CF-50426

An Elementary Proof of a Theorem on the Core of an N Person Game

by

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In a recent paper I described a condition which is sufficient to guarantee the existence of a point in the core for a general n-person game without the assumption of transferable utility [2]. In the present note I shall show that the theorem is a simple consequence of the existence of Nash equilibrium points for a finite 2-person nonzero sum game. Lemke and Howson have recently shown [1] that the Nash theorem, which is customarily demonstrated by means of the Kakutani fixed point theorem, does in fact have an elementary and constructive proof for the case of two players. This implies that the general theorem on the existence of the core does have a proof in which no fixed point theorems are required.

When the sufficient conditions are satisfied the algorithm of Lemke and Howson may be applied, with some minor modifications, to produce a simple iterative procedure for calculating a point in the core. I shall expand on this topic when the material is written out in a more leisurely form.

In the paper referred to above, I have shown that the sufficient conditions are an immediate consequence of the assumption of convex preferences in an exchange economy. We therefore have an elementary and constructive proof of the existence of the core in an exchange economy when the customary convexity conditions are assumed, and by letting the number of consumers tend to infinity this provides us with an elementary proof of the existence of equilibrium prices.

1. Definitions

I consider an n-person game without transferable utility defined as follows. N is the set of all players and S an arbitrary subset of N. E^S is the euclidean space of dimension equal to the number of players in S, and whose coordinates have as subscripts the members of S. For any vector $x \in E^N$, we define x^S to be the projection of x to E^S.

For each S we have a set V_S in E^S with the following assumptions:

a. V_S is closed, and not empty.

b. $x \in V_S$, $y \in E^S$ with $y \leqq x \Longrightarrow y \in V_S$

c. The points in V_N which give no person less than he could achieve by himself form a bounded set.

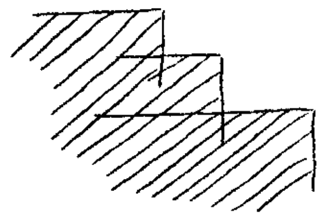
A point $x \in V_N$ is blocked by S, if there is a point $y \in V_S$ with $y_j > x_j$ for all $j \in S$. $x \in V_N$ is in the core of the game if it is blocked by no proper subset S.

A collection of subsets $T = \{S\}$ is "balanced" if the system $\sum_{S \supset j} \delta_S = 1$, $\delta_S \geq 0$ has a solution with $\delta_S = 0$ for all sets <u>not</u> in T.

The game is "balanced" if for any balanced collection T, a vector x with the property that $x^S \in V_S$ for each $S \in T$, must also be in V_N.

Theorem: A balanced n person game has a core.

I will prove the theorem in the case where each V_S is a finite union of "rectangles," as the figure indicates.



The limiting process to obtain the general result is trivial.

Lemma 1. Let C and A be $m \times n$ matrices such that $\{x \mid x \geqq 0 , Ax \leqq e\}$ is bounded, where e is the vector all of whose components are 1 . Then there is an $x \geqq 0$, with $\Sigma a_{ij} x_j \leq 1$ for each i , so that if we define $c_i = \min\limits_{x_j > 0} c_{ij}$, it will be true that for every k , there is at least one i with $\sum\limits_j a_{ij} x_j = 1$ and $c_i \geq c_{ik}$.

Proof: There is no loss in generality in assuming that $c_{ij} > 0$, since the Lemma is invariant if we raise all of the elements of C by the same amount. Let η be a positive number eventually tending to infinity and consider the 2-person game with payoff matrices A and $B = (b_{ij})$ where $b_{ij} = -(c_{ij})^{-\eta}$. Both players are assumed to be maximizing; the rows are strategies for the player with matrix A and the columns for B . Then by the Nash theorem there are probability distributions x^* and y^* such that

$$\Sigma\Sigma y_i^* a_{ij} x_j^* \geq \Sigma\Sigma y_i a_{ij} x_j^* \quad \text{and}$$

$$\Sigma\Sigma y_i^* (c_{ij})^{-\eta} x_j^* \leq \Sigma\Sigma y_i^* (c_{ij})^{-\eta} x_j$$

for any probability distributions x and y .

Then for each i , $\Sigma a_{ij} x_j^* \leq \Sigma\Sigma y_i^* a_{ij} x_j^*$ with $y_i^* = 0$ if there is strict inequality. Moreover, for any k

$$\Sigma\Sigma y_i^* c_{ij}^{-\eta} x_j^* \leq \sum_i y_i^* c_{ik}^{-\eta} .$$

There must, for this k , be some index i with $y_i^* > 0$ (and therefore $\Sigma a_{ij} x_j^* = \Sigma\Sigma y_i^* a_{ij} x_j^*$) for which

$$\sum_j c_{ij}^{-\eta} x_j^* \leq c_{ik}^{-\eta} \quad \text{or}$$

$$c_{ik} \leq \left(\sum_j c_{ij}^{-\eta} x_j^* \right)^{-1/\eta} .$$

Let j_1 be any index so that $x_{j_1}^* > 0$. Then

$$c_{ik} \leq \frac{c_{ij_1}}{(x_{j_1}^*)^{1/\eta}} \left[1 + \frac{(c_{ij_1})^{\eta}}{x_{j_1}^*} \sum_{j \neq j_1} \frac{x_j^*}{c_{ij}^{\eta}} \right]^{-1/\eta}$$

$$\leq \frac{c_{ij_1}}{(x_{j_1}^*)^{1/\eta}} .$$

Now we let $\eta \to \infty$. The particular choice of the i index associated with each k , and the vector x^* will of course depend on η , but we can select things in such a way that x^* converges to x , y^* converges to y and that the choice of the index i for each k remains constant, and continues to be selected from the set of indices for which $\Sigma a_{ij} x_j^* = \Sigma\Sigma y_i^* a_{ij} x_j^*$. This latter number will also converge to, say, a .

Now if $x_j > 0$, then $x_j^* > 0$ for large η and we have

$$c_{ik} \leq \frac{c_{ij}}{(x_j^*)^{1/\eta}} \longrightarrow c_{ij} . \quad \text{Therefore}$$

$$c_{ik} \leq \min_{x_j > 0} c_{ij} ,$$ with the index i selected from the set of indices for which $\Sigma a_{ij} x_j = a$. Since $a > 0$, for otherwise the set $\{x \mid x \geqq 0 , \; A x \leqq e\}$ would not be bounded, the vector x/a satisfies the conclusion of the lemma.

It should be remarked that for actual calculations the case "$\eta = \infty$" may be solved directly by the Lemke Howson algorithm without passing to the limit.

The next lemma involves a slight modification of the conclusion of Lemma 1, related to the possibility of some zero entries in the A matrix.

Lemma 2. The same assumptions as in the previous lemma. But now we define

$$c_i = \min_{\substack{x_j > 0 \\ a_{ij} \neq 0}} c_{ij} ,$$

and conclude that for each k there is an i with $\Sigma a_{ij} x_j = 1$ <u>and</u> $a_{ik} \neq 0$ for which $c_i \geq c_{ik}$.

The new c_i is possibly larger than the one of the previous lemma, but the selection of the index i to go with a column k is now from a smaller set.

The proof of the lemma is an immediate consequence of Lemma 1 applied to a matrix C^1 where $c_{ij}^1 = c_{ij}$ if $a_{ij} \neq 0$ and $c_{ij}^1 = M$

(some very large number) if $a_{ij} = 0$. There is then a vector $x \geqq 0$ with $\Sigma a_{ij} x_j \leq 1$ and such that if $c_i^1 = \min\limits_{x_j > 0} c_{ij}^1$, then for every k we have an i such that $\Sigma a_{ij} x_j = 1$ and $c_i^1 \geq c_{ik}^1$. But if $\Sigma a_{ij} x_j = 1$ for a particular index i, it must be true that $x_j > 0$ for some nonzero a_{ij} and therefore c_i^1 will be less than M, and in fact given by $c_i^1 = \min\limits_{\substack{x_j > 0 \\ a_{ij} \neq 0}} c_{ij}$.

In addition if $c_i^1 \geq c_{ik}^1$, then c_{ik}^1 is less than M and we must have $a_{ik} \neq 0$ and $c_{ik}^1 = c_{ik}$. This proves Lemma 2.

Now we turn to a proof of the main theorem. Assume that for each proper subset S, V_S is the union of ℓ_S sets V_S^1, V_S^2, ... $V_S^{\ell_S}$ with

$$V_S^j = \{x \in E^S | x_i \leq c_{ij}(S)\}$$

We define two matrices A and C, with n rows, indexed by i, the players in the game, and with a column corresponding to every set V_S^j as j ranges from 1 to ℓ_S, and as S ranges over all proper subsets of N. For the A matrix the entry in row i and column (j,S) will be 1 if $i \in S$ and zero otherwise, so that a column in A, corresponding to a set S, appears ℓ_S times.

For the C matrix the entry in row i and column (j,S) is $c_{ij}(S)$.

Now let us apply Lemma 2. We obtain a nonnegative vector $x = (x_{j,S})$ with $Ax \leqq e$ or

$$\sum_{S \ni i} \left(\sum_{j=1}^{\ell_S} x_{j,S} \right) \leq 1 \quad \text{for all } i .$$

The vector c_i is given by $\min\limits_{\substack{x_{j,S} > 0 \\ S \ni i}} c_{ij}(S)$.

Let us notice that $\sum\limits_{S \ni i} \left(\sum\limits_{j=1}^{\ell_S} x_{j,S} \right) = 1$ for all i , for if there were strict inequality for i^* , then considering a column (j,S) in which S is the single element set $\{i^*\}$, there would be no indices to which Lemma 2 could be applied, i.e., indices for which $a_{i,j,S} > 0$ and i corresponding to a zero slack.

We shall show that c is in the core of the game. Suppose that c is blocked by a set S . Then there is some k so that $c_i < c_{ik}(S)$ for all $i \in S$. But Lemma 2 says that $c_i \geq c_{ik}(S)$ for some i with $a_{i,j,S,} = 1$ (or $i \in S$) . This shows that c is not blocked by any set S . We need only show that $c \in V_N$ to finish the proof.

If we define $\delta_S = \sum\limits_{j=1}^{\ell_S} x_{j,S}$, we see that $\sum\limits_{S \ni i} \delta_S = 1$ for all i . Let us define T to be the collection of all S with $\delta_S > 0$. T is a balanced collection.

Now for each S with $\delta_S > 0$, $x_{j,S}$ must be positive for some j's . Define, for each such S , t^S to be the vector in E^S with $t_i^S = \min\limits_{x_{j,S} > 0} c_{ij}(S)$. t^S is clearly in V_S . But for each S , c^S , the projection of c to E^S , has coordinates no larger than those of t^S , and therefore $c^S \in V_S$. The hypothesis that the game is balanced implies that $c \in V_N$.

REFERENCES

[1] Lemke, C. E. and J. T. Howson, Jr., "Equilibrium Points of Bimatrix Games," J. Soc. Indust. Appl. Math., Vol. 12, No. 2, June, 1964.

[2] Scarf, Herbert, The Core of an N Person Game, Cowles Foundation Discussion Paper, No. 182.