ON DIFFERENTIAL GAMES WITH SURVIVAL PAYOFF

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§1. SURVIVAL GAMES

In a paper appearing elsewhere in this volume, a class of differential games with integral payoffs is discussed by Fleming [5]. The games are defined and it is shown that, under fairly stringent conditions, the game will have a value and each player will possess ϵ-effective strategies. We shall, in this paper, discuss differential games with a survival payoff. Basically, because of the fact that survival games may last for an infinite length of time, and the games discussed in the reference mentioned above are explicitly limited to a finite time, the techniques available for the treatment of survival games are more involved and comparable results are more difficult to obtain, than in the case of games with an integral payoff. For these reasons, the emphasis in this paper will be on proving a result which is somewhat weaker than the existence of a value for survival games. What we shall do is define a series of approximating games with a discrete time parameter, and show that under conditions similar to those given by Fleming, both the upper and lower values of the approximating games converge to the same limit, as the grid size for the time parameter tends to zero.

We shall not make any statements about the convergence of optimal strategies; but, inasmuch as the value functions for the discrete games characterize optimal play, their limit may be expected to give a reasonable indication of the limiting characteristics of optimal play.

The discrete games that we are interested in are readily seen to be generalizations of the survival games considered by Hausner, [8], Peisakoff, [12], Bellman, [1, 2], Blackwell [2], LaSalle [1], and Milnor and Shapley [11]. We consider a bounded n-dimensional region R, with boundary B. A bounded n-dimensional vector-valued function $g(x; y, z)$ is given, which for each x in R either is a continuous vector-valued function on the (y, z) unit square, or is a vector-valued matrix. The game is played as follows: A vector x_o, interior to the region R, is chosen. On the first move, player I chooses a particular value of y and player II simultaneously chooses a particular value of z, and subsequently a straight line is drawn from the point x_0 to the point $x_1 = x_0 + \delta g(x_0; y, z)$. The players then make another choice of y and z and we draw a straight line from x_1 to $x_2 = x_1 + \delta g(x_1; y, z)$. This process is repeated until the path penetrates the boundary, at which point the game is terminated. The payoff is defined as follows: We have a function $b(x)$, defined and continuous on the boundary B; if the game terminates at the point x, then the payoff to the first player is $b(x)$, and the second player receives the negative of this amount. In order to complete the definition of the game, we should define the payoff in the event that the game does not terminate, but, as we shall see, this is a matter of indifference to us. Of course, both players are permitted to use mixed strategies at every stage of the game. In order to indicate the dependence of this game upon the parameter δ we shall designate this game by G_δ.

As δ tends to zero, the motion of the game tends more and more to be described by the equations $\dot{x} = g(x; y, z)$, and these are the defining equations of a differential game. Let us assume for the moment that G_δ has a value $W_\delta(x)$, x being the initial starting point. We shall give sufficient conditions for this sequence of functions to converge, and also obtain a system of differential inequalities whose solution represents the limiting function.

Conditions for the existence of $W_\delta(x)$ are, at present, known only for the one-dimensional case, but as we shall see later in the paper, it is not necessary to assume the existence of the value. It will actually be true that both the upper and lower values (defined appropriately) will converge to the same limit. We define $W_\delta^+(x)$ to be the best that player I can guarantee himself, using mixed strategies, in G_δ. That is, W_δ^+ is the Sup Inf over the payoff, in mixed strategies. $W_\delta^-(x)$ is defined to be the Inf Sup of the payoff. It is true that $W_\delta^+(x) \leq W_\delta^-(x)$.

§2. SURVIVAL GAMES WITH FINITE TIME

As was mentioned in the previous section, one of the basic differences between survival games and the type of games with integral payoffs treated by Fleming in [5], is that the latter are played for a finite length of time, whereas, the former may continue indefinitely. It is possible to modify survival games so as to make them last for a finite length of time, and we shall devote this section to a discussion of games of this sort. The purpose will be to point out some important differences between infinite survival games and their finite counterparts.

In order to define the discrete analogues of a finite survival game, in addition to the data given in Section 1, we have to fix a time T, and also require that the boundary function $b(x)$ be extended continuously throughout the interior of the region R. The game

$$G_{\frac{T}{n}}(x_0, T)$$

is defined as follows: Starting from x_0, a broken line path is constructed from x_0 to x_1, and from x_1 to x_2, etc., in the same manner as in the infinite survival game described in Section 1, with $\delta = \frac{T}{n}$. If this path penetrates the boundary before time T, the payoff to player I is $b(x)$, and the negative of this amount to player II, where x is the point of penetration. If penetration does not occur before time T, then the game is stopped at time T, and the payoff is $b(x)$, where x is the position at time T.

It is easy to see that

$$G_{\frac{T}{n}}(x_0, T)$$

has a value, which we denote by

$$W_{\frac{T}{n}}(x_0, T).$$

We are interested in determining a set of conditions which are sufficient to insure that

$$W_{\frac{T}{n}}(x_0, T)$$

converges as n becomes infinite.

THEOREM 1. If $W(x, T)$ is a continuously differentiable solution of the equation

$$\frac{\partial W}{\partial t} = \mathrm{Val}\left(\frac{\partial W}{\partial x}, g\right),$$

for x in the closure of R and $t \geq 0$, which satisfies the boundary conditions

1. $W(x, t) = b(x)$ for x on the boundary, and any t, and
2. $W(x, 0) = b(x)$ for x in, or on the boundary, of R, then

$$\lim_{n \to \infty} W_{\frac{T}{n}}(x_o, T) = W(x_o, T) .$$

This result may be proved quite easily, either by an adaptation of the argument given by Fleming in [5], or by a simple argument using the theory of martingales, and we shall not present the proof here. Actually, one can go even further, and show that $W(x_o, T)$ is the value of the limiting finite-time survival game, if this game is defined correctly.

Theorem 1 suggests that the values of the discrete analogues to an infinite survival game converge to the solution of

$$\text{Val}\left(\frac{\partial W}{\partial x}, g\right) = 0,$$

which satisfies the boundary condition $W(x) = b(x)$ for x on the boundary. In general, this is false. The simplest way to see that this conjecture is false is to notice that, in general, the functional equation

$$\text{Val}\left(\frac{\partial W}{\partial x}, g\right) = 0,$$

with the associated boundary condition possesses more than one solution. More specifically, there is a large class of functions g, for which the equation

$$\text{Val}\left(\frac{\partial W}{\partial x}, g\right) = 0$$

is true for <u>every</u> continuously differentiable function W. The class of survival games which have this property are the furthest removed, in both technique and final results, from finite survival games, and it is this class of games which we shall discuss in the remainder of this paper.

§3. UNBIASED DIFFERENTIAL GAMES

In the previous section it was mentioned we shall restrict our attention to a specific class of survival games, which we shall call unbiased games. They are, roughly speaking, described by saying that at each point neither player can force any particular direction.

> DEFINITION 1. A differential game is said to be <u>unbiased</u> if for every x in R, and for every vector c, the scalar product $(c, g(x; y, z))$, when considered as a game over the (y, z) space, has value zero.

We shall discuss some examples of unbiased games in detail later on, but let us, for the moment, examine a specific game in the plane. Let the region and the boundary value be arbitrary, and let $g(x; y, z)$ be independent of x and equal to the matrix

$$\begin{bmatrix} (1, 0) & (-1, 0) & (0, 1) & (0, -1) \\ (0, -1) & (1, 0) & (-1, 0) & (0, 1) \\ (0, 1) & (0, -1) & (1, 0) & (-1, 0) \\ (-1, 0) & (0, 1) & (0, -1) & (1, 0) \end{bmatrix} .$$

This game is a specific example of a general class of unbiased games, i.e., when $g(x; y, z)$ is given by a cyclic matrix with row-sum equal to zero. The optimal strategy for either player in any projection of the above matrix, i.e., for any linear combination of the matrices, is to play each row, or column, with probability one-fourth. In the game G_δ, all of the elements of strategy customarily associated with a game are lacking; if both players play optimally, the resulting stochastic process is a simple random walk in which the point moves by an amount δ, with probability one-fourth in the north, east, south, or west directions. The value function $W_o(x_1, x_2)$ satisfies the equation

$$W_\delta(x_1, x_2) = \frac{1}{4} W(x_1, x_2 + \delta) + \frac{1}{4} W_\delta(x_1, x_2 - \delta) + \frac{1}{4} W_\delta(x_1 + \delta, x_2) + \frac{1}{4} W_\delta(x_1 - \delta, x_2) ,$$

and is close to $b(\mu)$ when x is near the boundary point μ. It is well-known that, as δ tends to zero, these functions converge. The limit function is harmonic in R and assumes the boundary value $b(x)$.

In the general case our results will be somewhat similar; the primary difference will be the replacement of the Laplacian by a considerably more complex differential operator. We need some definitions.

DEFINITION 2. We define D_{yz} to be the first-order linear differential operator

$$\sum_k g^k(x; y, z) \frac{\partial}{\partial x^k} .$$

The operator D^2_{yz} is defined to be

$$\sum_{k,\ell} g^k(x; y, z) g^\ell(x; y, z) \frac{\partial^2}{\partial x^k \partial x^\ell} .$$

DEFINITION 3. We define the operator $L(f)$ to be

$$\lim_{\delta \to 0+} \operatorname{Val} \left\| \left(D_{yz} + \frac{\delta}{2} D_{yz}^2 \right) f \right\| / \delta .$$

This latter definition is very important for us, and needs some comment. Let us first apply this definition to the cyclic game just discussed. In this case it is easy to verify that the matrix

$$\left\| \left(D_{yz} + \frac{\delta}{2} D_{yz}^2 \right) f \right\|$$

becomes

$$\begin{bmatrix} f_1 + \frac{\delta}{2} f_{11} & -f_1 + \frac{\delta}{2} f_{11} & f_2 + \frac{\delta}{2} f_{22} & -f_2 + \frac{\delta}{2} f_{22} \\ -f_2 + \frac{\delta}{2} f_{22} & f_1 + \frac{\delta}{2} f_{11} & -f_1 + \frac{\delta}{2} f_{11} & f_2 + \frac{\delta}{2} f_{22} \\ f_2 + \frac{\delta}{2} f_{22} & -f_2 + \frac{\delta}{2} f_{22} & f_1 + \frac{\delta}{2} f_{11} & -f_1 + \frac{\delta}{2} f_{11} \\ -f_1 + \frac{\delta}{2} f_{11} & f_2 + \frac{\delta}{2} f_{22} & -f_2 + \frac{\delta}{2} f_{22} & f_1 + \frac{\delta}{2} f_{11} \end{bmatrix}$$

which is itself cyclic and therefore has the value $\frac{\delta}{4}(f_{11} + f_{22})$. If we divide by δ and let δ tend to zero, we see that $L(f)$ is, aside from a constant factor, the Laplacian.

Another interesting case occurs when the functions $g(x; y, z)$ are again independent of x and are given by the two-dimensional matrix

$$\begin{bmatrix} (1, 0) & (-1, 0) \\ (-1, 0) & (1, 0) \\ (0, 1) & (0, -1) \\ (0, -1) & (0, 1) \end{bmatrix} .$$

This game is unbiased, and the matrix

$$\left\| \left(D_{yz} + \frac{\delta}{2} D_{yz}^2 \right) f \right\|$$

becomes

$$\begin{bmatrix} f_1 + \frac{\delta}{2} f_{11} & -f_1 + \frac{\delta}{2} f_{11} \\ -f_1 + \frac{\delta}{2} f_{11} & f_1 + \frac{\delta}{2} f_{11} \\ f_2 + \frac{\delta}{2} f_{22} & -f_2 + \frac{\delta}{2} f_{22} \\ -f_2 + \frac{\delta}{2} f_{22} & f_2 + \frac{\delta}{2} f_{22} \end{bmatrix} .$$

The maximizer can ensure himself of at least $\frac{\delta}{2}$ Max (f_{11}, f_{22}) by playing either the first two rows (if $f_{11} \geq f_{22}$) or the last two rows (if $f_{22} > f_{11}$) with equal probabilities, and the minimizer can hold him to at most this amount by playing the two columns with equal probabilities. It follows that $L(f) = \frac{1}{2}$ Max (f_{11}, f_{22}).

The definition of $L(f)$ may be put into another form by means of a result due independently to Gross [7] and Mills [10]. It is based on the observation that $L(f)$ is equal to the derivative of

$$\text{Val} \quad \left\| \left(D_{yz} + \frac{\delta}{2} D^2_{yz} \right) f \right\|$$

with respect to δ, when δ is zero.

LEMMA 1. Let $P(f, x)$ represent the class of optimal strategies for the maximizing player in the game $\|D_{yz}f\|$, and $Q(f, x)$ the corresponding class for the minimizing player. Then

$$L(f) = \underset{p \in P}{\text{Max}} \; \underset{q \in Q}{\text{Min}} \iint \frac{1}{2} D^2_{yz}(f) dp(y) dq(z).$$

There is an analogous statement for matrices. The proof of the statement for matrices may be found in the paper by Mills; the proof of the statement for the continuous case follows the same lines.

§4. THE MAIN RESULTS

We have given one example, the cyclic case, in which the limit of the value functions exists, and is equal to the solution of the Laplace equation which assumes the correct boundary values, that is, which is equal to $b(x)$ on the boundary. In general, the result will be analogous to this; the value functions, or, if these do not exist, the upper and lower values, will converge to the solution of $L(f) = 0$, which is equal to $b(x)$ on the boundary. Our actual technique will be to approach this result from above and below, in much the same manner that super- and subharmonic functions may be used to approximate harmonic functions.

THEOREM 2. Let $L(f)$ be the operator associated with an unbiased differential game. Let $f(x)$ be a function with continuous bounded derivatives up to the third order in some open set S containing the closure of R in its interior, and which satisfies the conditions:

1. $L(f) \geq c > 0$, for all interior points of R,

2. $f(x) \leq b(x)$ for x on the boundary B.

Then

$$\underline{\mathrm{Lim}}_{\delta \to 0} W_\delta^+(x) \geq f(x).$$

PROOF. Let us first of all choose δ so small that, if x is an interior point of R, then $x + \delta g(x; y, z)$ is in S. We now define a strategy for the first player in G_δ, by defining a set of probability distributions $dp(y; x, \delta)$. If the position of the game is x, then the first player is to play y with probability $dp(y; x, \delta)$. We define $dp(y; x, \delta)$ to be any optimal strategy for the first player in the game

$$\left(D_{yz} + \frac{\delta}{2} D_{yz}^2 \right) f \quad .$$

Let us examine the variations in the function $f(x)$ as this strategy develops. We know that

$$f(x + \delta g(x; y, z)) = f(x) + \delta D_{yz} f(x) + \frac{\delta^2}{2} D_{yz}^2 f(x) + \delta^3 r(x, \delta) ,$$

where $r(x, \delta)$ is bounded for all δ and x in the region. If we imagine the play to have started at x_o, and if (x_n) represents a typical sequence of plays, then we may say that

$$E \left(f(x_n) | x_o, \ldots, x_{n-1} \right) \geq f\left(x_{n-1} \right) + \delta \, \mathrm{Val} \left\| \left(D_{yz} + \frac{\delta}{2} D_{yz}^2 \right) f \right\| - \delta^3 M.$$

The way in which we have chosen the strategy for the first player implies that

$$E \left(f(x_n) | x_o, \ldots, x_{n-1} \right) \geq f\left(x_{n-1} \right) + c\delta^2 - \delta^3 M,$$

and for δ sufficiently small

$$E \left(f(x_n) | x_o, \ldots, x_{n-1} \right) \geq f\left(x_{n-1} \right) + \frac{c\delta^2}{2} \quad .$$

Since the function $f(x)$ is bounded inside the region R, we can deduce that with probability one the sequence (x_n) will leave the region. Let n^* represent the random variable which is the length of time that the process continues; that is, n^* is the number of steps, starting from x_o and continuing until the process penetrates the boundary for the first time. It is a standard result from the theory of martingales that $E(f(x_{n*})) \geq f(x_o)$ [4, p. 302]. The point x_{n*} is within δM of the boundary point x, if this is the point of penetration of the boundary; and since f is continuous, we may conclude that $E(b) \geq f(x_o) - \epsilon$, where ϵ tends to zero with δ. It follows that

$$\underline{\text{Lim}}_{\delta \to 0} W_\delta^+(x) \geq f(x).$$

It is not difficult to give conditions which guarantee that the class of functions satisfying the conditions of the theorem is not empty. One condition, which we shall not use later, is that $|g(x;\, y,\, z)|^2$ be bounded away from zero. Consider the function $f(x) = \frac{1}{2}|x|^2 - d$, where d is a positive constant chosen so large as to make $f(x)$ less than the minimum of $b(x)$. Then $D_{yz}^2(f)$ is equal to $|g(x;\, y,\, z)|^2$, and if c is positive and less than the minimum of $\frac{1}{2}|g(x;\, y,\, z)|^2$ we easily obtain

$$\iint \frac{1}{2} D_{yz}^2(f)dp(y)dq(z) \geq c$$

for any distributions p and q. Applying Lemma 1, we see that $L(f) \geq c$.

It is clearly possible to reverse the conditions of Theorem 2, and obtain an upper bound for

$$\overline{\text{Lim}}_{\delta \to 0} W_\delta^-(x) \quad .$$

We would, in this case, be dealing with the analogue of a superharmonic function. Continuing in this spirit, let us define two classes of functions.

DEFINITION 4. A function $f(x)$ will be said to be in class M^+, if it has continuous bounded derivatives up to the third order in some open set containing the closure of the region R, satisfies $L(f) \geq c > 0$ inside the region, and is less than or equal to b on the boundary.

DEFINITION 5. A function $f(x)$ will be said to be in class M^-, if it has continuous bounded derivatives up to the third order in some open set containing the closure of the region R, satisfies $L(f) \leq c < 0$ inside the region, and is greater than or equal to b on the boundary.

Our results so far may be summarized by saying that any function in M^+ is a lower bound for

$$\underline{\text{Lim}}_{\delta \to 0} W_\delta^+(x) .$$

It is also true that the pointwise Sup of all functions in M^+, which we denote by $W^+(x)$, is a lower bound for

$$\underline{\text{Lim}}_{\delta \to 0} W_\delta^+(x) ;$$

and similarly the function $W^-(x)$, defined to be the pointwise Inf of all functions in M^-, forms an upper bound for

$$\overline{\text{Lim}}_{\delta \to 0} W_\delta^-(x) .$$

Suppose that we can demonstrate that $W^+(x) = W^-(x) = W$. Then

$$W^+(x) \leq \underline{\text{Lim}}_{\delta \to 0} W_\delta^+(x) \leq \overline{\text{Lim}}_{\delta \to 0} W_\delta^-(x) \leq W^-(x) ;$$

and it follows that $W_\delta^+(x)$ and $W_\delta^-(x)$ both converge to W. Our goal will therefore be to impose a sufficient number of conditions on the differential game so that, first of all, the classes M^+ and M^- are each non-empty and second, that W^+ be equal to W^-. We shall, in the following theorem, present such a set of conditions. They are certainly unnecessarily strict, and it is quite probable that a closer analysis of the problem will uncover a more satisfactory set of sufficient conditions. We shall, in the discussion following the theorem, present several alternative sets of conditions.

THEOREM 3. Let the functions $g(x; y, z)$ have the property that any pair of optimal strategies in the scalar product game $(c, g(x; y, z))$, for c different from zero, produce $E\{(c, g(x; y, z))^2\} > 0$. Uniformly in x.

Let $W(x)$ be a function with continuous bounded derivatives up to the third order in some open set containing the closure of R in its interior, and which satisfies the conditions:

1. $L(W) = 0$, for all interior points of R,
2. $W(x) = b(x)$, for x on the boundary B,
3. The gradient of W is different from zero in R.

Then $W_\delta^+(x)$ and $W_\delta^-(x)$ both converge to $W(x)$.

PROOF. Let us first notice that the condition imposed on the functions $g(x; y, z)$ is sufficient to guarantee that the classes M^+ and M^- are both not empty. We use the same function as before, that is, $f(x) = \frac{1}{2}|x|^2 - d$, so that $D^2_{yz}(f) = |g(x; y, z)|^2$, which in turn is greater than $|(c, g(x; y, z))|^2/|c|^2$, for any non-zero c. It follows that if both players play optimally in $(c, g(x; y, z))$, then $E(D^2_{yz}f) > 0$, and this is sufficient, if we recall Lemma 1, to put f in M^+. In a similar manner, a member of M^- may be defined.

All that remains to be shown is that W^+ is equal to W^-. Let us consider the functions

$$W_\epsilon(x) = W(x) - \epsilon \operatorname*{Max}_{x \epsilon B} b^2(x) + \epsilon W^2(x) .$$

We wish to show that each one of these functions is in M^+. It is clear that $W_\epsilon(x) \leq b(x)$, for x on the boundary. Let us now show that $L(W_\epsilon) > 0$. If we refer to Lemma 1, we recall that $L(W_\epsilon)$ is equal to $\frac{1}{2}$ Max Min $E(D^2_{yz}W_\epsilon)$, when both players play optimally in the game $D_{yz}W_\epsilon$. But $D_{yz}W_\epsilon = (1 + 2\epsilon W)D_{yz}W$, so that if ϵ is chosen so small as to make $(1 + 2\epsilon W)$ positive, we see that $L(W_\epsilon)$ is equal to $\frac{1}{2}$ Max Min $E(D^2_{yz}W_\epsilon)$, when both players play optimally in $D_{yz}W$. But since $D^2_{yz}W_\epsilon$ is equal to $(1 + 2\epsilon W)D^2_{yz}W + 2\epsilon(D_{yz}W)^2$, we conclude that $L(W_\epsilon) = (1 + 2\epsilon W)L(W) + 2\epsilon E\{(D_{yz}W)^2\}$. Since $L(W) = 0$, and our assumptions about the nature of $g(x; y, z)$ and the non-vanishing of the gradient of W imply that $E\{(D_{yz}W)^2\} > 0$, we see that W_ϵ is in M^+.

W^+ was defined to be the sup of all elements in M^+, so that $W_\epsilon \leq W^+$. When ϵ tends to zero, W_ϵ tends to W so that $W \leq W^+$. In the same manner we conclude that $W \geq W^-$. We have remarked previously that $W^- \geq W^+$. It is therefore true that $W^+ = W^- = W$, and the theorem is proved.

§5. REMARKS

1. We would, first of all, like to comment upon some of the conditions imposed in Theorem 2. There are many examples in which the conditions of Theorem 2 are violated, and in which convergence occurs. If we examine the example in Section 3 in which $L(f)$ turned out to be $\text{Max}(f_1, f_2)$, we see that it violates the condition that optimal strategies in any projection actually produce non-zero motion in that projection (this is the verbal translation of the first condition in Theorem 3). It is, however, true that if $L(W) = 0$, then $L(W + \frac{1}{2}\alpha|x|^2)$ is greater than zero for positive α, and less than zero for negative α. This is sufficient to prove that there are functions arbitrarily close to W which lie in M^+, and ones which lie in M^-, and this is the basic idea of Theorem

2. It is also possible, in this case, to dispense with the condition that the gradient of W be different from zero. In general, if $L(f + \frac{\alpha}{2}|x|^2)$ is greater than $L(f)$ for α positive, and less than $L(f)$ for α negative, then the existence of a solution to $L(W) = 0$, with the correct boundary conditions, is sufficient to yield the conclusion of Theorem 2. It would be very interesting to see what classes of games produce operators with this property. It is, of course, the non-linearity of the operator $L(f)$ which makes it difficult to obtain any general results about its behavior.

2. We have restricted our treatment so far to unbiased differential games. In this type of game, neither player has the option of forcing the expected change of position to be in a favorable direction. He must, at each stage of the game, rest his hopes upon the variance of his choices. It is precisely this feature of unbiased games which gives rise to operators which resemble elliptic second-order differential operators.

In general, survival games are not unbiased. For example, if we expect the limiting game to have pure strategies for each player, then it seems unreasonable to impose a condition which makes the choice of the variance the only strategic element. It is quite possible, however, that in a game which is not unbiased, the players may, at various moments, be forced to pay some attention to the variance of their moves, so that the resulting play will be governed by a combination of first- and second-order operators.

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