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The Structure of the Complex of Maximal Lattice Free Bodies for a Matrix of Size $(n+1) \times n$

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To Laci Lovász on his 60th Birthday

The complex of maximal lattice free bodies associated with a well behaved matrix A of size $(n+1) \times n$ is generated by a finite set of simplicies, $K_0(A)$, of the form $\{0, h^1, \dots, h^k\}$, with $k \le n$, and their lattice translates. The simplicies in $K_0(A)$ are selected so that the plane $a_0 x = 0$, with a_0 the first row of A, passes through the vertex 0. The collection of simplicies $\{h^1, \dots, h^k\}$ is denoted by Top. Various properties of Top are demonstrated, including the fact that no two interior faces of Top are lattice translates of each other. Moreover, if g is a generator of the cone generated by the set of neighbors $\{h\}$ with $a_0 h > 0$, then the set of simplicies of Top which contain g is the union of linear intervals of simplicies with special features. These features lead to an algorithm for calculating the simplicies in $K_0(A)$ as a_0 varies and the plane $a_0 x = 0$ passes through the generator g.

1. Introduction

The purpose of this paper is to demonstrate some elementary structural properties of the simplicial complex of maximal lattice free bodies associated with a *"generic"* matrix A with $n+1$ rows and n columns. The properties are quite easy to establish, but they are not properties that I would have expected when I began studying this subject many years ago.

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The structure that is revealed permits an easy calculation of the simplicial complex associated with matrices obtained by perturbing A if the perturbed matrix is also generic. The calculation involves finding the generators of a cone, and comparing simplicies to see whether they are lattice translates of each other; no integer programs need be solved. The simplicity of the calculation suggests that homotopy methods may be useful in computing this simplicial complex and the test set for integer programming that is given by its edges. But it is imperative that both the initial matrix and the perturbed matrix be generic. The paper is silent if this is not correct.

Let A be a real matrix with $n+1$ rows and n columns satisfying the assumption:

$A1$. There exists a unique (aside from scale) strictly positive vector π such that $\pi A = 0$.

This assumption implies that the $n \times n$ minors of A are non-singular and that the bodies $K_b = \{x\colon Ax \geq b\}$ are bounded for any b. A *maximal lattice free body* is such a body K_b containing no lattice points in its interior, and such that any convex body that properly contains K_b does have a lattice point in its interior.

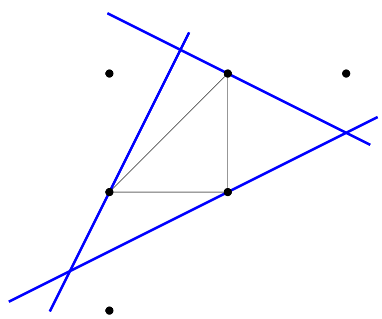

Fig. 1. A Maximal Lattice Free Body

Each of the $n+1$ faces of a maximal lattice free body will contain at least one lattice point. The possibility of several lattice points lying on a single face is a nuisance that can be avoided by perturbing the matrix A slightly. Let us assume that this has been done, and that the following assumption of *genericity* holds:

$A2$. Each face of a maximal lattice free body contains precisely one lattice point.

According to this assumption, each maximal lattice free body will be associated with $n+1$ lattice points $\{h^0, h^1, \ldots, h^n\}$ with, say, $a_i h^i = b_i$ and $a_i h^j > b_i$ for all i and all $j \neq i$. The simplicial complex $K(A)$, occasionally called the Scarf Complex, defined by A consists of these simplicies, and all of their proper subsimplicies.

The Scarf Complex was introduced, of course under a different name, as an essential ingredient in a constructive proof that a balanced n person game has a non empty core [12]. In this treatment, the set of vertices Y was an arbitrary finite set of points in R^{n+1} rather than the lattice $Y = \{Ah\colon h \in \mathbb{Z}^n\}$. The rule for finding an adjacent simplex, given in Section 3 of the current paper, was fully described. The term "Scarf Complex" was first used in [3], and [2].

The complex can be illustrated by a figure which is quite familiar to algebraic geometers. Let Y be the lattice generated by the columns of A. Append a negative orthant to each point $y \in Y$.

The union of these translated negative orthants has a "staircase" structure as in Figure 2 in which $n = 2$. The maximal points on the upper surface are the vectors in the lattice Y.

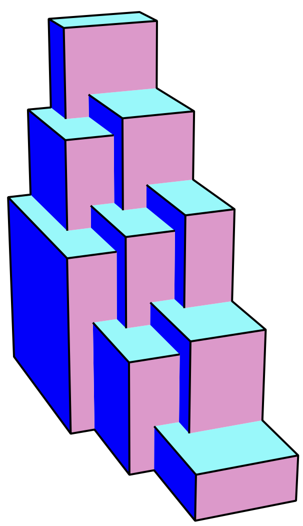

Fig. 2. The Staircase

In this example, each minimal point on the upper surface lies on 3 negative orthants associated with 3 different vectors, say $y^{j_1}, y^{j_2}, y^{j_3} \in Y$. Figure 3 shows a triangle whose vertices are one of these triples.

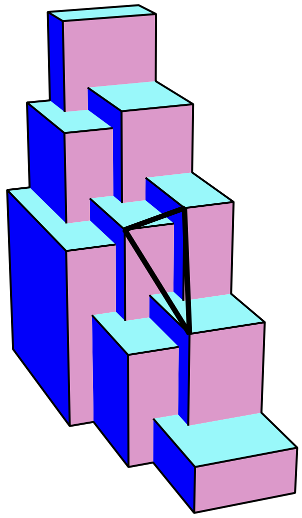

Fig. 3. A Simplex

If we construct the triangle associated with every minimal point on this surface we get a collection of triangles (**and edges and vertices**) that forms the simplicial complex associated with this set Y.

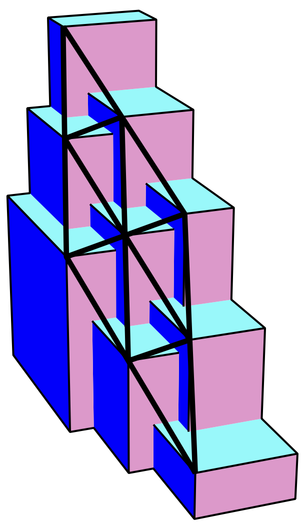

Fig. 4. The Complex

If A is an $(n+1)\times n$ matrix, then the simplicial complex will typically be of dimension n. It will consist of n dimensional simplicies and all of their faces of arbitrary dimensions. As in the example with $n = 2$, we append a negative orthant to each of the points

$$y \in Y$$

and take the union of these translated negative orthants. If A is generic, each minimal point on the upper surface will be associated with a set of $n + 1$ points

$$y^{j_0}, \dots, y^{j_n} \in Y.$$

The simplicies S of maximal dimension in $K(A)$ are these collections of $n+1$ vectors.

The edges of this complex provide the unique, minimal *test set* for the family of integer programs

$$\begin{aligned} \max \;\; & a_0 h \text{ subject to} \\ & a_i h \geq b_i, \text{ for } i = 1, \dots, n \text{ and } h \text{ integral,} \end{aligned}$$

obtained by selecting a single row of A, say row 0, as the objective function and imposing constraints derived from the remaining rows [13, 14]. More specifically, let $N(A)$ be the finite, symmetric set of those non-zero lattice points k that are contained in a maximal lattice free body, one of whose other vertices is the origin. The lattice points in $N(A)$ are called *neighbors of the origin.* (It is a consequence of Assumption $A2$ that $a_j h \neq 0$ for any neighbor h.) It can be shown that a lattice point h satisfying the constraints of this integer program is the optimal solution to the integer program if for every neighbor of the origin, k, with $a_0 k > 0$, the lattice point $h+k$ violates at least one of the constraints. Moreover, under the assumption of genericity this test set is minimal: If an arbitrary element, and its negative, of $N(A)$ are deleted, then some right hand side b, and some feasible solution h can be found, such that h is not optimal, but its lack of optimality cannot be detected using this smaller test set.

The minimal test set $N(A)$ for an alternative formulation

$$\begin{aligned} & \min a_0 h \text{ subject to} \\ & a_i h = b_i \;\text{ for }\; i = 1, \dots, n, \;\; h \in \mathbb{N}^n, \end{aligned}$$

is given in [19]. This test set is shown to be a particular Gröbner basis for a binomial ideal associated with the columns of A. Sturmfels and Thomas [18] study the relationship between the set of optimal solutions of two families of integer programming problems, with identical constraint sets, but with different objective functions a_0 and $\widehat{a}_0$. They show that the optimal solutions to the two problems will be the same for each right hand side b, if, and only if, the two associated matrices have indentical sets of neighbors.

Let $c = (c_1, \ldots, c_n)$ be a vector of positive integers with no common factor. The Frobenius problem is to find the largest integer b, termed the Frobenius number, which **cannot** be written as ch with $h \in \mathbb{Z}^n_+$. In [16] a direct relationship is exhibited between the Frobenius problem and the simplicial complex $K(A)$ where A is a matrix whose columns generate the lattice $L = \{h \in \mathbb{Z}^n \colon ch = 0\}$. There has recently been a considerable renewal of interest in the Frobenius problem. A variety of techniques have been developed to calculate the Frobenius number using Gröbner Bases [9], [11] and the entirely novel approach taken in [8].

The simplicial complex $K(A)$ has been carefully studied and its topological structure is quite well known [5], [7] and [4]. The complex (more precisely, its realization in R^n) is homeomorphic to R^n. The immediate presentation of the complex obtained by drawing all of its simplicies in R^n is quite intricate. Aside from a few elementary cases, many of the simplicies overlap with each other, so that the embedding into R^n is elaborately folded.

If B is another matrix with $n+1$ rows and n columns such that

$$\operatorname{sign}(a_i h) = \operatorname{sign}(b_i h) \text{ for all } i \text{ and } h \in N(A),$$

then B has the same set of neighbors and the same simplicial complex as A [6]. The result is also correct when the two matrices have m rows and n columns with $m > n$.

An arbitrary lattice translate of a simplex in $K(A)$ is also in the complex, and it is frequently convenient to choose one simplex from each class of lattice translates. We shall be concerned with the behavior of the collection of simplicies, of arbitrary dimension, as the first row of the matrix a_0 varies. For this purpose it will be useful to select a specific representative from each set of lattice translates by requiring the origin to lie on the plane $a_0 x = b_0 = 0$. In our notation, we consider the collection of simplicies in $K(A)$ of the form

$$\{0, h^1, \ldots, h^k\},$$

with $a_0 h^j > 0$ for $j = 1, \ldots, k$. We give the name $K_0(A)$ to this special subset of $K(A)$.

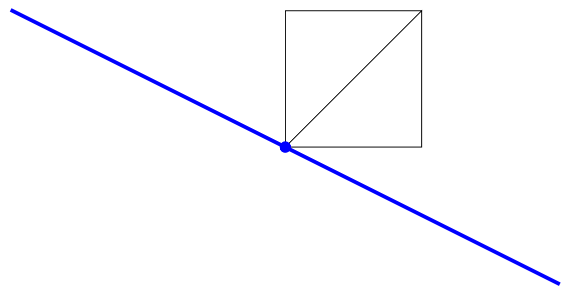

Fig. 5. $K_0(A)$

It will be useful for us to have a simple example of an $n+1 \times n$ matrix A for which $K_0(A)$ is easy to describe. Let A be an $(n+1) \times n$ matrix, with rows $0, 1, \ldots, n$, columns $1, \ldots, n$, with the sign pattern

$$A = \begin{bmatrix} + & + & \cdots & + \\ - & + & \cdots & + \\ + & - & \cdots & + \\ \vdots & \vdots & \ddots & \vdots \\ + & + & \cdots & - \end{bmatrix}$$

which satisfies

$$\sum_j a_{ij} < 0 \quad \text{for} \quad i = 1, \ldots, n.$$

Then the full dimensional simplicies in $K_0(A)$ are given by

$$S = \left(0, e_1, e_1 + e_2, \ldots, \sum_{j=1,n} e_j \right)$$

for $e_1, e_2, \ldots, e_n$ be an arbitrary permutation of the n unit vectors in R^n. The neighbors of the origin are the non-zero lattice points $h = \{h_1, \ldots, h_n\}$ with $h_j = 0, 1$, and their negatives [15].

2. Top

Let us define Top to be the collection of simplicies $\{h^1, \ldots, h^k\}$ such that $\{0, h^1, \ldots, h^k\}$ is a simplex in $K_0(A)$, with the plane $a_0 x = b_0$ passing through the origin.

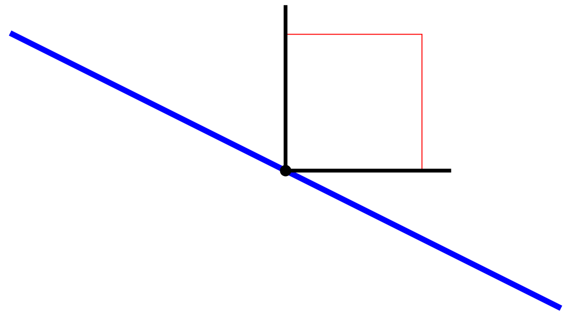

Fig. 6. Top

The vertices of Top consist of the neighbors h with $a_0 h > 0$. Let C be the cone generated by this half-set of neighbors. For each generator g of C, we define Top $[g]$ to be the collection of simplicies in Top, one of whose vertices is that generator. We use the notation Top / Top $[g]$ for the collection of $n-1$ simplicies in Top but not in Top $[g]$.

The number of generators of C may be shown to be polynomial in the bit size of A for fixed n [10].

Let us consider an example of Top based on the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -101 & 31 & 43 \\ 29 & -301 & 173 \\ 41 & 131 & -203 \end{bmatrix}$$

For this example, Top is the figure consisting of the following 6 triangles

$$\begin{array}{l} \{(1,0,0),(1,1,0),(1,1,1)\} \\ \{(1,0,0),(1,0,1),(1,1,1)\} \\ \{(0,1,0),(1,1,0),(1,1,1)\} \\ \{(0,1,0),(0,1,1),(1,1,1)\} \\ \{(0,0,1),(1,0,1),(1,1,1)\} \\ \{(0,0,1),(0,1,1),(1,1,1)\} \end{array}$$

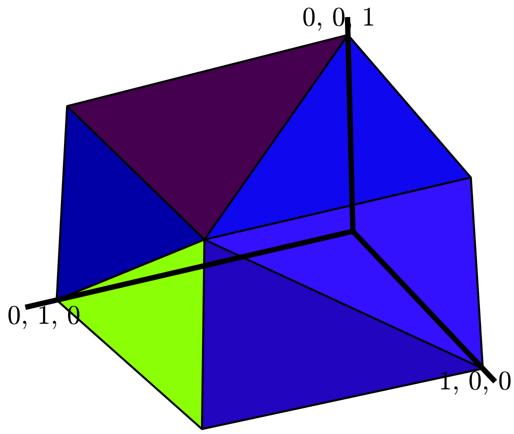

Fig. 7. Top

The generators of Top are given by the following three neighbors

$$(1, 0, 0)$$
$$(0, 1, 0)$$
$$(0, 0, 1)$$

Let $g = (1, 0, 0)$. Then Top $[g]$, the set of simplicies containing g, is shown in Figure 8.

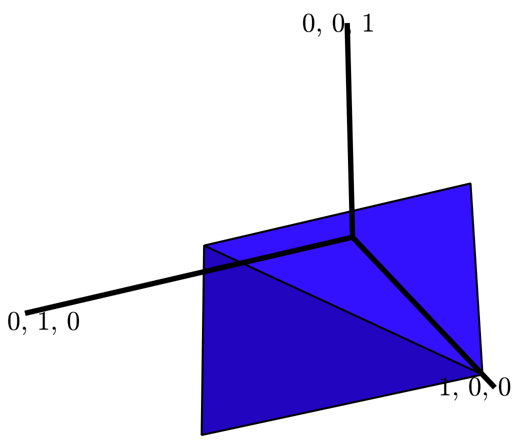

Fig. 8. Top $[(1, 0, 0)]$

The plane $a_0 x = 0$ passes through the origin. The complex $K(A)$ is unchanged by a perturbation of this plane as long as the plane does not touch one of the generators of Top. But as soon as the plane passes through one of the generators the entire complex is transformed into a new simplicial complex. For example, if the plane $a_0 x = 0$ passes through the generator $g = (1, 0, 0)$ the new version of Top, say Top*, can be shown to consist of the 12 triangles

$$\begin{array}{l}
\{\{1,1,0\},\{0,1,0\},\{1,1,1\}\} \\
\{\{2,1,1\},\{1,1,0\},\{1,1,1\}\} \\
\{\{4,0,1\},\{4,1,1\},\{3,0,1\}\} \\
\{\{5,1,1\},\{4,1,1\},\{4,0,1\}\} \\
\{\{1,1,1\},\{0,1,0\},\{-1,0,0\}\} \\
\{\{4,1,1\},\{3,1,1\},\{-1,0,0\}\} \\
\{\{3,1,1\},\{2,1,1\},\{-1,0,0\}\} \\
\{\{2,1,1\},\{1,1,1\},\{-1,0,0\}\} \\
\{\{4,1,1\},\{-1,0,0\},\{3,0,1\}\} \\
\{\{1,0,1\},\{-1,0,0\},\{0,0,1\}\} \\
\{\{2,0,1\},\{-1,0,0\},\{1,0,1\}\} \\
\{\{3,0,1\},\{-1,0,0\},\{2,0,1\}\}
\end{array}$$

shown in Figure 9.

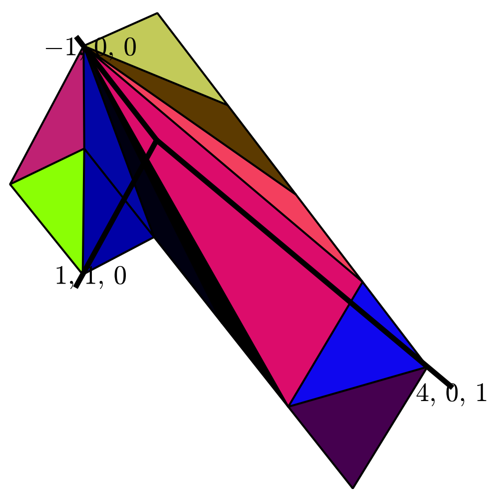

Fig. 9. Top*

The new generators are

$$
\begin{aligned}
&(1,1,0)\\
&(4,0,1)\\
&(-1,0,0)
\end{aligned}
$$

$\mathrm{Top}^*[-g]$ is the collection of simplicies in Top^* containing the new generator $-g = (-1,0,0)$. It is displayed in Figure 10.

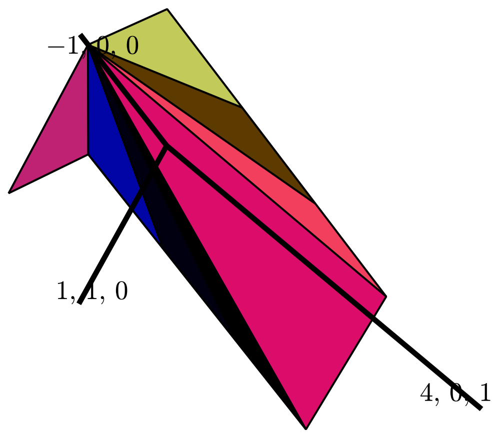

Fig. 10. $\mathrm{Top}^*[-g]$

As we see $\mathrm{Top}^*[-g]$ contains 8 distinct simplicies.

2.1. The Structure of Top. An Informal Presentation

I would like to present an informal summary of the basic structure of Top, reserving formal mathematical proofs for later sections. The two examples that we have constructed, Top and Top^*, are quite different and the properties we are about to discuss may be seen more readily in the more complex example.

Top is a connected $n-1$ dimensional piece-wise linear manifold: every $n-2$ dimensional face of Top is incident to either one or two $n-1$ dimensional simplicies. Top is homeomorphic to the $n-1$ ball [5].

An $n-2$ facet of Top incident to a single $n-1$ dimensional simplex is on the *boundary* of Top, which we denote by $\partial\,\mathrm{Top}$. An $n-2$ facet incident to two $n-1$ dimensional simplicies is *interior* to Top. The simplicies of Top have faces of dimension $0, 1, \ldots, n-2$. A $k-1$ dimensional face of Top is defined by a set $F = \{h^1, \ldots, h^k\}$ with $a_0 h^j > 0$ such that the $n-2$ facet

$$S = \{h^1, \ldots, h^k, \alpha^{k+1}, \ldots, \alpha^{n-1}\} \in \text{Top}$$

for some $\alpha^{k+1}, \ldots, \alpha^{n-1}$. There are, of course, many such $\alpha^{k+1}, \ldots, \alpha^{n-1}$ that can be used to complete F. (Section 4, and, in particular, Lemma 1)

- **The face F is in the boundary of** Top **if, and only if, there is some completion to a facet which is itself on the boundary of** Top**.**

The next set of features are unexpected (to me) properties of lattice translates of *interior* faces of Top. Two faces E and F are lattice translates if $F = E + h$ for some lattice point h.

- **No two interior k dimensional simplicies of** Top **are lattice translates of each other** (Lemma 3 in Section 5).

As a special case of this result, Top has a single interior vertex [4]. The single interior vertex may be shown to be the lattice point h^* where h^* is the solution to the integer program

$$\min a_0 h \quad \text{subject to}$$

$$a_i h < 0, \quad \text{for} \quad i = 1, \ldots, n, \quad \text{and} \quad h \quad \text{integral.}$$

- **Every k dimensional simplex on the boundary of** Top **has a lattice translate that is interior to** Top (Lemma 2 in Section 4).

Each $n-2$ simplex on the boundary of Top is a lattice translate of an interior simplex, but many of them may also be lattice translates of other boundary simplicies. Of particular interest are those $n-2$ simplicies that are translates by a generator of the cone C. Let g be such a generator. Two $n-2$ simplicies, E and F, of Top will be said to be congruent $\bmod (g)$, if $F = E + jg$, for some integral j.

- **Lattice Translates of $n-2$ simplicies by a generator. The following properties are demonstrated in Theorem 4 of Section 6.**

Let g be a generator of the cone C. The set of $n-2$ simplicies of Top $[g]$ that are congruent $\bmod (g)$ to a specific $n-2$ simplex consists of an interval

$$E + g, \ldots, E + ug,$$

with $E = \{h^1, \ldots, \text{not } i, \ldots, h^n\}$ a simplex in Top / Top $[g]$. ($u = 0$, if there are no simplicies of this form.) Moreover

1. All of the $n-2$ simplicies $E, E+g, \ldots, E+(u-1)g$ are in the boundary of Top.
2. All of the $n-1$ simplicies $\{g, E+g\}, \ldots, \{g, E+ug\}$ are in Top $[g]$.
3. The first $n-2$ simplex E is contained in an n simplex $S = \{0, h^1, \ldots, \alpha, \ldots, h^n\}$ with $\alpha \neq g$.

The last simplex, $E+ug$, in such an interval may, or may not be *interior* to Top. It is important to differentiate between these two cases; let us call such a sequence of $n-2$ simplicies *interior* if the last simplex in the interval is interior to Top, and call it a *boundary* sequence if the last simplex is on the boundary. If $u < t$ the interval is definitely a boundary interval. If $u = t$ the interval may be of either type.

Figure 11 illustrates a boundary sequence of Top$^*[-g]$, with $g = (1, 0, 0)$, derived from our basic example using the matrix A. As we see, the first such simplex is not in Top$^*[-g]$ and the last simplex is on the boundary of Top*. I have taken the liberty, in this and in subsequent drawings, of including the $n-1$ simplicies containing the members of the sequence and also the initial $n-1$ simplex which does not belong to Top$^*[-g]$.

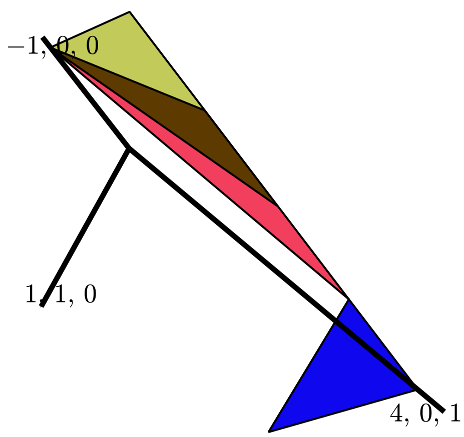

Fig. 11. A Boundary Sequence

In this example, Top$^*[-g]$ has three interior sequences which are shown in the next three figures. Again I have included the $n-1$ simplicies containing the edges of the sequence and the pair of $n-1$ simplicies at the two ends of the interior sequence.

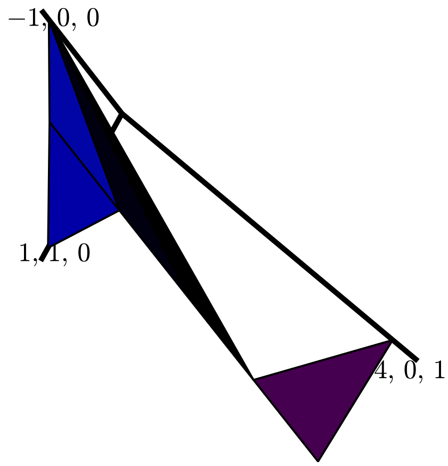

Fig. 12. Interior Sequence 1

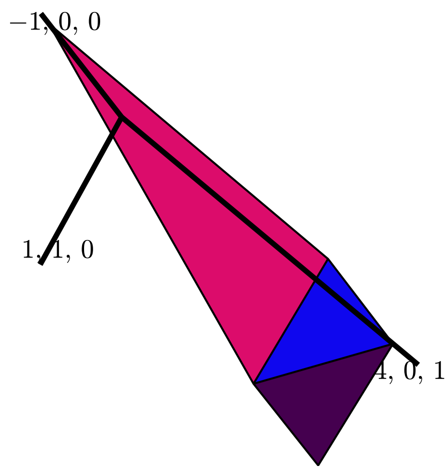

Fig. 13. Interior Sequence 2

If $\{g, E\}$ is an $n-1$ simplex in $\text{Top}\,[g]$, then E belongs either to an interior or a boundary sequence. **It would be extremely interesting if the number of sequences in** $\text{Top}\,[g]$ **could be shown to be small in fixed dimension.**

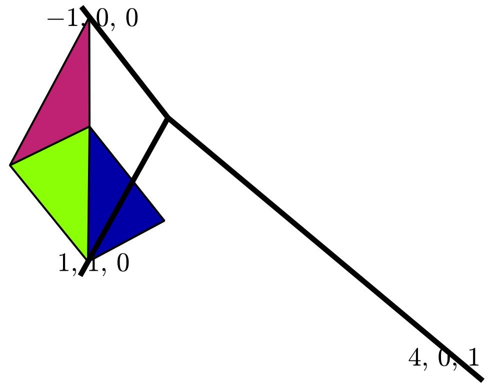

Fig. 14. Interior Sequence 3

3. Adjacent n Simplicies

In order to differentiate the faces of Top which are interior to Top from those which are on the boundary we begin by examining adjacent n simplicies in $K(A)$.

Let $S = \{h^0, h^1, \ldots, h^n\}$ be an n simplex in $K(A)$. For each such S we introduce the $(n+1)$ by $(n+1)$ matrix

$$AH = \left[a_i h^j\right], i, j = 0, \ldots, n.$$

If the matrix A is *generic,* then the row minima of AH lie in different columns. We have agreed to name the vectors $\{h^j\}$ so that the row minima lie on the main diagonal of AH. If this is done then the smallest body of the form $Ax \geq b$, containing S has

$$b_i = a_i h^i.$$

Let

$$\{h^0, h^1, \ldots \text{ not } h^i, \ldots, h^n\}$$

be a particular $n-1$ face of S. We remind the reader of the simple rule for finding the unique, different maximal lattice free body which shares this $n-1$ face with S. Define $j = j(i)$ to be the index j which gives the **second**

smallest value of $a_i h^j$. Then the replacement for h^i is that unique lattice point h which solves

$$\begin{aligned}&\max a_j h \;\text{ subject to}\\ &a_k h > a_k h^k \;\text{ for }\; k \neq i, j \;\text{ and}\\ &a_i h > a_i h^j.\end{aligned}$$

4. Which $k-1$ Faces are Interior to Top?

A $k-1$ dimensional face of Top is defined by a set $F = \{h^1, \ldots, h^k\}$ with $a_0 h^j > 0$, for $j = 1, \ldots, k$, such that

$$S = \{h^1, \ldots, h^k, \alpha^{k+1}, \ldots, \alpha^n\} \in \text{Top}$$

for some $\alpha^{k+1}, \ldots, \alpha^n$ with $a_0 \alpha^l > 0$, for $l = k+1, \ldots, n$. There are, of course, many such $\alpha^{k+1}, \ldots, \alpha^n$ that can be used to complete F. If we take any such completion

$$\{h^1, \ldots, h^k, \alpha^{k+1}, \ldots, \alpha^j, \ldots, \alpha^n\}$$

and replace any particular α^j, then if F is *interior* to Top, the resulting simplex will also be in Top. If F is not *interior*, there is some completion so that if we replace a particular α^j the resulting simplex will not be in Top.

We have the following useful conclusion:

Lemma 1. *Let* $F = \{h^1, \ldots, h^k\}$ *be a* $k-1$ *dimensional face of* Top*.* F *is* interior *if, and only if,*

$$\min_{j=1,\ldots,k} a_i h^j < 0 \;\; \textit{for all} \;\; i = 1, \ldots, n.$$

Proof. Let us consider an arbitrary completion of F given by the simplex in $K_0(A)$:

$$S = \{0, h^1, \ldots, h^k, \alpha^{k+1}, \ldots, \alpha^n\}.$$

According to our convention the row minima of the matrix associated with S lie on the main diagonal, so that in particular

$$\min_{j=1,\dots,k} a_i h^j = a_i h^i < 0 \quad \text{for} \quad i = 1, \dots, k \quad \text{and}$$

$$\min_{j=k+1,\dots,n} a_i \alpha^j = a_i \alpha^i < 0 \quad \text{for} \quad i = k+1, \dots, n.$$

Our first task is to show that if F is an interior $k-1$ face then

$$\min_{j=1,\dots,k} a_i h^j < 0 \quad \text{for} \quad i = k+1, \dots, n \quad \text{as well.}$$

Assume to the contrary that F is an interior face and $\min_{j=1,\dots,k} a_i h^j > 0$ for some particular row i, with $k+1 \leq i \leq n$. Let us remove column α^i from the simplex S and replace it with column $\widehat{\alpha}^i$, obtaining a new simplex $\widehat{S}$. According to the general properties of the replacement step, the smallest entry in row i of the matrix associated with $\widehat{S}$ is equal to the *second* smallest element in row i of the corresponding matrix for S. There are two possible cases:

1. The second smallest element in row i of the matrix associated with S does not lie in any of the columns $k+1, \dots, n$. But since $\min_{j=1,\dots,k} a_i h^j > 0$ the second smallest element in row i must therefore lie in column 0. It follows that the replacement for α^i leads to a simplex not in $K_0(A)$ and therefore F is not an interior face.
2. The second smallest element in row i of the matrix associated with S does lie in one of the columns $k+1, \dots, n$. In this case we reach a new completion $\widehat{S}$ in which the minimum entry in row i of the associated matrix has strictly increased. This can only occur for a finite number of steps; we therefore ultimately return to the previous case and obtain a contradiction to the assumption that F is an interior face and $\min_{j=1,\dots,k} a_i h^j > 0$.

On the other hand if

$$\min_{j=1,\dots,k} a_i h^j < 0$$

for all $k+1 \leq i \leq n$, then the second smallest entry in row $i \geq k+1$ of any completion

$$S = \{0, h^1, \dots, h^k, \alpha^{k+1}, \dots, \alpha^n\}$$

of F will not lie in column 0 of the associated matrix. The replacement for α^i will lead to a new simplex in $K_0(A)$ for any completion and F is therefore an interior face. ■

The same argument can be used to produce the following property of Top.

Lemma 2. *Let* $F = \{h^1, \ldots, h^k\}$ *be a* $k-1$ *dimensional boundary face of* Top. *Then* F *has a lattice translate* $F + h$, *with* $a_0 h > 0$, *which is interior to* Top.

Proof. There may be several lattice translates of F on the boundary of Top. Let us assume that F has been selected from this set so that no lattice translate $F + h$ with $a_0 h > 0$ is on the boundary of Top. Since F is on the boundary there must be a completion $S = \{0, h^1, \ldots, h^k, \alpha^{k+1}, \ldots, \alpha^n\}$ with the replacement for a particular α^j leading to a simplex not in $K_0(A)$, say

$$S' = \{-h, h^1, \ldots, h^k, \alpha^{k+1}, \ldots, 0, \ldots, \alpha^n\}$$

with $a_0 h > 0$. But then $F + h \in$ Top, and it must therefore be interior. ■

5. Lattice Translates of Interior Simplicies

We have the following remarkably simple conclusion:

Lemma 3. *Let* $F = \{h^1, \ldots, h^k\}$ *with* $a_0 h^j > 0$ *be a* $k-1$ *dimensional* interior *face of* Top *and let* $h \in Z^n$ *with* $a_0 h > 0$. *Then* $\{h^1 + h, \ldots, h^k + h\}$ *is* ***not*** *a face of* Top.

Proof. The smallest body of the form $Ax \geq b$, containing $0, h^1 + h, \ldots, h^k + h$ has

$$b_0 = 0 \quad \text{and}$$
$$b_i \leq \min_{j=1,\ldots,k} a_i(h^j + h) < a_i h \quad \text{for} \quad i = 1, \ldots, n,$$

because $\min_{j=1,\ldots,k} a_i h^j < 0$. But then h is contained in this body and $\{h^1 + h, \ldots, h^k + h\}$ is not a face in Top.

This result implies that if F and $F+h$, with $a_0h>0$ are both faces of Top, then $F\in\partial\,\mathrm{Top}$. There cannot be a pair of lattice translates of a $k-1$ face both of which are interior simplicies of Top. ■

6. Intervals in Top $[g]$

Let g be a generator of the smallest cone containing the set of neighbors $\{h\}$ with $a_0h>0$.

Theorem 4. *Let $E+g$ be the first $n-2$ face of the form $E+lg$ contained in* Top $[g]$ *and $E+tg$, with $t\geq 1$, the last such face. Then the set of $n-2$ simplicies $E+lg\in$* Top $[g]$, *consists of an interval*

$$E+g,\dots,E+tg.$$

Moreover the faces

$$E,E+g,\dots,E+(t-1)g$$

*are all in $\partial\,$*Top. *The last face in the interval, $E+tg$, may be interior to* Top *or on the boundary. If $E+tg$ is a boundary interval, the value of t is given by*

$$t=\Big\lfloor\min_j\big(-a_jh^j/a_jg\big):j>0,\ a_jg>0\big)\Big\rfloor.$$

Proof. A Preliminary Observation. Let $\{g,E+g\}$ be the first $n-1$ simplex of the form $\{g,E+ug\}$ in Top $[g]$. We assume, as customary, that the row minima of

$$\{0,g,E+g\}=\{0,h^1+g,\dots,g,\dots,h^n+g\}\in K_0(A)$$

are on the main diagonal (with "g" in column i). Let us show that E is also in Top. To obtain this conclusion we first observe that a_0 can be varied – without changing the simplicial complex – so that a_0g, while positive, is arbitrarily close to 0. This tells us that the second smallest entry in row 0 of the matrix associated with $\{0,g,E+g\}$ is in column i. If column 0 is removed from this simplex we obtain a new simplex

$$\{g,h,E+g\}=\{g,h^1+g,\dots,h,\dots,h^n+g\},$$

with row minima on the main diagonal and with g the smallest entry in row 0. It follows that

$$\{0, h - g, E\} = \{0, h^1, \ldots, h - g, \ldots, h^n\}$$

is a maximal lattice free body, again with row minima on the main diagonal. E is therefore in Top.

From the assumption that the row minima of $\{0, g, E + g\}$ lie on the main diagonal we also know that $a_i g < 0$.

Our next observation is that a completion of $E + tg$ can be found with row minima on the main diagonal. Let $T = \{0, h^1 + tg, \ldots, \beta, \ldots, h^n + tg\}$ be a completion of $E + tg$. There are two cases to consider:

1. $E + tg$ is *interior* to Top. In this case, according to Lemma 2, $\min_{j \neq i} a_k(h^j + tg) < 0$ for $k = 1, \ldots, n$. For $k \neq i$ the minimum is reached at $j = k$ and we have

$$\min_{j \neq i} a_k(h^j + tg) = a_k(h^k + tg) < 0.$$

   On the other hand, if the smallest element in row i of T is in column i then the row minima of T lie on the main diagonal. If the row minimum in row i of T is in column k then the smallest entry in row k of T must be in column i and the second smallest entry in row k is $a_k(h^k + tg)$. If we then replace β in T, we obtain a new $T^* = \{0, h^1 + tg, \ldots, \beta^*, \ldots, h^n + tg\} \in K_0(A)$ with row minima on the main diagonal. In this event let the completion be T^* rather than T.

2. $E + tg$ is on the *boundary* of Top. Then from Lemma 2 we must have $\min_{j \neq i} a_k(h^j + tg) > 0$ for some $k = 1, \ldots, n$. We shall show that this k is equal to i. If k is different from i, the smallest entry in row k of the matrix associated with T, being negative, must be in column i and therefore $a_k(h^k + tg) > 0 > a_k g$. But this is impossible since the row minima of $\{0, g, E + g\}$ lie on the main diagonal implying that $a_k(h^k + g) < 0$. It follows that $k = i$ and the row minima of T lie on the main diagonal.

Now let us turn to the main argument. Let $0 < s < t$ and define

$$T_s = \{0, h^1 + sg, \ldots, g, \ldots h^n + sg\}.$$

We want to show by induction, starting with $s = 1$ and continuing to $s = t$, that T_s is a simplex in $K_0(A)$, and that the row minima of T_s lie on the main diagonal.

Let us assume that these two features are correct for $T_1, \ldots, T_{s-1}$ and show that they are also correct for T_s. Since they hold for $s = 1$, by assumption, this demonstrates the theorem.

The row minima of the matrix associated with $\{0, h^1, \ldots, h - g, \ldots, h^n\}$ lie on the main diagonal. Therefore for any $k \neq 0, i$, we have

$$a_k h^k < a_k h^j \quad \text{for} \quad j \neq 0, k, i$$

It follows that

$$a_k(h^k + sg) < a_k(h^j + sg) \quad \text{for} \quad j \neq 0, k, i.$$

We also know that

$$\begin{aligned} a_k h^k < 0 \quad &\text{for} \quad k \neq 0, i \quad \text{and} \\ a_k(h^k + tg) < 0 \quad &\text{for} \quad k \neq 0, i. \end{aligned}$$

If we average these inequalities we see that $a_k(h^k + sg) < 0$ and also $a_k\big(h^k + (s-1)g\big) < 0$, (from which it follows that $a_k(h^k + sg) < a_k g$), for $0 < s < t$ and $k \neq 0, i$. Therefore the smallest element in row $k \neq 0, i$ of T_s is in column k.

By induction $E + (s-1)g$ is a face of Top so that it must be in $\partial\,$Top and its associated matrix must have one row which is entirely positive. But this must be row i, since, by induction,

$$a_k\big(h^k + (s-1)g\big) < 0 \quad \text{for} \quad k \neq 0, i.$$

It follows that row i is the row with positive entries and

$$\begin{aligned} a_i\big(h^j + (s-1)g\big) > 0 \quad &\text{for} \quad j \neq 0, i \text{ and therefore} \\ a_i(h^j + sg) > a_i g & \end{aligned}$$

Combining this observation with the previous remark that $a_k(h^k + sg) < a_k g$ for $k \neq 0, i$ we see that the row minima of T_s lie on the main diagonal.

In order to show that T_s is a simplex in $K_0(A)$, we need to show that there are no lattice points interior to the smallest body $Ax \geq b$ containing the vertices of T_s. The vector b for this body is given by

$$\begin{aligned} b_0 &= 0 \\ b_k &= a_k(h^k + sg) \quad \text{for} \quad k \neq 0, i \quad \text{and} \\ b_i &= a_i g. \end{aligned}$$

If the lattice point ξ, not a multiple of "g", is strictly contained in this body then

$$\begin{aligned} a_0\xi &> 0 \\ a_k\xi &> a_k(h^k + sg) \quad \text{for} \quad k \neq 0, i \quad \text{and} \\ a_i\xi &> a_i g. \end{aligned}$$

But then $\xi - sg$ is strictly contained in the body

$$\{0, h - g, E\} = \{0, h^1, \ldots, h - g, \ldots, h^n\},$$

since

$$\begin{aligned} a_0(\xi - sg) &> 0 \quad \text{for} \quad a_0 g \quad \text{close to} \quad 0 \\ a_k(\xi - sg) &> a_k h^k \quad \text{for} \quad k \neq 0, i, \quad \text{and} \\ a_i(\xi - sg) &> (1 - s)a_i g \geq 0 > a_i(h - g) \end{aligned}$$

This contradicts the fact that $\{0, h - g, E\} \in K_0(A)$ and demonstrates that T_s is a simplex in $K_0(A)$. The induction is complete.

In order to complete the proof of Theorem 4, we need to show that if $E + tg$ is a boundary interval then

$$(*) \qquad t = \Big\lfloor \min_j \big(-a_j h^j / a_j g \big) : j > 0,\ a_j g > 0 \Big\rfloor.$$

First of all, we know that $T_t = \{0, h^1 + tg, \ldots, g, \ldots, h^n + tg\} \in K_0(A)$, with row minima on the main diagonal. Therefore $a_j(h^j + tg) < 0$ for $j \neq 0, i$. Since $a_j h^j < 0$, for such j, this inequality is certainly correct if $a_j g < 0$. But if $a_j g > 0$, the inequality implies that $t < \big(-a_j h^j / a_j g \big)$, and therefore

$$t \leq \Big\lfloor \min_j \big(-a_j h^j / a_j g \big) : j > 0,\ a_j g > 0 \Big\rfloor.$$

We shall show that if the inequality is strict then the row minima of the matrix associated with

$$T_{(t+1)} = \{0, h^1 + (t+1)g, \ldots, g, \ldots, h^n + (t+1)g\}$$

lie on the main diagonal and that $T_{(t+1)}$ is a simplex in $K_0(A)$, contradicting the assumption of Theorem 4.

If the inequality is strict so that

$$t + 1 \leq \left\lfloor \min_j \left(-a_j h^j / a_j g \right) : j > 0,\ a_j g > 0 \right\rfloor,$$

then $a_j\big(h^j + (t+1)g\big) < 0$ for $j \neq 0, i$. Since $a_j(h^j + tg) < 0$ for $j \neq 0, i$, we see that $a_j\big(h^j + (t+1)g\big) < a_j g$ for $j \neq 0, i$, and therefore the row minima of the matrix associated with $T_{(t+1)}$ lie on the main diagonal for all rows other than possibly row i.

But since $E + tg$ is on the boundary of Top, it follows from Lemma 1 that $\min_{k \neq 0,i}(a_j h^k + tg) > 0$ for some row j and this can only be row i. Therefore the row minimum in row i in the matrix associated with $T_{(t+1)}$ is in column i, so that the row minima lie in different columns. The smallest body $Ax \geq b$ containing the vertices of $T_{(t+1)}$ is therefore given by

$$\begin{aligned} b_0 &= 0 \\ b_k &= a_k\big(h^k + (t+1)g\big) \quad \text{for} \quad k \neq 0, i \quad \text{and} \\ b_i &= a_i g. \end{aligned}$$

Arguments identical to those previously given show that there are no lattice points interior to this body. Therefore $T_{(t+1)}$ is a simplex in $K_0(A)$, again contradicting the assumption of Theorem 4. ■

Note: We cannot have

$$\min_j \left(-a_j h^j / a_j g \right) : j > 0,\ a_j g > 0 \text{ equal to an integer } t,$$

because then

$$a_j(h^j + tg) = 0 \text{ for some } j.$$

Since $h^j + tg$ is a neighbor, this violates the assumption of genericity $A2$.

7. Recovering Top from Top / Top $[g]$

It is very easy to recover all of the simplicies in Top from a knowledge of the simplicies in Top / Top $[g]$. It suffices, of course, to recover the $n-1$ simplicies. We assume that a list, say L, of the $n-1$ simplicies of Top / Top $[g]$ is known. Figure 17 shows these simplicies in our standard example. Let us begin by recovering the boundary intervals in Top.

7.1. Boundary Intervals in Top $[g]$

Begin by making a list of those $n-2$ faces E of Top / Top $[g]$, which are on the boundary of Top. (We remind the reader that Lemma 1 permits us to determine if a particular face E is on the boundary of Top, even if we do not yet have a full list of the simplicies in Top.) Each such E is contained in a simplex

$$\{0, h^1, \ldots, h, \ldots, h^n\} = \{0, h, E\} \in \text{Top} / \text{Top}\,[g],$$

with h in column i, arranged so that the row minima are on the main diagonal. Because each such face is on the boundary the second smallest entry in row i of the matrix associated with $\{0, h, E\}$ is in column 0. We are looking for those boundary faces E, of Top / Top $[g]$ which initiate a boundary interval

$$\{h, E\}, \{g, g+E\}, \ldots, \{g, tg+E\}$$

with each $E + ug$ in $\partial\,\text{Top}$. In order to find the length of the interval we use $(*)$ to calculate the appropriate value of

$$t = \Big\lfloor \min_j \big(-a_j h^j / a_j g \big) : j > 0,\ a_j g > 0 \Big\rfloor .$$

If $t = 0$, E does not initiate a boundary interval.

7.2. Interior Intervals in Top $[g]$

Examine the $n-2$ faces of Top / Top $[g]$ to see if there is a pair E, $E+tg$ (with $t>0$) which are congruent mod (g). Such a pair will give rise to an interior interval

$$\{h,E\},\{g,g+E\},\dots,\{g,tg+E\}$$

with $\{g,g+E\},\dots,\{g,tg+E\}\in$ Top $[g]$ and $tg+E$ interior to Top.

After the interior intervals have been added there will be no remaining pairs E, $E+ug\in$ Top / Top $[g]$.

8. The Behavior of Top under Continuous Changes in a_0

The plane $a_0x=0$ supports the cone C generated by the set of neighbors with $a_0h>0$. As we vary the normals to this plane, $K_0(A)$ will remain the same until the plane touches one of the generators of the cone, say, g. After the plane passes through g (and no other neighbor) the generator g is replaced by $-g$; the other generators have more dramatic replacements. In this section, I will describe, first without proofs, how to calculate the changes in the entire simplicial complex Top after this perturbation. Let us use the notation Top* for the simplicial complex after the perturbation, and Top$^*[-g]$ for the collection of simplicies in Top* containing $-g$ as a vertex. We shall recover Top$^*[-g]$ using the techniques of the last section.

- Every $n-1$ simplex in Top $[g]$ disappears without an image in Top*.

Let $S=\{0,h^1,\dots,g,\dots,h^n\}$ be a simplex in $K_0(A)$, with row minima on the main diagonal. After the plane passes through g (and no other neighbor), a_0g changes sign. The smallest body of the form $Ax\geq b$ containing $0,h^1,\dots,g,\dots,h^n$ now has $b_0=a_0g$ so that 0 is properly contained in this body and $\{0,h^1,\dots,g,\dots,h^n\}$ is not a simplex.

- Each of the remaining $n-1$ simplicies in Top / Top $[g]$ (in this instance 4 simplicies) is translated by an integral amount. Specifically, if $F=\{h^1,\dots,h^n\}$ is an $n-1$ simplex in Top / Top $[g]$ before this perturbation, it is replaced by $F+tg$ after the perturbation, where

$$t=\Big\lfloor\min_j\big(-a_jh^j/a_jg\big):j>0,\ a_jg>0\Big\rfloor.$$

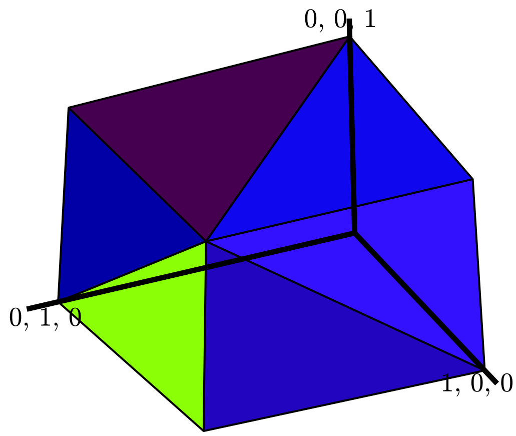

Fig. 15. Top

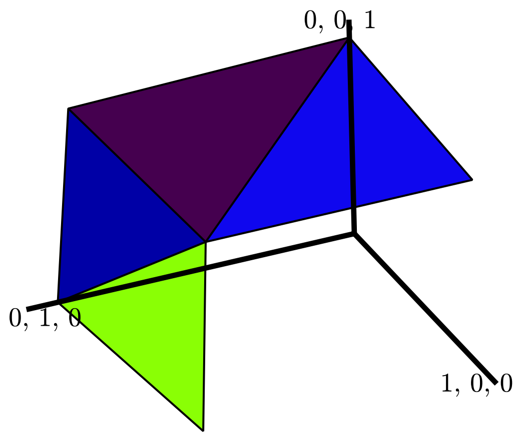

Fig. 16. Top / Top $[g]$

Let $F = \{h^1, \ldots, h^n\} \in$ Top / Top $[g]$, and let $Ax \geq b$ be the smallest body of this form containing $S = \{0, F\}$, with row minima of the matrix associated with S on the main diagonal. The right hand side b does not change after the plane passes though g, but $-a_0 g > 0$, and lattice points of the form $-ug$ (with $u > 0$) may now be contained in this body if

$$-ua_j g > a_j h^j \quad \text{for} \quad j = 1, \ldots, n, \ \text{ or}$$

$$u \leq t = \Big\lfloor \min_j \big(-a_j h^j / a_j g \big) : j > 0, \ a_j g > 0 \Big\rfloor.$$

Since there are no other lattice points in this body, it follows that $\{-tg, h^1, \dots, h^n\} \in K^*(A)$ and therefore $F + tg \in \mathrm{Top}^*$.

As before, we cannot have

$$t = \min_j(-a_j h^j / a_j g) \text{ for some } j,$$

for this would violate the assumption that the matrix A is generic, after $a_0 g$ changes sign.

The above rule requires us to examine every simplex in $\mathrm{Top} / \mathrm{Top}\,[g]$. But it is very easy to work with and gives us a complete description of $\mathrm{Top}^* / \mathrm{Top}^*[-g]$. How can we find the simplicies in $\mathrm{Top}^*[-g]$?

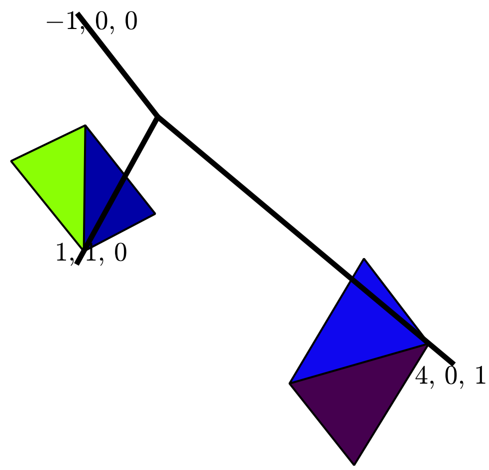

Fig. 17. $\mathrm{Top}^* / \mathrm{Top}^*[-g]$

Boundary Intervals We make a list of those $n-2$ faces E of $\mathrm{Top}^* / \mathrm{Top}^*[-g]$ which are on the boundary of Top^* and such $E + u(-g)$ is not in $\mathrm{Top}^* / \mathrm{Top}^*[-g]$ for $u > 0$. Each such E initiates a boundary interval

$$\{h, E\}, \{-g, -g + E\}, \dots, \big\{ -g, t(-g) + E\big\}$$

of $\mathrm{Top}^*[-g]$, with

$$t = \Big\lfloor \min_j \big(a_j h^j / a_j g\big) : j > 0,\ a_j g < 0\big) \Big\rfloor.$$

This is our earlier t with g replaced by $-g$.

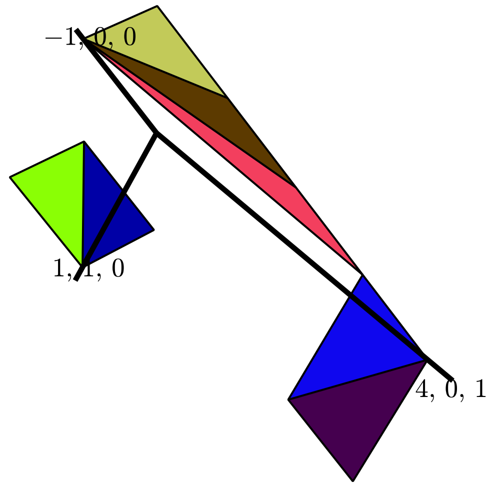

Fig. 18. Adding the Boundary Interval

Our example contains a unique boundary interval which we add to the previous figure.

Interior Intervals As previously described, an interior interval of $\text{Top}^*[-g]$ is determined by a sequence

$$E + tg, E + (t-1)g, \ldots, E$$

with

1. The $n-2$ simplicies $E+tg, E+(t-1)g, \ldots, E+g$ in the boundary of Top^*,
2. The $n-1$ simplicies $\{-g, E+(t-1)g\}, \ldots, \{-g, E\}$ in $\text{Top}^*[-g]$,
3. E interior to Top^*,
4. The first simplex $E + tg$ is not in $\text{Top}^*[-g]$.

We have the full list of simplicies in $\text{Top}^* / \text{Top}^*[-g]$. In order to find an interior interval, we examine the list to find pairs of $n-2$ simplicies

$$E + tg, E$$

in $\text{Top}^* / \text{Top}^*[-g]$. Any such pair will generate an interior interval. Our example has three interior intervals which are added one at a time in each of the following figures.

This is the essential step in a homotopy algorithm.

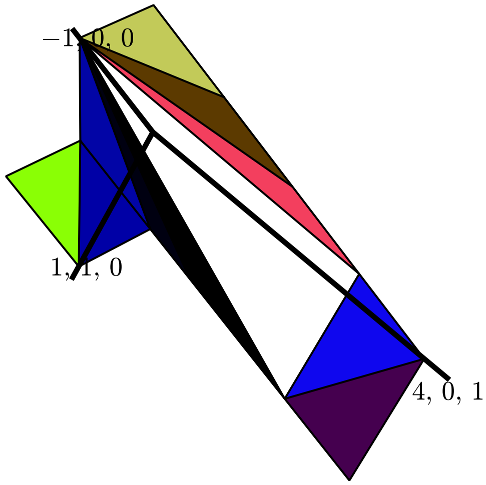

Fig. 19. Adding Interior Interval 1

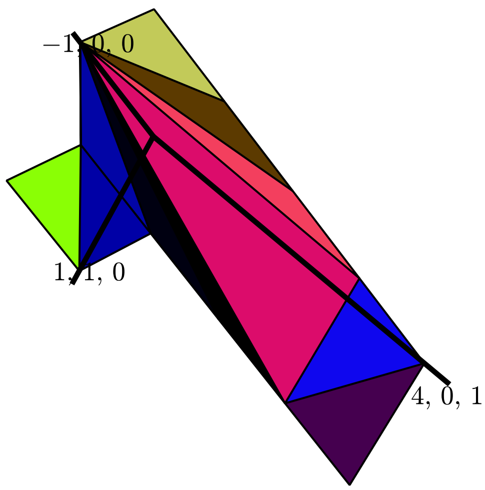

Fig. 20. Adding Interior Interval 2

9. What is Next?

Let us return to the Frobenius problem based on the positive integer vector $c = (c_1, c_2, \ldots, c_n)$. Let

$$G = \{b\colon b = ch, \text{ for some } h \in Z_+^n\},$$

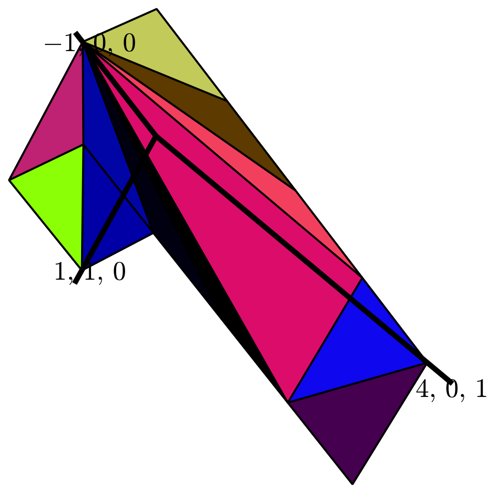

Fig. 21. Adding Interior Interval 3

and define the generating function

$$f(x) = \sum_{b \in G} x^b.$$

There are two remarkable results about $f(x)$, which seem to be at arm's length. It would be extremely interesting to unite them.

In [1] the authors show that – for fixed n – the generating function $f(x)$ can be written as the sum of a polynomial number of rational functions.

To state the second result, let A be an $n \times (n-1)$ integral matrix whose columns generate the lattice $L = \{h = (h_1, \ldots, h_n) \in \mathbb{Z}^n \colon ch = 0\}$ and let $K_0(A)$ be the associated set. Let S be a j dimensional simplex $S \in K_0(A)$ defined by $j+1$ lattice points $\{0, h^1, \ldots, h^j\}$ with $a_0 h^k > 0$ for $k = 1, \ldots, j$. The smallest body of the form $Ax \geq b$ containing $\{0, h^1, \ldots, h^j\}$ will have

$$b(S) = \mathrm{Min}\,[0, Ah^1, \ldots, Ah^j].$$

[2] and [17] showed that if A is generic, then

$$f(x) = \frac{\sum_{S \in K_0(A)} (-1)^{\dim(S)} x^{-c \cdot b(S)}}{\prod_i (1 - x^{c_i})}.$$

What is surprising about these two results is that in general dimension, the number of simplicies in $K_0(A)$ is definitely not small in the size of the matrix A. This suggests that the complex $K(A)$ has sufficient structure so that we can combine the terms in the generating function into a small number of rational functions. What can this structure be? Can it be related to the collection of interior and boundary intervals that characterize Top $[g]$?

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