TEST SETS FOR INTEGER PROGRAMS

BY

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Abstract

In this paper I discuss various properties of the simplicial complex of maximal lattice free bodies associated with a matrix A. If the matrix satisfies some mild conditions, and is *generic*, the edges of the complex form the minimal test set for the family of integer programs obtained by selecting a particular row of A as the objective function, and using the remaining rows to impose constraints on the integer variables. © 1997 The Mathematical Programming Society, Inc. Published by Elsevier Science B.V.

Keywords: Test sets; Integer programming; Simplicial complexes; Groebner bases

1. Introduction

Let $A = [a_{ij}]$ be a matrix with $m+1$ rows $(i = 0,\dots,m)$ and n columns $(j = 1,\dots,n)$. In this talk I will discuss *test sets* for the family of integer programs

$$\max \sum_{j=1}^{n} a_{0j}h_j, \qquad \text{subject to}$$

$$\sum_{j=1}^{n} a_{ij}h_j \geqslant b_i, \qquad i = 1,\dots,m, \quad h \in Z^n, \tag{1}$$

obtained by fixing the objective function and the constraints a_i, $i = 0,\dots,m$ and selecting arbitrary values of the right-hand side b_i.

In order to introduce the basic ideas of test sets it may be useful to step back from the lattice structure of integer programming and replace the set of vectors $\{y = Ah \mid h \in Z^n\}$ with an arbitrary finite set of vectors $Y = \{y^j \mid j = 0,\dots,k\}$ in R^{m+1} [8]. The members of Y are assumed to satisfy the following:

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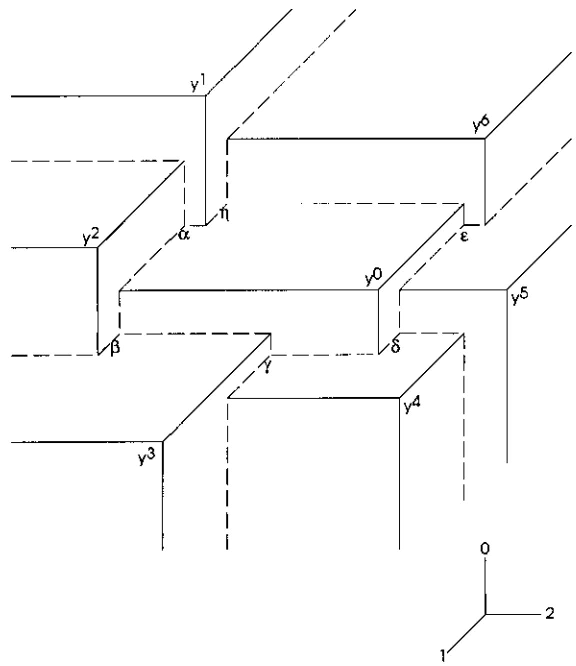

Fig. 1. The set Y.

Non-Degeneracy Assumption. No two members of Y have the same ith coordinate for any $i = 0, \ldots, m$.

Let us consider the collection of discrete programming problems of the form

$$\begin{aligned} &\max y_0 \quad \text{subject to} \\ &y_i \geqslant b_i \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad y \in Y. \end{aligned} \tag{2}$$

A *test set* for $y^* \in Y$ is a subset $N(y^*)$ of Y with the property that if y^* satisfies the constraints $y_i^* \geqslant b_i$, for $i = 1, \ldots, m$ but is not optimal, then one of the members of $N(y^*)$ is feasible and has a larger 0th coordinate. A test set provides a proof that a feasible solution is not optimal.

Test sets are based on a simplicial complex $C(Y)$ of dimension m which is defined in a canonical fashion for the set Y. We begin the construction of $C(Y)$ by adding a negative orthant N to each of the points $y^j \in Y$ obtaining the set $\hat{Y} = \bigcup_j (y^j + N)$ as in Fig. 1. The upper boundary of $\hat{Y}$ is piece-wise linear, with each linear piece parallel to one of the coordinate hyperplanes.

Definition 1. The set $\{y^{j_0}, y^{j_1}, \ldots, y^{j_m}\}$ is an *m-dimensional simplex* of $C(Y)$ if there is no point $y \in Y$ with $y > \min[y^{j_0}, y^{j_1}, \ldots, y^{j_m}]$ in each coordinate. More generally,

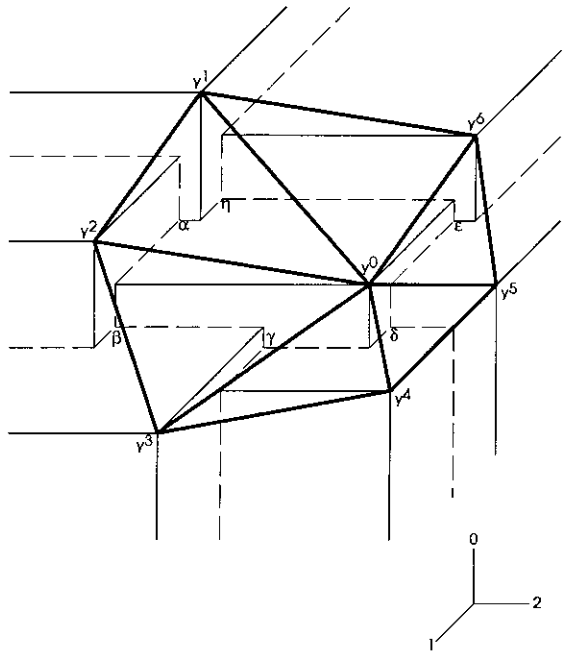

Fig. 2. The simplicial complex.

a set $\{y^{j_0}, y^{j_1}, \ldots, y^{j_l}\}$ with $l \leqslant m$ is an l-dimensional simplex of $C(Y)$ if there is no point $y \in Y$ with $y > \min[y^{j_0}, y^{j_1}, \ldots, y^{j_l}]$ in each coordinate.

An m-simplex of $C(Y)$ is obtained by translating the positive orthant of R^{m+1} parallel to itself until it lies fully above the set $\hat{Y}$, i.e., looking at the set $\{y \mid y_i \geqslant b_i\}$ for some b. We then translate the positive orthant downwards until no further translation is possible without passing through one of the points $y \in Y$. At such a position each of the $m+1$ coordinate hyperplanes of the translated orthant will contain an element of Y. Given the *non-degeneracy* assumption, no coordinate hyperplane will contain more than a single member of Y. The translation will therefore be stopped by $m+1$ points which define a simplex in $C(Y)$. Fig. 2 displays the 6 simplices of dimension 2 arising from the 7 points in Fig. 1.

There are clearly some translations of the positive orthant which can be lowered indefinitely without passing through a vector in Y. If the orthant sits above the horizontal line between y^1 and y^2, then it can be translated along that line by lowering its 2nd coordinate, without penetrating the upper surface of $\hat{Y}$. The pair $\{y^1, y^2\}$, and other pairs as well, form *boundary simplices* of $C(Y)$. The boundary simplices can also be captured by Definition 1 if the set Y is augmented by the introduction of $m+1$ "ideal" vectors $\xi^0, \xi^1, \ldots, \xi^m$ with

$$\xi^i_j = \infty \text{ for } i \neq j \quad \text{and} \quad \xi^i_i = -\infty.$$

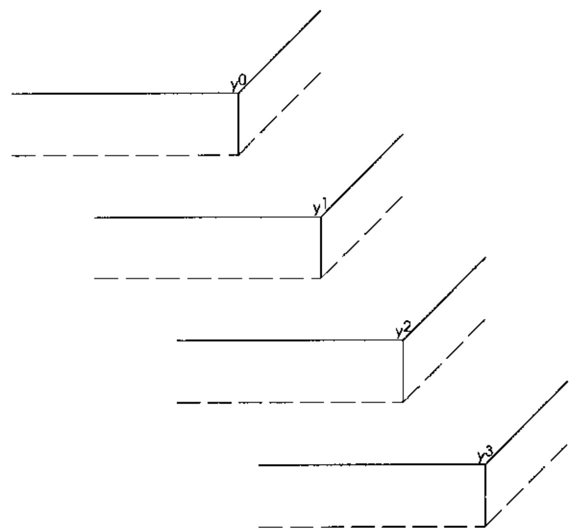

Fig. 3. An example in which $K(Y)$ has no interior vertices.

If Definition 1 is applied to this augmented set, we see that the triples $\{\xi^2, y^1, y^2\}$ and $\{\xi^2, y^2, y^3\}$ are both simplices in $C(Y)$ and together they constitute a face, say F_2 of the simplicial complex. More generally, if $\{i_1, \dots, i_t\}$ is a *proper* subset of $\{0, 1, \dots, m\}$, then the $m - t$ dimensional simplices in the *boundary* face $F_{i_1,\dots,i_t}$ are those collections of $m - t + 1$ points in Y which satisfy Definition 1 in conjunction with the ideal vectors $\xi^{i_1}, \dots, \xi^{i_t}$. We say that a point y is on the *boundary* of $C(Y)$ if it is contained in a simplex sitting in a boundary face. If y is not on the boundary of $C(Y)$ it is said to be *interior* to the complex. As Fig. 3 illustrates, there are examples in which every point in Y is on the boundary of $C(Y)$.

Definition 2. Let y^* be in Y. Then $N(y^*)$, the set of *neighbors* of y^*, consists of those $y \in Y$ such that y and y^* are members of a common simplex in $C(Y)$.

Theorem 3. *A feasible solution $y^* \in Y$ to Problem* (2) *is optimal if none of its neighbors is feasible and yields a higher value for the* 0*th coordinate.*

Theorem 3 is quite ancient; I have known about it for more than twenty years. An equally venerable result is the following topological characterization of the complex $C(Y)$ [9].

Theorem 4. *The complex $C(Y)$ is a manifold: every interior $m - 1$ dimensional face is contained in two m-simplices, and every boundary $m - 1$ face in a single m-simplex. Moreover, if $C(Y)$ contains an interior point, it is topologically an m-dimensional simplex.*

2. Take Y to be a lattice

Let us return to integer programming by taking Y to be the lattice $\{y \mid y = Ah,\ h \in Z^n\}$. The additional structure afforded by the lattice permits a much deeper analysis of the simplicial complex, which we denote by $K(A)$ to emphasize its dependance on the matrix A. But I should remark that it is difficult to calculate this simplicial complex, and the set of neighbors, $N(A)$ – much more difficult than solving a single instance of the integer program (1). In my opinion, test sets are useful if one intends to solve, or analyze a large family of integer programs based on the same matrix, with varying right-hand sides, as in the Frobenius problem [7].

Some conditions have to be imposed on the matrix A in order for the complex to be well-behaved [2].

The Main Assumption. $\exists \pi > 0$ with $\pi A = 0$. Moreover, if $\pi \geqslant 0$ (but not $= 0$) and $\pi A = 0$, then $\pi_i > 0$ for at least $n + 1$ coordinates.

The *non-degeneracy* assumption asserts that no two vectors $y = Ah$ have the same ith coordinate for any i. This is a particularly unpleasant assumption which rules out the possibility that A is a matrix of integers. For expository purposes, however, I shall retain this assumption for the moment, and postpone until later the question of how it can be relaxed.

We begin by asking when a collection of $m + 1$ vectors

$$y^{i_0} = Ah^{i_0},\ \ldots,\ y^{i_m} = Ah^{i_m}$$

is a simplex in the simplicial complex.

As before, we append a negative orthant to each $y = Ah$, and call the resulting set $\hat{Y}$. We then take the positive orthant P and translate it to $b+P$ so that it no longer intersects $\hat{Y}$. Such a translation can be described in R^n, the space of variables (see Fig. 4). The statement that $b + P$ does not intersect $\hat{Y}$ is identical with the statement that

the body $P_b = \{x \mid Ax \geqslant b\}$ contains no lattice points.

When we lower the translated positive orthant, the hyperplanes defining the body P_b are relaxed and the body is enlarged. If we reach a position in which no further relaxation is possible without introducing a lattice point, then each of the $m + 1$ hyperplanes will contain a lattice point. According to the non-degeneracy assumption, the ith such hyperplane will contain precisely one lattice point, say h^{j_i}. This collection of $m+1$ lattice points defines a simplex in the complex $K(A)$. It is useful to introduce the following notation.

Notation. For $h^0, h^1, \ldots, h^k$ in Z^n let $\langle h^0, h^1, \ldots, h^k\rangle$ be the smallest body of the form $\{x \mid Ax \geqslant b\}$ containing these lattice points. For this body, b is the coordinatewise minimum of $Ah^0, Ah^1, \ldots, Ah^k$.

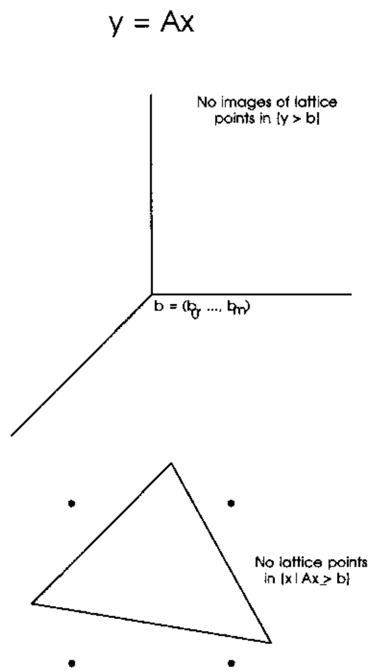

Fig. 4. The view in R^n.

Definition 1 can then be translated into R^n to define the simplices in $K(A)$:

Revised definition. The set of lattice points $\{h^{i_0}, \ldots, h^{i_m}\}$ is an m-simplex in $C(A)$ if the body $\langle h^{i_0}, \ldots, h^{i_m} \rangle$ contains no lattice points other than $\{h^{i_0}, \ldots, h^{i_m}\}$. More generally, $\{h^{i_0}, \ldots, h^{i_\ell}\}$, with $\ell \leqslant m$ is an ℓ-simplex in $C(A)$ if the body $\langle h^{i_0}, \ldots, h^{i_\ell} \rangle$ contains no lattice points other than $\{h^{i_0}, \ldots, h^{i_\ell}\}$.

There will now be an infinite number of simplices in $K(A)$; if $\{y^{j_0}, y^{j_1}, \ldots, y^{j_l}\}$ is one such simplex, then so is $\{y^{j_0}, y^{j_1}, \ldots, y^{j_l}\} + Ah$, for any lattice point h. It follows that y is a neighbor of y^* if, and only if, $y - y^*$ is a neighbor of 0; the test set is fully described by the set of neighbors of the origin, which we denote by $N(A)$. The lattice point h is a neighbor of the origin if the body $\langle 0, h \rangle$ contains no lattice points other than 0 and h.

Consequences of the Main Assumption. It can be shown that the Main Assumption implies that $N(A)$ is finite, non-empty and symmetric. In addition, one can obtain a bound on the size of the neighbors of the origin. Specifically, if the rows of A are normalized so as to have length 1, and if d is the smallest non-vanishing n by n minor of A in absolute value, then

$$\|h\| \leqslant n^2/d \qquad \text{for any neighbor } h.$$

The set of neighbors $N(A)$ is a test set for the family of integer programs (1) in the sense that a lattice point z satisfying the inequalities $a_i z \geqslant b_i$ for $i = 1, \ldots, m$ will be an optimal solution to (1) if $z + h$ is infeasible for every neighbor $h \in N(A)$, with $a_0 h > 0$. Under the non-degeneracy assumption, $N(A)$ is a *minimal* test set in the sense that if a single neighbor h with $a_0 h > 0$ is discarded from $N(A)$ then a right-hand side b and a feasible, but non-optimal solution z can be found, whose lack of optimality cannot be detected by the smaller test set.

3. Some examples

Let us consider the case in which the matrix A has 3 rows and 2 columns. A simplex in $K(A)$ will be given by 3 lattice points, say, $\{h^0, h^1, h^2\}$, such that the body

$$\{x \mid Ax \geqslant b\} \qquad \text{with} \quad b_i = \min[a_i h^0, a_i h^1, a_i h^2]$$

contains no other lattice points. It follows that the triangle with vertices h^0, h^1, h^2 contains no other lattice points, and must therefore have area $1/2$. By a lattice translation, we can bring h^0 to the origin, and then find a unimodular transformation so that

$$U(0, h^1, h^2) = ((0,0), (1,0), (1,1)).$$

If A is multiplied on the right by U^{-1}, the lattice points defining the simplex for the matrix AU^{-1} will be given by $((0,0), (1,0), (1,1))$.

It is easy to argue [10] that, aside from lattice translates, the simplicial complex associated with a matrix with 3 rows and 2 columns has precisely 2 distinct simplices (see Fig. 5), and by a unimodular transformation they can be brought into the form:

$$((0,0), (1,0), (1,1))$$
$$((0,0), (0,1), (1,1)).$$

We see that a matrix of this size has 6 neighbors. A family of examples in which the number of neighbors is bounded by a function of the dimension alone is given by matrices with $n + 1$ rows, n columns, and the following sign pattern

$$\begin{bmatrix} - & - & \cdots & - \\ + & - & \cdots & - \\ - & + & \cdots & - \\ \vdots & \vdots & \ddots & \vdots \\ - & - & \cdots & + \end{bmatrix}$$

and with the additional property that $\sum_j a_{ij} > 0$ for $i = 1, \ldots, m$. In this case it can be shown that $N(A)$ consists of the $2^n - 1$ non-zero lattice points on the unit cube, and their negatives [12].

But generally, the cardinality of $N(A)$ depends on the entries of A and not just on its dimensions. For example, if A has 4 rows and 2 columns, the simplices in $K(A)$ will

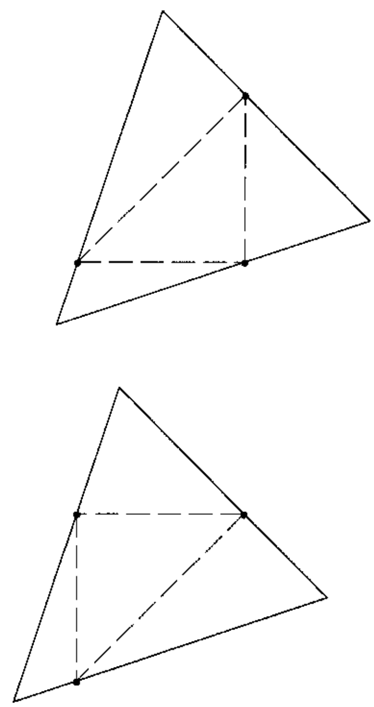

Fig. 5. The two simplices for a matrix of size 3 by 2.

consist of 4 lattice points, say $\{0, h^1, h^2, h^3\}$, such that the body $\langle 0, h^1, h^2, h^3\rangle$ contains no additional lattice points. It follows that these 4 lattice points must be the vertices of a parallelogram of unit area. But in this case, by selecting A properly, the simplicial complex can contain arbitrarily many parallelograms of unit area which are not lattice translates of each other, and, therefore, arbitrarily many neighbors.

In spite of its large cardinality, the collection of simplices associated with a matrix of size 4 by 2 has a simple structure. If we adopt the notation that h^1 is *above* the diagonal $[0, h^3]$ and h^2 *below* this diagonal, then the parallelograms representing 3-simplices in $K(A)$ can be ordered linearly so that the successor of a parallelogram is obtained by reflecting either the lattice point above the main diagonal through h^3 or the lattice point below the diagonal through h^3 (see Fig. 6). The linear set of parallelograms is given by a sequence of symbols $a, a, \dots, a, b, b, \dots, b, a \dots$ where a stands for reflecting the lattice point above the diagonal, and b for reflecting the point below. In the case in which A is an integral matrix, the number of alternations between a and b – and conversely – in the sequence is bounded by a linear function of the bit-size of A. For a non-integral matrix, the bound is linear in the logarithm of the condition number. While there are many simplices in $K(A)$ they have a polynomial description.

The simplicial complex associated with a matrix of size 4 by 3 has been studied in great detail [11]. Each simplex is described by 4 lattice points in 3 space, $\{0, h^1, h^2, h^3\}$,

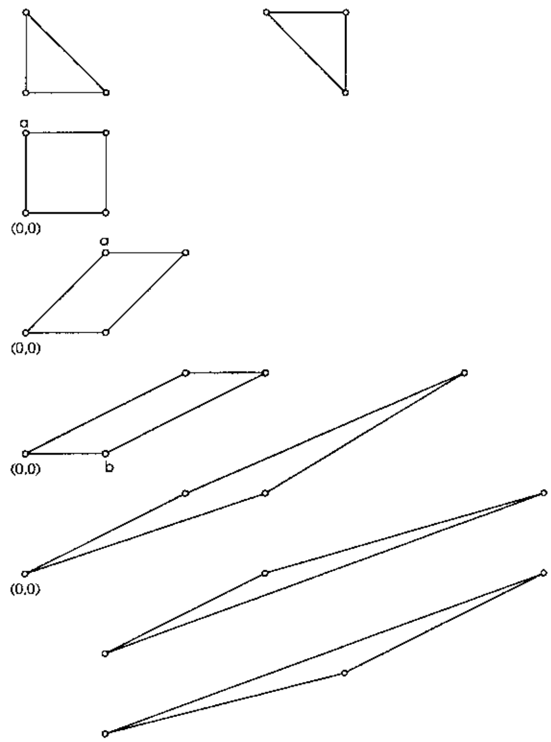

Fig. 6. The sequence of parallelograms.

such that the body $\langle 0, h^1, h^2, h^3 \rangle$ contains no additional lattice points. The convex hull of the 4 points will contain no additional lattice points, but in distinction to the case of 2 dimensions, such a tetrahedron may have an arbitrarily high volume. Moreover, the number of tetrahedra that are not lattice translates of each other, and therefore the number of neighbors, can be made arbitrarily large by a proper selection of the matrix A. Nevertheless there is a surprising structure to the collection of simplices.

A special structure when A has size 4 by 3. There exists a non-zero, integral, linear function $L(x)$ such that $|L(h)| \leqslant 1$ for every neighbor h of the origin.

A great number of computational examples strongly suggest that, in this case, the cones $C_i(A)$ have either 3 or 4 generators. If there are 3 generators, they have a determinant of ± 1. If there are 4 generators, they form the vertices of a planar parallelogram of unit area. A partial proof of this fact appears in the previously cited paper by Bárány and Scarf.

In order to illustrate the complexity that arises in this case, I include in Fig. 7 the collection of non-translation equivalent tetrahedra arising from a particular matrix. They form a 3-Torus.

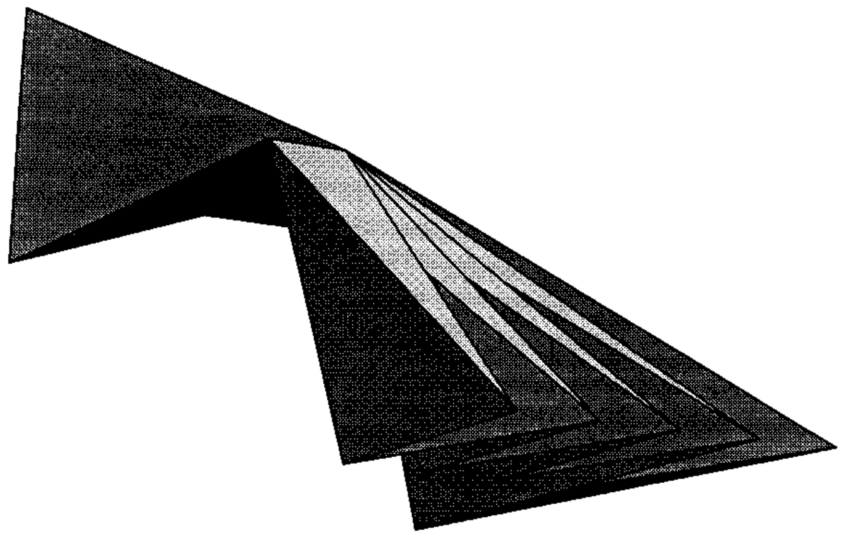

Fig. 7. The collection of tetrahedra.

4. Varying the matrix

As I shall indicate, the collection of matrices A satisfying the Main Assumption can be partitioned into a set of closed convex cones with non-empty interiors, such that any two matrices in the same cone have precisely the same simplicial complex, and, as a consequence, the same set of neighbors. The result implies that degeneracy does not influence the features of the simplicial complex unless the matrix lies on the boundary of one of these cones.

The discussion requires a definition of the simplicial complex for degenerate matrices satisfying the Main Assumption. If A is degenerate the difficulty in deciding whether h is a neighbor is that $\langle 0, h\rangle$, the smallest body of the form $\{x \mid Ax \geqslant b\}$ containing 0 and h, may have no lattice points in its interior, but may contain other lattice points k on its boundary. It may happen that $\langle 0, h\rangle$ and $\langle 0, k\rangle$ are identical for two distinct lattice points h and k, and it is unclear which of the pair is to be designated as a neighbor.

For the purpose of the present discussion, let us take the most ample definition of neighbors: i.e., the lattice point h will be a neighbor of A if $\langle 0, h\rangle$ contains no lattice points in its *interior*. And more generally, the collection of lattice points $\{h^{i_0}, \ldots, h^{i_l}\}$ will be a face of $K(A)$ if there are no lattice points interior to $\langle h^{i_0}, \ldots, h^{i_l}\rangle$. With this definition, $N(A)$ will be non-empty, finite and symmetric about the origin.

Now let us introduce the concept of a *generic* matrix. Generic matrices will be precisely those matrices interior to one of the cones referred to above.

Definition 5. The matrix A is defined to be *generic* if $a_i h \neq 0$ for all i and all $h \in N(A)$.

Of course a non-degenerate matrix will be generic, since genericity requires $a_ih \neq 0$ only for the neighbors of A and not for all lattice points. It is easy to see that if A is generic and $h \in N(A)$, then the body $\langle 0, h\rangle$ contains no lattice points in its interior or on its boundary. Moreover, if A is generic, $K(A)$ is a manifold.

Theorem 6. *Let A be a generic matrix satisfying the Main Assumption and let B be a matrix of the same size as A. Assume that*

$$\text{sign}(b_ih) = \text{sign}(a_ih) \qquad \textit{for all } i \textit{ and all } h \in N(A).$$

Then B is generic, satisfies the Main Assumption and has precisely the same simplicial complex and set of neighbors as A.

We see that a generic matrix A is interior to the polyhedral cone $C(A)$ consisting of those matrices B such that $\text{sign}(b_ih) = \text{sign}(a_ih)$ for all i and all $h \in N(A)$. The cone is the product of $m+1$ cones $C_i(A)$, one for each row of A. The cone for row i is $C_i(A) = \{b_i \mid b_ih > 0 \text{ for all } h \in N(A) \text{ with } a_ih > 0\}$. It is the *dual* to the cone generated by the set of neighbors with $a_ih > 0$. The facets of $C_i(A)$ have normals given by the extreme rays of the cone generated by the set of neighbors with $a_ih > 0$. (Computational experience suggests that the number of extreme rays is small; perhaps bounded by a function of n.)

It is easy to describe the changes in the set of neighbors that arise from a passage through a facet of $C_i(A)$. This information can be used to construct an algorithm for calculating the set of neighbors $N(A)$. Take a matrix B whose neighbors are known, and examine the linear set of matrices $A(t) = (1-t)B + tA$. It is easy to find the first value of t such that the matrix $A(t)$ passes through a facet of $C_i(B)$ for some i into a second cone. We then calculate the new collection of neighbors on the other side of this facet. Knowing the facets of the second cone, we can determine the value of the parameter t for the transition to the next cone. We continue until the matrix A, whose neighbors we wish to obtain, is reached. Of course, some care is required in order to guarantee that the sequence does not have a limit point at some value of t less than 1.

5. The global structure of $K(A)$

It was mentioned in Section 1 that the simplicial complex $K(Y)$, based on a finite set of points Y in R^{m+1}, is homeomorphic to the m-simplex. The corresponding result when Y is a lattice is that $K(A)$ is homeomorphic to R^m [3]. I shall indicate the outline of the argument in the special case in which $m = n$ [1].

Let π be the unique (up to scale) positive vector with $\pi A = 0$. We define a mapping from R^n to R^{n+1} as follows:

$$y_0 = \exp(\pi_0 t a_0 x), \quad y_1 = \exp(\pi_1 t a_1 x), \quad \ldots, \quad y_n = \exp(\pi_n t a_n x)$$

for t a large positive parameter. It is easy to see that R^n is mapped *onto* the hyperboloid sheet

$$\prod_{i=1}^{n} y_i = 1.$$

Let V be the image of Z^n under the mapping, and H, the boundary of the convex hull of V. In order to show that the complex $K(A)$ is homeomorphic to H (itself homeomorphic to R^n) one argues that, for large t, the set $\{h^{i_0}, \dots, h^{i_m}\}$ is a simplex in $K(A)$, if and only if its images are the vertices of an m-dimensional face of H. The details of the argument may be found in the papers cited above.

6. The relationship to Groebner Bases

One of the most exciting recent developments in the theory of test sets is the recognition, by Thomas and others [4,15,14], of a close relationship between test sets and Groebner Bases for certain polynomial ideals. I am far from expert in this area, but I would like to take this opportunity to provide a brief summary of the basic theme.

We are required to restrict ourselves to integral matrices in the discussion of Groebner Bases. The problem of degeneracy will arise and be treated by introducing a complete ordering on the lattice points $y = Ah$, for $h \in Z^n$. It will also be useful if, in this section, we take our inequalities in the form $Ah \leqslant b$. Then $\langle 0, h \rangle$, the smallest body of the form $\{x \mid Ax \leqslant b\}$ containing 0 and the lattice point h, will have $b_i = \max[0, a_i h]$.

For each $y = Ah$, with $h \in Z^n$, let $y^+ = \max[0, y]$ and $y^- = -\min[0, y]$. We associate with h the binomial in the variables $x_0, x_1, \dots, x_m$

$$f(h; x_0, x_1, \dots, x_m) = \prod_{i=0}^{m} x_i^{y_i^+} - \prod_{i=0}^{m} x_i^{y_i^-}.$$

Let $I[x_0, x_1, \dots, x_m]$ be the ideal of polynomials with rational coefficients in the variables $x_0, x_1, \dots, x_m$ generated by all of these binomials, as h ranges over Z^n. We then take a specific complete ordering on the integers in R^{m+1}, say the lexicographic ordering, and say that the lattice point h is lex positive if $y = Ah$ is lexicographically positive. The *leading term* $\mathrm{LT}(f(h, x))$ is given by the *monomial* $\prod_{i=0}^{m} x_i^{y_i^+}$ if h is lex positive, and $\prod_{i=0}^{m} x_i^{y_i^-}$ if h is lex negative.

The monomial $m_1(x) = \prod_{i=0}^{m} x_i^{a_i}$ is divisible by the monomial $m_2(x) = \prod_{i=0}^{m} x_i^{b_i}$ if $a_i \geqslant b_i$ for all i. If h is lex positive, then the leading term of $f(h; x_0, x_1, \dots, x_m)$ has the exponent $\max[0, a_i h]$. If k is also lex positive, then the leading term of $f(h; x_0, x_1, \dots, x_m)$ will be divisible by the leading term of $f(k; x_0, x_1, \dots, x_m)$ if and only if

$$\max[0, a_i h] \geqslant \max[0, a_i k] \quad \text{for all } i, \text{ or } \langle 0, h \rangle \supseteq \langle 0, k \rangle.$$

Division of the leading terms of the binomials in I is identical with inclusion of the corresponding bodies $\langle 0, h\rangle$.

Definition 7. A *Groebner Basis* for the binomial ideal I is a finite set of binomials $f^1, f^2, \ldots, f^k$, associated with the lex positive lattice points $h^1, h^2, \ldots, h^k$, such that the leading term of every binomial in the ideal is divisible by the leading term of one of the f^i. The Groebner Basis is *minimal* if no proper subset is also a Groebner Basis.

Consider the partial ordering, say $\succeq_A$, on lex positive lattice points h given by

$$h \succeq_A k \quad \text{if and only if} \quad \langle 0, h\rangle \supseteq \langle 0, k\rangle.$$

h is a *least* element under this partial ordering if there is no lex positive k with $k \prec_A h$. The set of least elements may be partitioned into disjoint sets $L_1, \ldots, L_k$ such that lattice points in the same L_i are equivalent under $\succeq_A$ (i.e., they have the same bodies $\langle 0, h\rangle$) and lattice points in different L_i are incomparable. A minimal Groebner Basis is given by a selection of one lattice point from each of these sets. The unique *reduced* Groebner Basis selects from each L_i the lexicographically largest lattice point. (It is this point where degeneracy is resolved.)

There is a slight difference between neighbors and members of a minimal Groebner Basis. The set $\langle 0, h\rangle$ for an element h of a minimal basis contains no lex positive lattice points in its interior. But it may contain lex negative lattice points and not qualify as a neighbor. This difference is relatively minor in the discussion of test sets, but it offers a substantial problem for the construction of the simplicial complex $K(A)$ whose edges are the neighbors $N(A)$. I see no way to construct a useful simplicial complex all of whose edges are the elements in a reduced Groebner Basis.

The elements of the reduced Groebner Basis may be calculated by the Buchberger algorithm (see, for example, [5]). The algorithm can be translated directly into the language of our polyhedra $\{x \mid Ax \leqslant b\}$. It is an interesting question whether some variant of the Buchberger algorithm can be adapted to calculate neighbors for non-integral, generic matrices.

The Graver test was introduced in 1975 [6]. (Also see [13].) In our formulation the Graver test set consists of the set of lattice points $h \in Z^n$ which cannot be written as the sum of two lattice points h^1 and h^2 such that $\operatorname{sign}(a_i h^j) = \operatorname{sign}(a_i h)$ for all i and both j. It is easy to see that the Graver test contains *all* least elements under the partial ordering $\succeq_A$, for any term order.

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