \cdot b^*/p \cdot b = (f^* + \Sigma p_i)/(p \cdot b)$. □

3. Maximal lattice free bodies for $n = 3$. Relatively little is known about the set of maximal lattice free bodies associated with a general matrix A with n rows and $n - 1$ columns. It is not clear to us how to use the analysis given by Scarf (1985) for the case $n = 4$ to solve the corresponding Frobenius problem. When $n = 3$, Scarf (1981) has demonstrated—under the assumptions that the entries in each row of A have an irrational ratio, that $\pi A = 0$ for a strictly positive vector π and that no two rows are proportional—that there are two maximal lattice free bodies of the form $\{x: Ax \leqslant b\}$, up to a lattice translation, and that these bodies are easy to find. Specifically, Scarf shows that there is a unimodular coordinate transformation so that the matrix A has the sign pattern

$$\begin{bmatrix} - & - \\ + & - \\ - & + \end{bmatrix},$$

with the sum of the second and third rows strictly positive, and that the two maximal lattice free bodies are given by

$$b^1 = (0, a_{2,1}, a_{3,1} + a_{3,2})$$

and

$$b^2 = (0, a_{2,1} + a_{2,2}, a_{3,2}).$$

But the assumption that the entries in each row of A have an irrational ratio is, of course, not satisfied in our case, and the analysis to be presented becomes somewhat more complex; in particular, some of the strict inequalities given above may become weak inequalities and there may be only one maximal lattice free body, aside from integral translations.

We shall describe an algorithm which yields a unimodular transformation of coordinates such that the matrix A has the sign pattern

$$\begin{bmatrix} - & \leqslant \\ + & - \\ \leqslant & + \end{bmatrix},$$

with the sum of the entries in the second row greater than or equal to zero, and the sum in the third row strictly positive; and then demonstrate that this pattern is sufficient to characterize the maximal lattice free triangles. (The symbol $\leqslant$ appearing in the matrix signifies that the corresponding entry is less than or equal to zero.)

We begin with a particular form for the matrix A. Let γ be the greatest common divisor of p_2 and p_3, and write $\gamma = m_3 p_2 - m_2 p_3$, with m_2 and m_3 integers satisfying $0 \leqslant m_2 < p_2/\gamma$ and $0 < m_3 \leqslant p_3/\gamma$. Then the columns of

$$A = \begin{bmatrix} -\gamma & 0 \\ m_3 p_1 & -p_3/\gamma \\ -m_2 p_1 & p_2/\gamma \end{bmatrix}$$

generate the lattice of integers satisfying $p \cdot h = 0$. The matrix has the sign pattern described above, but without any specific signs for the sums of the second and third rows. We shall systematically add integral multiples of one of the columns of A to the other column, retaining the signs of the entries in A and ultimately achieving the desired signs for these row sums.

The algorithm alternates between two steps:

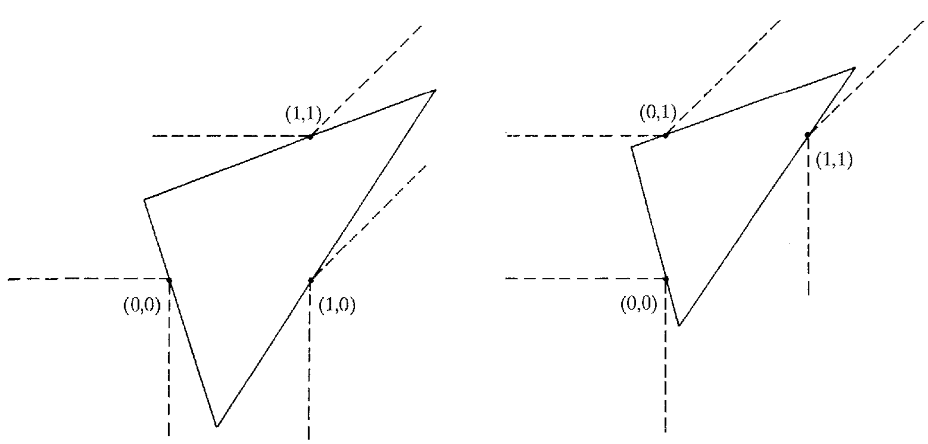
1. adding the largest integral multiple of column 2 to column 1 so as to preserve the sign pattern $(-, +, \leqslant)$ in column 1, and

2. adding the largest integral multiple of column 1 to column 2 so as to preserve the sign pattern $(\leqslant, -, +)$ in column 2.

After a step of type 1, the sum of the two columns of A will have the sign pattern $(-, \geqslant, +)$, in which case we terminate, or $(-, -, +)$ and we move to a step of type 2. After a step of type 2, the sum of the two columns of A will have the sign pattern $(-, \geqslant, +)$, in which case we terminate, or the pattern $(-, +, \leqslant)$ and we continue with a step of type 1. In both of these observations, we use the fact that entries in the column sum cannot be all less than or equal to zero, since the pair of columns in a matrix arising from repeated applications of steps 1 and 2 will generate the lattice of integers satisfying $p \cdot h = 0$. The algorithm clearly terminates in a number of steps bounded above by the bit size of p.

Now let us argue that the sign pattern which has just been established is sufficient to identify the maximal lattice free bodies associated with A—in particular, that if we let $b^1 = (0, a_{2,1}, a_{3,1} + a_{3,2})$, and $b^2 = (0, a_{2,1} + a_{2,2}, a_{3,2})$, then the triangle $\{x\colon Ax \leqslant b^1\}$ and its integral translates are always maximal lattice free bodies, the triangle $\{x\colon Ax \leqslant b^2\}$ and its integral translates are maximal lattice free bodies when the inequalities $a_{1,2} \leqslant 0$, $a_{2,1} + a_{2,2} \geqslant 0$, and $a_{3,1} \leqslant 0$ are strict, and there are no other maximal lattice free bodies. In the cases in which $\{x\colon Ax \leqslant b^2\}$ is not a maximal lattice free body $pb^2 < pb^1$. Thus $f^* = \max\{pb^1, pb^2\} - p_1 - p_2 - p_3$.

First we show that any maximal lattice free body $\{x\colon Ax \leqslant b\}$ has an integral translate such that either $b \leqslant b^1$ or $b \leqslant b^2$. This immediately implies that only $\{x\colon Ax \leqslant b^1\}$ and $\{x\colon Ax \leqslant b^2\}$ and their integral translates can be maximal lattice free bodies. Given a particular maximal lattice free body, translate it so that the origin is in the interior of the facet given by the constraint $a_1 x \leqslant b_1$, and so that no other integer point on the relative interior of that same facet has a positive second coordinate. Then $b_1 = 0$, $b_2 > 0$, and $b_3 > 0$. Only the constraint $a_2 x \leqslant b_2$ can exclude the point $(1,0)$ from the interior, so that $b_2 \leqslant a_{2,1}$. Only the constraint $a_3 x \leqslant b_3$ can exclude $(0,1)$ from both the interior of the triangle and the relative interior of the facet $a_1 x = b_1$, so that $b_3 \leqslant a_{3,2}$. Either of the last two constraints can exclude $(1,1)$, so we must have either $b_2 \leqslant a_{2,1} + a_{2,2}$ or $b_3 \leqslant a_{3,1} + a_{3,2}$. Depending on this choice, either $b \leqslant b^1$ or $b \leqslant b^2$.



Second, we show that $\{x\colon Ax \leqslant b^1\}$ is a maximal lattice free body. All points in Z^2 lie either in the set $\{x\colon x_1 \leqslant 0,\ x_2 \leqslant 0\}$, which satisfy $a_1x \geqslant b_1^1$, the set $\{x\colon x_1 \geqslant 1,\ x_1 - x_2 \geqslant 1\}$, which satisfy $a_2x \geqslant b_2^1$, or the set $\{x\colon x_1 - x_2 \leqslant 0,\ x_2 \geqslant 1\}$, which satisfy $a_3x \geqslant b_3^1$. Thus, the triangle contains no interior lattice points. The inequalities

$$a_{1,1} < 0, \qquad 0 < a_{2,1}, \quad 0 < a_{3,1} + a_{3,2},$$

$$a_{1,1} + a_{1,2} < 0, \quad a_{2,2} < 0, \qquad 0 < a_{3,2},$$

guarantee that $(0,0)$, $(1,0)$, and $(1,1)$ are each in the relative interior of one of the three facets of the triangle. Thus, b^1 gives a maximal lattice free body.

Finally, we show that $\{x\colon Ax \leqslant b^2\}$ is a maximal lattice free body if the inequalities $a_{1,2} \leqslant 0$, $a_{2,1} + a_{2,2} \geqslant 0$, and $a_{3,1} \leqslant 0$ are strict, but is not a maximal lattice free body if any of these inequalities are satisfied with equality. The points in Z^2 lie either in the set $\{x\colon x_1 \leqslant 0,\ x_2 \leqslant 0\}$, which satisfy $a_1 \geqslant b_1^2$, the set $\{x\colon x_1 \geqslant 1,\ x_1 - x_2 \geqslant 0\}$, which satisfy $a_2x \geqslant b_2^2$, or the set $\{x\colon x_1 - x_2 \leqslant -1,\ x_2 \geqslant 1\}$, which satisfy $a_3 \geqslant b_3^2$. Thus the triangle contains no lattice points in its interior. The inequalities

$$a_{1,2} \leqslant 0, \qquad 0 < a_{2,1}, \quad 0 \leqslant a_{2,1} + a_{2,2},$$

$$a_{1,1} + a_{1,2} < 0, \quad a_{3,1} \leqslant 0, \qquad 0 < a_{3,2},$$

guarantee that the points $(0,0)$, $(1,1)$, and $(0,1)$ lie on the boundary of the triangle given by b^2. If all of these inequalities are strict, then these points lie in the relative interiors of their respective facets of the triangle. Otherwise, one of these points will lie on a corner of the triangle, and there will be a facet without integral points in its relative interior preventing the triangle from being a maximal lattice free body. A maximal lattice free body containing this triangle must be an integral translate of $\{x\colon Ax \leqslant b^1\}$, so that $pb^2 < pb^1$.

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H. E. Scarf: The Cowles Foundation, Yale University, New Haven, Connecticut 06520

D. F. Shallcross: Bell Communications Research, Morristown, New Jersey 07962