PANEL 6
INVENTORY CONTROL

SOME REMARKS ON BAYES SOLUTIONS TO THE INVENTORY PROBLEM*

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A functional equation is described for the inventory problem in which the demand distribution contains an unknown parameter with a known *a priori* distribution. If the demand distribution is a member of the exponential family, then the minimum cost is a function of two variables: current stock and past mean demand. Some properties of the optimum policies are described, and some conditions are given which imply that the functional equation can be reduced to one involving functions of *one* variable only. Under these conditions, the computation of optimal policies becomes no more difficult than the corresponding computation with a known demand distribution.

INTRODUCTION

In [2] we discussed a technique for the computation of optimal inventory levels in the case where the demand distribution contains an unknown statistical parameter. It was assumed that initially the parameter could be described by a Bayes distribution, which would be subsequently revised on the basis of additional demand information. By assuming that the demand distribution is a member of the exponential family, so that the cumulative observed demand is a sufficient statistic for the unknown parameter, it was shown that the optimal inventory levels could be obtained by the recursive computation of a sequence of functions of *two* variables. The first of these variables—as in the case of complete knowledge of the demand distribution—is the current stock level x; the second variable is s, the sum of the observed demands up to the present time.

Functions of two variables are, of course, difficult to compute recursively. We shall show, in this paper, that if several additional assumptions are made, then it is possible to determine the optimal levels by the recursive computation of functions of *one* variable alone (See Equations 5 and 6).

The first of these assumptions is that the purchase cost be proportional to the amount purchased, in other words, that there be no set up cost. This is undoubtedly a severe restriction. On the other hand, whatever applications are made of the technique described in this paper will most probably be to expensive, low-demand items; and for these items the set-up cost can frequently be incorporated in the unit cost.

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The second assumption will be that the demand distribution be a member of the Γ family, that is

$$\phi(x|w) = \frac{w(wx)^{a-1}\, e^{-wx}}{\Gamma(a)}, \tag{1}$$

with w as the unknown parameter and a as a fixed constant. (For this family of distributions, the ratio of mean to standard deviation is $\sqrt{a}$. Therefore, all possible means are permitted, but the standard deviation is to be in a fixed ratio to the mean.) We shall assume moreover that the a priori distribution of w also is from the Γ family.

$$f(w) = \frac{\lambda(\lambda w)^{b-1}\, e^{-\lambda w}}{\Gamma(b)}, \tag{2}$$

with the constants λ and b known.

The one remaining assumption is that the holding cost is proportional to the excess of stock and the shortage cost proportional to the deficit of stock.

The main idea of the argument presented in this paper is that if we make the above assumptions the optimal policies will be given by single critical numbers which are actually proportional to λ plus cumulative past demand. The only quantities to be computed are the proportionality constants, and these may be obtained by techniques which are virtually identical to those used in the case of a known demand distribution.

THE COMPUTATION TECHNIQUE

Let us begin by an examination of the functional equation derived in [2]. In this paper we shall make the same stipulation as in [2] concerning the length of the time-lag in delivery; that is, we shall take it to be zero. If there is a time-lag in delivery, and excess demand is backlogged, then the procedure described in Chapter 9 of [1] may be used in conjunction with the method described here to obtain a recursive computation based on functions of a single variable. We define a sequence of functions $C_n(x, s)$ to be the minimum, discounted, expected cost for the remainder of the program, if at the beginning of the n^{th} period the current stock level is x, and the cumulative past observed demand is s. The functional equation is, as usual, obtained by examining the cost consequences of a purchasing decision for the present period and for the remainder of the program.

If we decide to raise our stock from x to y units, then we incur a cost of $c(y-x)$, where c is the unit cost. We must then take account of the expected holding and shortage costs at the end of the present period. If the demand during the period is ξ, then these costs are given by the expectation of

$$h \max(0, y - \xi) + p \max(0, \xi - y)$$

This expectation is to be taken with respect to the a posteriori distribution of demand in the present period when given that the total past demand was s. It is a simple calculation to show that the a posteriori distribution is given by:

$$\phi_n(x|s) = \frac{\Gamma(na + b)}{\Gamma(a)\Gamma[(n-1)a+b]} \frac{\frac{1}{(s+\lambda)}\left(\frac{x}{s+\lambda}\right)^{a-1}}{\left(1 + \frac{x}{s+\lambda}\right)^{na+b}} ,$$

and therefore the expected holding and shortage costs are given by

$$L_n(y|s) = h\int_0^y \left(y - \xi\right)\phi_n\left(\xi\middle|s\right)d\xi + p\int_y^\infty \left(\xi - y\right)\phi_n\left(\xi\middle|s\right)d\xi,$$

for $y \geq 0$ and

$$p\int_0^\infty \left(\xi - y\right)\phi_n\left(\xi\middle|s\right)d\xi \qquad \text{for } y \leq 0 .$$

The state of the system at the beginning of the next period, if the demand during this period is ξ, is given by $(y - \xi, s + \xi)$, and therefore the discounted future costs will be

$$\alpha\int_0^\infty C_{n+1}\left(y - \xi, s + \xi\right)\phi_n\left(\xi\middle|s\right)d\xi .$$

These considerations lead to the following functional equation:

$$(3)\quad C_n(x, s) = \operatorname*{Min}_{y \geq x}\left[c(y - x) + L_n(y \mid s) + \alpha\int_0^\infty C_{n+1}\left(y - \xi, s + \xi\right)\phi_n\left(\xi\middle|s\right)d\xi\right],$$

with $C_N \equiv 0$ if the program lasts a total of N periods.

The important fact to be used in simplifying this equation is that

$$\phi_n(x \mid s) = \frac{1}{(s+\lambda)}\phi_n\left(\frac{x}{s+\lambda}\right) ,$$

where $\phi_n(x)$ is defined to be

$$\phi_n(x) = \frac{\Gamma(na + b)}{\Gamma(a)\Gamma[(n-1)a+b)]} \frac{x^{a-1}}{(1 + x)^{na+b}}$$

Let us examine the consequences of this relation for the function $L_n(y \mid s)$. We have

$$h \int_0^y \left(y - \xi\right) \phi_n(\xi|s)\, d\xi = \frac{h}{s + \lambda} \int_0^y (y - \xi)\, \phi_n\left(\frac{\xi}{s + \lambda}\right) d\xi \quad .$$

Make the change of variable $\xi = (s + \lambda)t$ and we obtain

$$h \int_0^{\frac{y}{s+\lambda}} [y - (s + \lambda)t]\, \phi_n(t)\, dt = (s + \lambda)\, h \int_0^{\frac{y}{s+\lambda}} \left(\frac{y}{s + \lambda} - t\right) \phi_n(t)\, dt.$$

We do the same sort of thing to the shortage cost component, and we see that

$$L_n(y \mid s) = (s + \lambda)\, L_n\left(\frac{y}{s + \lambda}\right),$$

where

$$L_n(y) = h \int_0^y (y - \xi)\phi_n(\xi)\, d\xi + p \int_y^\infty (\xi - y)\phi_n(\xi)\, d\xi \ ,$$

for $y \geq 0$.

This relationship tells us that the expected holding and shortage cost function can very well be represented by a function of a single variable. It seems intuitively clear that a similar sort of representation is valid for the cost functions $C_n(x, s)$ and, as we shall show, this is indeed the case. Specifically, we shall show that

$$C_n(x, s) = (s + \lambda)\, C_n\left(\frac{x}{s + \lambda}\right) \ , \tag{4}$$

where C_n of a single variable is a new function satisfying a functional equation that we shall indicate. The proof of Equation 4 is by induction backwards on n. Let us assume that this is true for $(n + 1)$. Then

$$C_n(x, s) = \min_{y \geq x} \left[c(y - x) + (s + \lambda)\, L_n\left(\frac{y}{s + \lambda}\right) + \alpha \int_0^\infty C_{n+1}(y - \xi\ , s + \xi)\, \phi_n(\xi \mid s)\, d\xi \right].$$

The last term is equal (by the induction assumption) to

$$\alpha \int_0^\infty C_{n+1}\left(\frac{y - \xi}{s + \xi + \lambda}\right) (s + \xi + \lambda)\, \phi_n\,(\xi | s)\, d\xi$$

$$= \alpha \int_0^\infty C_{n+1}\left(\frac{y - \xi}{s + \xi + \lambda}\right) \frac{(s + \xi + \lambda)}{s + \lambda}\ \phi_n\left(\frac{\xi}{s + \lambda}\right) d\xi\ .$$

Now make a change of variable $\xi = (s + \lambda)t$, and we obtain

$$(s + \lambda)\ \alpha \int_0^\infty C_{n+1}\left(\frac{\frac{y}{s + \lambda} - t}{1 + t}\right) (1 + t)\ \phi_n\,(t)\ dt\ .$$

Inserting this in the functional equation, we obtain

$$C_n(x, s) = (s + \lambda) \min_{\frac{y}{s + \lambda} > \frac{x}{s + \lambda}} \left[c\left(\frac{y}{s + \lambda} - \frac{x}{s + \lambda}\right) + L_n\left(\frac{x}{s + \lambda}\right)\right.$$

$$\left. + \alpha \int_0^\infty C_{n+1}\left(\frac{\frac{y}{s + \lambda} - t}{1 + t}\right) (1 + t)\ \phi_n\,(t)\ dt \right] .$$

Since we also know in [2] that the optimal policy is defined by a single critical level for each s, we conclude that there exists a number a_n, with the property that the optimal policy is

$$\text{(5)} \qquad \begin{cases} \text{if } x \leq a_n(\lambda + s), \text{ select } y = a_n(\lambda + s) \\ \text{if } x \geq a_n(\lambda + s), \text{ select } y = x \end{cases}$$

It is also clear that if we define $C_n(x)$ by the equation

$$\text{(6)} \qquad C_n(x) = \min_{y \geq x} \left[c(y - x) + L_n(y) + \alpha \int_0^\infty C_{n+1}\left(\frac{y - t}{1 + t}\right) (1 + t)\, \phi_n\,(t)\ dt \right] .$$

then $C_n(x, s) = (s + \lambda)\ C_n\left(\frac{x}{s + \lambda}\right)$ and the induction step is complete.

REFERENCES

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[2] Scarf, H., "Bayes Solutions of the Statistical Inventory Problem," Annals of Mathematical Statistics, Vol. 30, 1959, pp. 490-508.

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