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INVENTORY MODELS OF THE ARROW-HARRIS-MARSCHAK TYPE WITH TIME LAG

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A factor of considerable importance in the study of inventory or production processes is the time lag involved in effecting policies. Time lags, which generally represent an inability of the process to respond instantly, occur in a number of different forms. In the inventory problem the most important type of time lag is that which represents the time between placing an order and its subsequent delivery. There are situations in which this lag is sufficiently small so that it may be disregarded. However, in many practical cases the lag is of fundamental significance and its neglect would result in the selection of inappropriate inventory policies. In this chapter we shall assume that the time lag is constant. Inventory models in which the time lag is assumed to be a random variable with a known probability distribution are investigated in [5] and [6].

Our model will be essentially that of Arrow, Harris, and Marschak, with the addition of the lag factor. We shall attempt to characterize the optimal inventory policies and indicate their dependence upon the lag. We shall also examine a version of a stationary model associated with an inventory problem involving a lag. The basic model is as follows: The three principal cost factors as usual are ordering cost $c(z)$, handling costs $h(x)$, and penalty or shortage costs $p(x)$. The distribution of demand is given by a positive continuous density function $\varphi(\xi)$. This simplification is introduced in order to expedite the analysis and is definitely not crucial so far as the qualitative nature of the results is concerned. At the expense of tedious detail, it is possible to modify our methods and extend our results to an arbitrary distribution of demand. Between delivery and order we assume there is a lag of λ periods of time with λ fixed. When delivery is made it takes place at the start of each period.

Let x represent current stock size. This includes all stock brought in from past orders. Let $y_1, y_2, \cdots, y_{\lambda-1}$ represent the outstanding orders such that y_1 is due in at the start of the next period, y_2 is to be delivered two periods hence, etc. Define z to be equal to the stock to be ordered at the start of the present period. Finally, let $f(x, y_1, y_2, \cdots, y_{\lambda-1})$ = minimum expected loss following an optimal policy where $(x, y_1, y_2, \cdots, y_{\lambda-1})$ expresses all the information about the current stock level as well as the amounts of goods whose orders have been submitted and are to be delivered during the following $\lambda - 1$ periods.

In discussing this model, it is necessary for us to specify more completely how excess demand is to be taken care of. Two possibilities are usually considered. First, if demand exceeds supply, the extra demand can be thought of as satisfied immediately through priority shipments. Another interpretation of excess demand, which leads to the same model, is to consider such demands as lost sales. We shall refer to this model as model I, as distinguished from model II, which embodies an alternative assumption about excess demand.

In model I, the functional equation satisfied by f is easily seen to be

$$(1)\quad f(x, y_1, y_2, \cdots, y_{\lambda-1}) = \min_{z\geq 0}\Big\{c(z) + L(x) + \alpha f(y_1, y_2, \cdots, y_{\lambda-1}, z)\int_x^\infty \varphi(\xi)\, d\xi + \alpha\int_0^x f(x-\xi+y_1, y_2, \cdots, y_{\lambda-1}, z)\varphi(\xi)\, d\xi\Big\}$$

where

$$L(x) = \int_0^x h(x-\xi)\varphi(\xi)\, d\xi + \int_x^\infty p(\xi - x)\varphi(\xi)\, d\xi\ ,$$

the total expected cost, exclusive of the purchasing cost, for one period when x is available and where α represents the discount factor. The general existence theory tells us that the minimum of (1) is attained [4]. The final integral terms of $L(x)$ correspond to the two contingencies according as demand exceeded supply or supply was sufficient to meet the demand during the first period. These terms express all future expected cumulative costs based on the ordering policy z. Through analysis of (1) we shall be able to characterize the form of the optimal ordering policy.

A second possible way of treating excess demand is to allow for deferring this demand to a later period. This makes it necessary for the current stock level variable x to assume both negative and positive values. A negative value of x means an amount of goods owed to consumption. We assume that the penalty cost of keeping a negative stock level will cumulate if goods are owed for several periods. The functional equation for $f(x, y_1, y_2, \cdots, y_{\lambda-1})$ may be derived as before. We obtain

$$(2)\qquad f(x, y_1, y_2, \cdots, y_{\lambda-1}) = \min_{z\geq 0}\left[c(z) + L(x) + \alpha\int_0^\infty f(x-\xi+y_1, y_2, \cdots, y_{\lambda-1}, z)\varphi(\xi)\,d\xi\right],$$

where $L(x)$ is the same as before when $x > 0$ and

$$L(x) = \int_0^\infty p(\xi - x)\varphi(\xi)\,d\xi$$

for x negative. It is important to take note of the differences between (1) and (2). We shall refer to the case in which negative inventories are permitted as model II. The other situation, where excess demand is considered as lost sales, we have called model I.

A third case of the lag problem would correspond to a situation where some of the excess demand in any given period may be deferred to a later period, while the remaining sales represent lost profits or are satisfied by other means. This is probably the most realistic case. However, in our present discussion we restrict attention to the cases described in models I and II.

Basic differences in the form of the optimal solution in these versions of the inventory problem will be demonstrated.

The following two results manifest one difference:

THEOREM 1. *For model II (equation 2), the optimal policy*

$$z(x, y_1, y_2, \cdots, y_{\lambda-1})$$

is a function of the sum $x + y_1 + y_2 + \cdots + y_{\lambda-1}$ *only.*

THEOREM 2. *If in model I (equation 1), it is necessary to order a positive amount when the stock is small, and unprofitable to order a positive amount when the stock is large, then no optimal policy is a function of the sum.*

One way to interpret Theorem 1 in the language of management is as follows: One should not view stock level exclusively in terms of the physical quantity of stock on hand. Explicitly, stock size shall consist of stock on hand plus stock coming in. In the case where excess demands can be deferred, the decision as to how much to order is to be based on this definition of stock size. A moment's reflection will convince the reader that this is indeed plausible because of the fact that the extra demands can be made up in later periods. This is definitely not the situation of model I. It will be shown later that the optimal policy $z(x, y_1, y_2, \cdots, y_{\lambda-1})$ in model I does not depend functionally in any simple way on its arguments.

The validity of Theorem 1 has been recognized previously. Nevertheless, no proofs have been given in the literature and, generally speaking, little care has been exerted in distinguishing the inventory problems corresponding to the two models described above. We shall continue to emphasize the distinction in view of the differences in the form of the optimal policy.

We turn now to a discussion of the nature of the optimal ordering rule. To this end, let us assume that the ordering cost is linear—i.e., $c(z) = c \cdot z$. The principal assertion for model II is

THEOREM 3. *If $h(x)$ and $p(x)$ are convex increasing and $c(z) = c \cdot z$, the optimal policy $z^* = z^*(x, y_1, y_2, \cdots, y_{\lambda-1})$ is of the form: There exists an $\bar{x}$ such that*

$$z^* = \max\,(0,\ \bar{x} - (x + y_1 + \cdots + y_{\lambda-1}))\,.$$

The critical value $\bar{x}$ may be determined by solving an appropriate transcendental equation (see equation 33). The statement of Theorem 3 is very similar to the results obtained in our general dynamic studies for the case of linear ordering cost [3].

For model I, the solution is strikingly different. We indicate the result for model I, under the assumption that $\lambda = 1$.

THEOREM 4. *If all the cost functions are linear, then the optimal policy $z^*(x)$ in model I, with $\lambda = 1$, has the property that $z^*(x)$ is continuous and of the form*

$$\begin{aligned} z^*(x) &> 0 \qquad x < \bar{x} \\ z^*(x) &= 0 \qquad x \geqq \bar{x}\,. \end{aligned}$$

Moreover, $z^*(x)$ is strictly decreasing for $x < \bar{x}$ while the difference quotient of $z^*(x)$ is in absolute magnitude strictly smaller than 1.

For $0 < x < \bar{x}$ the optimal policy calls for a positive ordering but the amount ordered decreases as a function of current stock level. Nevertheless, the level to which ordering is done is an increasing function of x. On comparison with [2], Section 2, the reader will observe that the optimal policy $z^*(x)$ possesses the same properties as the optimal ordering rule for the one stage inventory model with convex ordering costs. The same is true when one deals with lags involving more than one period. This suggests that the way in which a lag in delivery influences the decision process is analogous to that of a convex ordering cost. This is in sharp contrast to the belief that a lag factor must propose the use of an (s, S) policy. We have pointed out elsewhere [2] that (s, S) policies are intrinsically bound to the case where ordering costs are concave functions of quantity of goods ordered. This appears to be the only reason for employing policies of the (s, S) type.

The conditions of Theorems 3 and 4 may be weakened so as to allow $h(x)$ increasing while $p(x)$ is either concave or convex. In that case, the density of demand $\varphi(\xi)$ must be restricted to be a Pólya frequency function. The analysis for these cases follows along the lines set forth in [2] and [3].

The first four sections are devoted to demonstrating the assertions of Theorems 1 through 4, as well as some extensions and refinements of these results. The final section is concerned with the stationary distribution generated by policies of the form indicated in Theorem 3.

However, we investigate these policies only for the interesting circumstance of model I where excess demand in any period results in lost sales and profits, so that the implication of Theorem 4 is that these policies are not optimal. The stationary distribution in the case of one period lag is completely determined. Some special results are obtained when the duration of lag is more than one period. The special results are of sufficient interest to suggest possible general consequences for the stationary structure of inventory models with lag in delivery. We also carry out the optimization for a linear cost structure with respect to the parameter S which is the level at which stock is kept.

1. Proof of Theorems 1 and 2

In this section we present the proof of Theorem 1. The analysis is elementary and consists of deriving the qualitative properties of the optimal policy by an examination of the functional equation (2). Indeed, inspection of (2) shows that this equation may be written in the form

$$(3)\qquad \begin{aligned} &f(x, y_1, y_2, \cdots, y_{\lambda-1}) \\ &\quad = \min_{z\geq 0}\left[c(z) + a(x) + \alpha\int_0^\infty f(x + y_1 - \xi, y_2, \cdots, y_{\lambda-1}, z)\varphi(\xi)\, d\xi\right] \\ &\quad = a(x) + \min_{z\geq 0}\left[c(z) + \alpha\int_0^\infty f(x + y_1 - \xi, y_2, \cdots, y_{\lambda-1}, z)\varphi(\xi)\, d\xi\right]. \end{aligned}$$

When the minimum of (3) is performed, it is evident that z^* is a function of the form $z^* = z^*(x + y_1, y_2, y_3, \cdots, y_{\lambda-1})$. Inserting this value into (3), we conclude

$$(4)\qquad f(x, y_1, y_2, \cdots, y_{\lambda-1}) = a(x) + b(x + y_1, y_2, y_3, \cdots, y_{\lambda-1})\,.$$

This equation shows that the optimal loss depends on y_1 only through the sum of $x + y_1$. The representation (4) for f may be used in (3) again and we find

$$(5)\qquad \begin{aligned} &f(x, y_1, y_2, \cdots, y_{\lambda-1}) \\ &\quad = a(x) + \min_{z\geq 0}\left\{c(z) + \alpha\int_0^\infty\left[a(x + y_1 - \xi) \right.\right. \\ &\qquad\qquad \left.\left. + b(x + y_1 + y_2 - \xi, y_3, y_4, \cdots, y_{\lambda-1}, z)\right]\varphi(\xi)\, d\xi\right\} \\ &\quad = a(x) + a_1(x + y_1) \\ &\qquad + \min_{z\geq 0}\left[c(z) + \alpha\int_0^\infty b(x + y_1 + y_2 - \xi, y_3, \cdots, y_{\lambda-1}, z)\varphi(\xi)\, d\xi\right]. \end{aligned}$$

This expression of the functional equation for f shows that $z^*(x, y_1, y_2, \cdots, y_{\lambda-1})$ is of the form $z^* = z^*(x + y_1 + y_2, y_3, \cdots, y_{\lambda-1})$. Inserting this result into (5) gives

$$f(x, y_1, y_2, \cdots, y_{\lambda-1}) = a(x) + a_1(x + y_1) + b_1(x + y_1 + y_2, y_3, \cdots, y_{\lambda-1})\,.$$

Iteration of this idea yields the results:

$$(6)\quad f(x, y_1, y_2, \cdots, y_{\lambda-1}) = a(x) + a_1(x + y_1) + a_2(x + y_1 + y_2) + \cdots + a_{\lambda-1}(x + y_1 + y_2 + \cdots + y_{\lambda-1}),$$

and $z^*(x, y_1, y_2, \cdots, y_{\lambda-1})$ is of the form $z^* = z^*(x + y_1 + y_2 + \cdots + y_{\lambda-1})$. This completes the proof of Theorem 1.

The expression of (6) and its derivation possess an additional interpretation of interest for the inventory problem. The function $a(x)$ represents the expected costs, excluding the ordering cost factor $c(z)$, incurred through the first period as a function of the present stock level. All the future deliveries $y_1, y_2, \cdots, y_{\lambda-1}$ cannot evidently affect the present costs, and it is obvious that the amount ordered z cannot be relevant until λ periods later.

Examination of (5) implies that $a_1(x + y_1)$ expresses (except for the ordering cost) the costs incurred during the second period. It is intuitively clear that such costs are functions of $x + y_1$, the total stock delivered by the start of period two.

Analogously, $a_r(x + y_1 + \cdots + y_r)$ represents the expected costs, excluding ordering, incurred in the future during the $(r + 1)$st period. The fact that a_r is a function of $x + y_1 + \cdots + y_r$ is again logical because of the natural meaning ascribed to the quantities $y_1, y_2, \cdots, y_r$ which represent goods delivered prior to the rth period. Finally, projecting λ periods into the future, we find that $a_{\lambda-1}(x + y_1 + \cdots + y_{\lambda-1})$ represents the total expected costs for the λth period and beyond, including ordering cost in the first period. The aggregation of the ordering costs incurred for the first period with the total expected costs due to handling and shortage for the λth and subsequent periods seems natural in view of the fact that the quantity of goods ordered influences the inventory control problem for the first time at the λth period. This means that the optimal ordering rule has the property that we order that amount now which minimizes total expected costs projecting λ periods into the future and thereafter. The procedure of making the decision now by examining costs starting λ periods in the future is a recognized practice. The previous discussion substantiates this mode of operation. We shall see later that this is not a valid method in dealing with a model of type I.

We now turn to a proof of Theorem 2. Let us assume, to the contrary, that there is an optimal policy z which is of the form $z(x + y_1 + \cdots + y_{\lambda-1})$. We shall show as a consequence that the policy of never ordering is also optimal, thereby contradicting the assumptions of Theorem 2. We shall demonstrate this fact for $\lambda = 2$; the proof, however, is perfectly general.

Let us assume that the function $z(u)$ has only a finite number of discontinuities (this is correct if the policy is unique; if the policies are not unique, then a particular one of this form may be selected).

LEMMA 1. *Let $z(\tilde{u}) = \tilde{z} > 0$. Then*

$$\frac{\partial}{\partial y} f(x, y),$$

evaluated at $y = \tilde{z}$, is independent of x, for $x \leqq \tilde{u}$.

PROOF. Let us select any value of (x, y) with $x + y = \tilde{u}$. If we define

$$G(x, y; z) = \left\{c(z) + L(x) + \alpha f(y, z)\int_x^\infty \varphi(\xi)\, d\xi + \alpha\int_0^x f(x + y - \xi, z)\varphi(\xi)\, d\xi\right\},$$

it follows from equation (1) that $\tilde{z}$ is a root of

$$\frac{\partial G}{\partial z} = 0 .$$

Inasmuch as the optimal policy is a function of the sum, it follows that $\tilde{z}$ is also a root of the same equation with (x, y) replaced by $(x + h, y - h)$, and therefore we obtain

$$\frac{\partial^2 G}{\partial z \partial x} - \frac{\partial^2 G}{\partial z \partial y} = 0, \quad \text{for } z = \tilde{z} .$$

If this quantity is evaluated directly from its definition, it is seen to be

$$-\alpha \frac{\partial^2 f(y, z)}{\partial y \partial z}\int_x^\infty \varphi(\xi)\, d\xi ,$$

and we therefore obtain

$$\frac{\partial^2 f(y, z)}{\partial y \partial z} = 0 \quad \text{for } z = \tilde{z} ,$$

this relation being valid for any $y \leqq \tilde{u}$. The proof of the lemma follows by integrating with respect to y.

This lemma may be applied in the following way. Since $z(u)$ has only a finite number of discontinuities, it follows that the range of z consists of a finite number of disjoint intervals. Let z_1 and z_2 be any two points in the same interval (we assume that z_1 is optimal for $x + y = u_1$ and z_2 for u_2, with $u_1 \leqq u_2$). It follows from Lemma 1 that

$$\begin{aligned} f(y, z_1) - f(y, z_2) &= \int_{z_2}^{z_1} \frac{\partial f(y, z)}{\partial z}\, dz \\ &= K(z_1, z_2) , \end{aligned}$$

for any $y \leqq u_1$. Therefore, for $x + y \leqq u_1$,

$$G(x, y; z_1) = G(x, y; z_2) + \{c(z_1) - c(z_2)\} + K(z_1, z_2) .$$

Now, since z_1 is optimal for $x + y = u_1$ we see that z_1 minimizes $c(z) + K(z, z_2)$ for all z in the particular interval in question. Therefore if (x, y) is a point with $x + y < u_1$ and which has an optimal policy in the same interval, this policy can do no better than z_1. If u_1 is chosen close to the max $(x + y)$ for all (x, y) which have an optimal policy in the interval in question, we obtain

LEMMA 2. *If there is an optimal policy in model I which is a function of the sum, then there exists a function $z(u)$ which is constant over a finite number of intervals, such that $z(x+y)$ is also optimal.*

Let us now assume that we are discussing a policy of the sort described in Lemma 2, and let u be a point at which the function z has a jump, from z_1 to z_2. From considerations of the continuity of G, it follows that both z_1 and z_2 are optimal at u. In other words, if $x+y=u$, then $G(x, y; z_1) \equiv G(x, y; z_2)$, and consequently

$$\frac{\partial G(x, y; z_1)}{\partial x} - \frac{\partial G(x, y; z_1)}{\partial y} \equiv \frac{\partial G(x, y; z_2)}{\partial x} - \frac{\partial G(x, y; z_2)}{\partial y} .$$

Evaluating this equation by a direct reference to the definition of G, we see that

$$\frac{\partial f(y, z_1)}{\partial y} = \frac{\partial f(y, z_2)}{\partial y} \quad \text{for } y \leqq u .$$

Therefore $f(y, z_1) - f(y, z_2)$ is independent of y, and we obtain, as in the proof of Lemma 2,

$$G(x, y; z_1) = G(x, y; z_2) + c(z_1) - c(z_2) + [f(y, z_1) - f(y, z_2)] ,$$

for all $x + y \leqq u$. It follows that

$$c(z_2) - c(z_1) + [f(y, z_2) - f(y, z_1)] = 0$$

for $x + y = u$; but this condition is independent of y, and therefore $G(x, y; z_1) \equiv G(x, y; z_2)$ for all $x + y \leqq u$.

This argument shows us that z_2 is also optimal in the interval below the discontinuity u, and permits us to form a new optimal policy with one less jump. We have therefore demonstrated

LEMMA 3. *If model I has an optimal policy which is a function of the sum, then it also has an optimal policy which tells us to order the same amount regardless of the stock levels.*

The assumptions of Theorem 2 are inconsistent with Lemma 3, which proves Theorem 2.

2. Characterizations of the Optimal Policy for Model II

The proof of Theorem 3 will be presented in stages. We first deal with the case of a one stage lag. The proof of this theorem is carried out by truncating the number of periods to n and subsequently letting $n \to \infty$. Let $f_n(x)$ represent the minimum expected cost if x is available now, given that only n future periods are to be taken into account. Analogous to (2), we find that

$$(7) \qquad f_n(x) = \min_{z \geqq 0} \left\{ c \cdot z + L(x) + \alpha \int_0^\infty f_{n-1}(x + z - \xi) \varphi(\xi)\, d\xi \right\}$$

where

$$(8)\qquad L(x) = \int_0^x h(x-\xi)\varphi(\xi)\,d\xi + \int_x^\infty p(\xi - x)\varphi(\xi)\,d\xi, \qquad x > 0$$

$$\text{and} \quad \int_0^\infty p(\xi - x)\varphi(\xi)\,d\xi, \qquad x < 0 .$$

In order to avoid an unnecessary enumeration of cases, we impose the assumption that

$$(9)\qquad \lim_{x\to\infty} \; p'(x) > \frac{1-\alpha}{\alpha} c .$$

This is an exceedingly mild restriction. For example, if the penalty costs are linear $p \cdot x$, then (9) states

$$p > \frac{1-\alpha}{\alpha} c .$$

Since α is usually very close to 1, this is undoubtedly satisfied.

Let $z_n^*(x)$ represent the optimal policy for the n-stage problem. The general theory asserts that $f_n(x) \to f(x)$ where $f(x)$ satisfies (2). We shall establish that $z_n^*(x)$ converges to an optimal policy $z^*(x)$ for the full dynamic problem, with $z^*(x)$ explicitly determined. The proof is by induction on the number of periods n.

Suppose we have shown that

$$(10)\qquad z_n^*(x) = \begin{cases} \bar{x}_n - x & x < \bar{x}_n \\ 0 & x > \bar{x}_n , \end{cases}$$

where $\bar{x}_n$ is the unique solution of the equations

$$(11)\qquad c + \alpha \int_0^\infty f'_{n-1}(\bar{x}_n - \xi)\varphi(\xi)\,d\xi = 0 .$$

Suppose further that $\bar{x}_n$ has the properties that

(i) $\bar{x}_n \geqq \bar{x}_{n-1}$;

(ii) $$f'_n(x) = \begin{cases} -c + L'(x) & x < \bar{x}_n \\ L'(x) + \alpha \int_0^\infty f'_{n-1}(x-\xi)\varphi(\xi)\,d\xi & x > \bar{x}_n ; \end{cases}$$

(iii) $f_n(x)$ is a convex function of x. Moreover, the second derivative of f_n exists everywhere except possibly for $x = \bar{x}_n$ where the right- and lefthand second derivatives exist;

(iv) $-f'_n(x) \geqq -f'_{n-1}(x)$ for all x .

Assuming that properties (i) through (iv) and equations (8) and (9) are correct, and that $z_n^*(x)$ is determined by means of (10) and (11), we establish that these same properties are valid for the ordering rule for $z_{n+1}^*(x)$. Indeed,

$$(12)\qquad f_{n+1}(x) = \min_{z\geq 0} \left\{ c \cdot z + L(x) + \alpha \int_0^\infty f_n(x + z - \xi)\varphi(\xi)\,d\xi \right\} .$$

Differentiating the bracketed expression with respect to z yields

$$(13)\qquad c + \alpha \int_0^\infty f'_n(x + z - \xi)\varphi(\xi)\,d\xi .$$

We replace $x + z$ by ω and regard (13) as a function of ω. Explicitly set

$$H_n(\omega) = c + \alpha\int_0^\infty f_n'(\omega - \xi)\varphi(\xi)\,d\xi\ . \tag{14}$$

From (iii), it follows that $H_n(\omega)$ is increasing, and by (ii) and assumption (9), $H_n(\omega)$ has a negative value for $\omega \to -\infty$. But

$$\lim_{\omega\to\infty} H_n(\omega) > 0$$

so that $H_n(\omega)$ has at least one zero. Since φ is strictly positive, it follows, because of the special form of $H_n(\omega)$, that $H_n(\omega)$ is at worst necessarily constant for a half interval stretching to $-\infty$ and then strictly increases as ω traverses the infinite line from left to right. Hence, $H_n(\omega)$ has a unique zero, $\bar{x}_{n+1}$. Because of (iv) we find that $H_n(\omega) \leqq H_{n-1}(\omega)$ and hence $\bar{x}_{n+1} \geqq \bar{x}_n$. Inspection of (13) in conjunction with (14) implies that if $x < \bar{x}_{n+1}$, then the minimum of the bracketed quantity of (12) is achieved for $z^*_{n+1}(x) = \bar{x}_{n+1} - x$ where $\bar{x}_{n+1}$ is the unique solution of $H_n(\omega) = 0$. To continue the induction we must check all the properties (i), (ii), (iii), and (iv) for f_{n+1}. The truth of (i) $\bar{x}_{n+1} \geqq \bar{x}_n$ has already been indicated. Since

$$f_{n+1}(x) = \begin{cases} c\cdot[\bar{x}_{n+1} - x] + L(x) + \alpha\int_0^\infty f_n(\bar{x}_{n+1} - \xi)\varphi(\xi)\,d\xi & x < \bar{x}_{n+1} \\ L(x) + \alpha\int_0^\infty f_n(x - \xi)\varphi(\xi)\,d\xi & x > \bar{x}_{n+1}\ , \end{cases} \tag{15}$$

we obtain (ii). We next verify condition (iv). For $x > \bar{x}_{n+1} \geqq \bar{x}_n$, using the induction step, we get

$$\begin{aligned} -f_{n+1}'(x) &= -L(x) - \alpha\int_0^\infty f_n'(x - \xi)\varphi(\xi)\,d\xi \geqq -L(x) - \alpha\int_0^\infty f_{n-1}'(x - \xi)\varphi(\xi)\,d\xi \\ &= -f_n'(x)\ . \end{aligned}$$

For $x \leqq \bar{x}_n$, it follows that $-f_{n+1}'(x) = -f_n'(x)$. Finally, let $\bar{x}_n < x < \bar{x}_{n+1}$, then $-f_{n+1}'(x) = c - L'(x)$. But $c > -\alpha\int f_{n-1}'(x - \xi)\,d\xi$ in this range. Combining, we get $-f_{n+1}'(x) \geqq -f_n'(x)$ and the proof of (iv) is complete. Since $L(x)$ is convex and $f_n'(x)$ is continuous, it follows readily that $f_{n+1}''(x) \geqq 0$ for all $x \neq \bar{x}_{n+1}$. From this we conclude easily that f_{n+1} is convex and behaves as indicated in (iii). The induction step from n to $n+1$ is finished. To complete the proof it is necessary to verify all the properties of (i) through (iv) for f_2 and f_1 where $f_1(x) = L(x)$. This requires the same kind of reasoning as in the general case and we omit the details.

Allowing n to tend to infinity, we deduce the existence of $\bar{x}$, necessarily finite, such that

$$z^*(x) = \begin{cases} \bar{x} - x & x < \bar{x} \\ 0 & x \geqq \bar{x} \end{cases} \tag{16}$$

and such that $\bar{x}$ is the unique solution of the equation

$$c + \alpha\int_0^\infty f'(\bar{x} - \xi)\varphi(\xi)\,d\xi = 0\ , \tag{17}$$

where f is convex.

We shall now derive a transcendental equation from which $\bar{x}$ may be computed. The functional equation (2) with z^* described in (16) becomes for $x < \bar{x}$,

$$f(x) = c[\bar{x} - x] + L(x) + \alpha\int_0^\infty f(\bar{x} - \xi)\varphi(\xi)\,d\xi\ .$$

Hence

$$f'(x) = -c + L'(x) \quad \text{for the interval } x < \bar{x}\ . \tag{18}$$

The expression (17) involves $f'(u)$ for $u < \bar{x}$ which is known through (18). Consequently, $\bar{x}$ is the unique solution of the equation

$$c[(1 - \alpha)] + \alpha\int_0^\infty L'(\bar{x} - \xi)\varphi(\xi)\,d\xi = 0\ . \tag{19}$$

The uniqueness is assured by employing an argument analogous to the method by which we demonstrated the fact that $H_n(\omega)$ possessed a unique solution. To sum up:

THEOREM 3a. *If the assumptions of Theorem 3 are satisfied and the length of the lag is $\lambda = 1$, then the optimal policy is of the form*

$$z^*(x) = \begin{cases} \bar{x} - x & x < \bar{x} \\ 0 & x \geqq \bar{x}\ , \end{cases}$$

where $\bar{x}$ is determined as the unique solution of (19).

To further illustrate the methodology, we continue by presenting an analysis of the two stage lag problem. This case already embodies the features of the general proof. The functional equation in the two stage lag truncated to n periods ($n > 2$) can be expressed in the form

$$f_n(x, y_1) = \min_{z\geq 0}\left\{c \cdot z + L(x) + \alpha\int_0^\infty f_{n-1}(x + y_1 - \xi,\ z)\varphi(\xi)\,d\xi\right\}\ . \tag{20}$$

Suppose we have shown that f_{n-1} possesses the following properties:

(i) $f_{n-1}(x,\ y_1) = L(x) + b_{n-1}(x + y_1)$;

(ii) $z^*_{n-1}(x,\ y_1) = \bar{x}_{n-1} - x - y_1$ for $x + y_1 < \bar{x}_{n-1}$,
$= 0$ otherwise,

where $\bar{x}_{n-1}$ is the unique solution of the equation

$$c + \alpha\int_0^\infty b'_{n-2}(\omega - \xi)\varphi(\xi)\,d\xi = 0\ ; \tag{21}$$

(iii) $\bar{x}_{n-1} \geqq \bar{x}_{n-2}$;

(iv) $$b'_{n-1}(x + y_1) = \begin{cases} -c + \alpha\int_0^\infty L'(x + y_1 - \xi)\varphi(\xi)\,d\xi & y_1 + x < \bar{x}_{n-1} \\ \alpha\int_0^\infty L'(x + y_1 - \xi) + \alpha\int_0^\infty b'_{n-2}(x + y_1 - \xi)\varphi(\xi)\,d\xi & y_1 + x \geqq \bar{x}_{n-1} \end{cases}$$

and is continuous in the variable $x + y_1$;

(v) b_{n-1} is convex in its argument and $b''_{n-1}(u) > 0$ for all u except $u = \bar{x}_{n-1}$ where right- and lefthand derivatives exist ;

(vi) $-b'_{n-1}(u) \geqq -b'_{n-2}(u)$ for all u .

We advance the induction by demonstrating that all these properties persist as we pass from $n-1$ to n. Inserting (i) into (20), we deduce that

$$f_n(x,\ y) = L(x) + \alpha\int_0^\infty L(x + y_1 - \xi)\varphi(\xi)\,d\xi + \min_{z\geq 0}\left\{c \cdot z + \alpha\int_0^\infty b_{n-1}(x + y_1 + z - \xi)\varphi(\xi)\,d\xi\right\} = L(x) + b_n(x + y_1)\ , \tag{22}$$

where

$$b_n(x + y_1) = \alpha\int_0^\infty L(x + y_1 - \xi)\varphi(\xi)\,d\xi + \min_{z\geq 0}\left\{c \cdot z + \alpha\int_0^\infty b_{n-1}(x + y_1 + z - \xi)\varphi(\xi)\,d\xi\right\}\ . \tag{23}$$

The derivative of the bracketed term of (23) with respect to z yields

$$c + \alpha\int_0^\infty b'_{n-1}(x + y_1 + z - \xi)\varphi(\xi)\,d\xi = K_n(\omega) \tag{24}$$

with $\omega = x + y_1 + z$.

Since $b_{n-1}(\omega)$ is convex, we conclude that $K_n(\omega)$ is increasing. Analogous to the analysis of (11), we find that $K_n(\omega)$ possesses a unique zero which we denote by $\bar{x}_n$. By use of (vi) we deduce that $\bar{x}_n \geqq \bar{x}_{n-1}$. The derivation of the analog of (iv) for $b'_n(x + y_1)$ is straightforward. Because of the definition of $\bar{x}_n$ we observe also that b'_n is continuous. The truth of (vi) is demonstrated as previously by considering three cases : $x + y_1 < \bar{x}_{n-1}$, $x + y_1 > \bar{x}_n$, and $\bar{x}_{n-1} < x + y_1 < \bar{x}_n$. We illustrate only the third :

$$b'_n(x + y_1) = -c + \alpha\int_0^\infty L'(x + y_1 - \xi)\varphi(\xi)\,d\xi\ .$$

But, $\omega = x + y_1 > \bar{x}_{n-1}$ implies $-c < \alpha\int b'_{n-2}(\omega - \xi)\varphi(\xi)\,d\xi$. Hence,

$$b'_n(x + y_1) < \alpha\int_0^\infty L'(x + y_1 - \xi)\varphi(\xi)\,d\xi + \alpha\int_0^\infty b'_{n-2}(x + y_1 - \xi)\varphi(\xi)\,d\xi = b'_{n-1}(x + y_1)\ .$$

Property (v) is verified for b_n by straight differentiation. This completes the induction step. The truth of (i) through (vi) for f_3 is obtained by direct calculation with $f_1(x)$ defined as $L(x)$, and $f_2(x,\ y_1)$ set to equal

$$\alpha\int_0^\infty L(x + y_1 - \xi)\varphi(\xi)\,d\xi\ .$$

Letting n tend to infinity, we deduce that the optimal policy is of the form

$$z^*(x,\ y_1) = \begin{cases} \bar{x} - x - y_1 & x + y_1 < \bar{x} \\ 0 & x + y_1 \geqq \bar{x}\ . \end{cases}$$

It remains to show how to evaluate $\bar{x}$. The limit on n in (i) yields

$$f(x,\ y_1) = L(x) + b(x + y_1)\ , \tag{25}$$

where

$$b'(\omega) = \begin{cases} -c + \alpha\int_0^\infty L'(\omega - \xi)\varphi(\xi)\,d\xi & \omega < \bar{x} \\ \alpha\int_0^\infty L'(\omega - \xi)\varphi(\xi)\,d\xi + \alpha\int_0^\infty b'(\omega - \xi)\varphi(\xi)\,d\xi & \omega > \bar{x}\ , \end{cases} \tag{26}$$

and $\bar{x}$ is the unique solution of

$$c + \alpha\int_0^\infty b'(\bar{x} - \xi)\varphi(\xi)\,d\xi = 0. \tag{27}$$

Observe that to evaluate (27) we need to know $b'(u)$ only for $u < \bar{x}$. But by (26), over this range

$$b'(u) = -c + \alpha\int_0^\infty L'(u - \xi)\varphi(\xi)\,d\xi\ .$$

Substituting this value of $b'(u)$ into (27), we deduce that $\bar{x}$ is determined as the unique solution of

$$c(1 - \alpha) + \alpha^2\int_0^\infty\int_0^\infty L'(\bar{x} - \xi - \eta)\varphi(\xi)\varphi(\eta)\,d\xi\,d\eta = 0. \tag{28}$$

The analysis of the two stage lag case is complete. We state the results for the general λ lag model. The induction proof in this case is the natural extension of the preceding arguments. We have

$$\begin{aligned} f(x,\ y_1,\ y_2,\ \cdots,\ y_{\lambda-1}) &= L(x) + L_1(x + y_1) + L_2(x + y_1 + y_2) + \cdots \\ &\quad + L_{\lambda-2}(x + y_1 + y_2 + \cdots + y_{\lambda-2}) + b(x + y_1 + y_2 + \cdots + y_\lambda) \end{aligned} \tag{29}$$

and

$$z^*(x,\ y_1,\ y_2,\ \cdots,\ y_{\lambda-1}) = \begin{cases} \bar{x} - (x + y_1 + y_2 + \cdots + y_{\lambda-1}) & \\ \qquad\qquad \text{for } x + y_1 + \cdots + y_{\lambda-1} < \bar{x} \\ 0 \quad \text{otherwise,} \end{cases} \tag{30}$$

where

(31) $L_r(u)$ is defined as equal to $\int_0^\infty L_{r-1}(u - \xi)\varphi(\xi)\,d\xi\ ,\quad L_0(u) = L(u)\ ,$

and

$$b'(\omega) = \begin{cases} -c + \alpha\int_0^\infty L'_{\lambda-2}(\omega - \xi)\varphi(\xi)\,d\xi & \omega < \bar{x} \\ L'_{\lambda-2}(\omega - \xi) + \alpha\int_0^\infty b'(\omega - \xi)\varphi(\xi)\,d\xi & \omega > \bar{x}\ . \end{cases} \tag{32}$$

The critical value $\bar{x}$ is computed as the unique solution of the equation

$$(33)\quad c(1-\alpha) + \alpha^{\lambda}\int_0^{\infty}\cdots\int_0^{\infty} L'(\bar{x}-\xi_1-\xi_2-\cdots-\xi_\lambda)\varphi(\xi_1)\varphi(\xi_2)\cdots\varphi(\xi_\lambda)\,d\xi_1\cdots d\xi_\lambda = 0\,.$$

The principal consequence of our discussion is the following important theorem:

THEOREM 3b. *If the hypotheses of Theorem 3 are satisfied and the length of lag is λ periods, then the optimal policy is of the form described in (30) where $\bar{x}$ is determined satisfying (33), or equivalently*

$$(34)\qquad c(1-\alpha) + \alpha^{\lambda}\int_0^{\infty} L'_{\lambda-1}(\bar{x}-\xi)\varphi(\xi)\,d\xi = 0\,.$$

It is interesting to observe that $\bar{x}$ as a function of the length of the lag is increasing, provided α is chosen sufficiently close to 1. Let y_r denote the unique zero of

$$\int_0^{\infty} L'_{r-1}(y_r-\xi)\varphi(\xi)\,d\xi = 0\,.$$

Since $L_{r-1}(u)$ is convex, we obtain

$$L'_{r-1}(y_r-\xi) \geqq L'_{r-1}(y_r-\xi-\eta)$$

for any ξ and η positive, and strict inequality holds somewhere. Therefore,

$$(35)\qquad 0 = \int_0^{\infty} L'_{r-1}(y_r-\xi)\varphi(\xi)\,d\xi > \int_0^{\infty}\int_0^{\infty} L'_{r-1}(y_r-\xi-\eta)\varphi(\xi)\varphi(\eta)\,d\xi\,d\eta = \int_0^{\infty} L'_r(y_r-\xi)\varphi(\xi)\,d\xi\,.$$

Since

$$\int_0^{\infty} L'_r(y-\xi)\varphi(\xi)\,d\xi$$

is an increasing function of y, we may conclude because of (35) that

$$y_{r+1} > y_r\,.$$

The critical values $\bar{x}_r$ are the solutions of the equation

$$(36)\qquad c(1-\alpha) + \alpha^{r}\int L'_{r-1}(\bar{x}_r-\xi)\varphi(\xi)\,d\xi = 0\,.$$

Obviously, as $\alpha \to 1$, $\bar{x}_r \to y_r$. Consequently, for α sufficiently close to 1,

$$\bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_{\lambda-1}\,.$$

The accompanying tabulation of the critical values for $\lambda = 1, 2$, where

$$\varphi(\xi) = \frac{\mu^k \xi^{k-1} e^{-\mu\xi}}{k!}\,,$$

is of some interest. Here $\alpha = 1$ and $h = p$. In particular, we see that if the demand density is an exponential with unit average demand, the optimal level of stock for one lag period is 1.68 and for two lag periods is 2.67. On the other hand, if the demand has a density which is a gamma of order one with average demand unity, the optimal quantity of stock to hold if delivery involves a lag of one period is 1.83, and

Optimal x^*

$\varphi(\xi)$	Lag of One Period	Lag of Two Periods
$\mu e^{-\mu\xi}$	$\frac{1.68}{\mu}$	$\frac{2.67}{\mu}$
$\mu^2\xi e^{-\mu\xi}$	$\frac{2}{\mu}(1.83)$	$\frac{2}{\mu}(2.83)$

for a two stage lag, 2.83. This might be explained as due to the fact that the median as a function of the order of the gamma distribution is increasing. In fact, as the order of the gamma family becomes infinite, the demand distribution concentrates at a point and is therefore predictable. In this case, the optimal policy for a λ-stage lag would be $\lambda + 1$ times mean demand.

3. Solution of One Stage Lag, Model I

We devote this section to a discussion of the proof of Theorem 4. A recapitulation of the set-up of the infinite stage inventory model in the presence of a time lag of one period should be helpful.

The purchasing cost is assumed to be linear $c \cdot z$. The penalty cost is also taken to be linear ($p(x) = p \cdot x$), while the storage may be an arbitrary convex increasing function. The costs at each succeeding stage will be discounted by a factor α.

The distribution of demand is given as usual by a positive continuous density function $\varphi(\xi)$. Let $f(x)$ equal the minimum expected loss following an optimal policy given that x is the current available stock. The functional equation satisfied by f is a special circumstance of (1) and reduces to

$$(37)\quad f(x) = \min_{z \geq 0}\left[c \cdot z + L(x) + \alpha f(z)\int_x^\infty \varphi(\xi)\,d\xi + \alpha\int_0^x f(x + z - \xi)\varphi(\xi)\,d\xi\right]$$

where, as before,

$$L(x) = \int_0^x h(x - \xi)\varphi(\xi)\,d\xi + \int_x^\infty p(\xi - x)\varphi(\xi)\,d\xi\ .$$

The variable x takes on only positive values as distinguished from the situation of model II, since excess demand in any single period may be regarded as satisfied by other means. Whenever demand exceeds x in a given stage, the quantity z which has been ordered will represent the totality of stock available at the start of the following period. Otherwise, $x + z - \xi$ is the stock size of the second period. These two contingencies are expressed by the last two terms of (37). As before, our method of analysis approaches the dynamic problem through a study of a sequence of truncated models. We first solve the problem when we consider only looking ahead for a total of n periods. Then, allowing n to become infinitely large, we can deduce the form of the optimal policy for the dynamic model. For $n = 1$,

$$f_1(x) = \min_{z \geq 0} \{c \cdot z + L(x)\} .$$

Clearly, the solution is $z = 0$ and f_1 has the properties $-f_1'(x) \leqq p$ and $f_1''(x) > 0$. We advance an induction argument for the truncated problem of n periods to that of $n+1$ periods. We suppose the following properties regarding the solution of the n-stage problem:

(i) If $z_n^*(x)$ represents the quantity ordered according to the optimal policy, taking account of all possibilities that may arise for a totality of n periods in the future, then

$$(38) \qquad -1 < \frac{dz_n^*(x)}{dx} < 0 \text{ for } x < \bar{x}_n \quad \text{and} \quad z_n^*(x) = 0 \text{ for } x \geqq \bar{x}_n ;$$

(the value zero for $\bar{x}_n$ is not excluded);

(ii) $-f_n'(x) \leqq p$ ($f_n'(x)$ exists and is continuous);

(iii) $f_n''(x) > 0$ for all $x > 0$;

(iv) $\lim_{x \to \infty} f_n'(x) \geqq 0$.

The functional equation for the $(n+1)$-period problem is

$$(39) \qquad f_{n+1}(x) = \min_{z \geq 0} \Big[c \cdot z + L(x) + \alpha \int_0^x f_n(x + z - \xi)\varphi(\xi)\, d\xi + \alpha f_n(z) \int_x^\infty \varphi(\xi)\, d\xi \Big] .$$

To perform the minimum operation of (39), we differentiate the bracketed formula with respect to z. This gives

$$(40) \qquad c + \alpha \int_0^x f_n'(x + z - \xi)\varphi(\xi)\, d\xi + \alpha f_n'(z) \int_x^\infty \varphi(\xi)\, d\xi = K_n(z\,;\ x) .$$

In view of (iii), $K_n(z\,;\,x)$ is a strictly increasing function in each variable when the other variable is held fixed. By (iv), $K_n(z\,;\,x)$ is positive for x sufficiently large. Let $z_{n+1}^*(x)$ denote the unique zero of (40) if such exists; otherwise set $z_{n+1}^*(x) = 0$. $z_{n+1}^*(x)$ is bounded by (iv). Furthermore, since $z_{n+1}^*(x)$ is obtained as the unique solution of the relation $K_n(z\,;\,x) = 0$ (x fixed), it follows that $z_{n+1}^*(x)$ is a continuous function of x. Since $K_n(z\,;\,x)$ strictly increases with x, we deduce also that z_{n+1}^* has a continuous derivative and

$$(41) \qquad \frac{dz_{n+1}^*(x)}{dx} < 0 \quad \text{for } x < \bar{x}_{n+1}, \quad \text{with } z_{n+1}^*(x) = 0$$

for x beyond $\bar{x}_{n+1}$. Differentiation of $K_n(z_{n+1}^*(x);\, x) = 0$ with respect to x produces the relation

$$(42) \qquad 0 = \Big(1 + \frac{dz_{n+1}^*}{dx}\Big) \int_0^x f_n''(x + z_{n+1}^* - \xi)\varphi(\xi)\, d\xi + f_n''(z_{n+1}^*) \int_x^\infty \varphi(\xi)\, d\xi \, \frac{dz_{n+1}^*}{dx} .$$

By (iii) and (41) we conclude that

$$(43) \qquad 1 + \frac{dz^*_{n+1}(x)}{dx} > 0 \,.$$

Insertion of the optimal ordering rule of $z^*_{n+1}(x)$ into (39) yields

$$(44) \quad f_{n+1}(x) = c \cdot z^*_{n+1}(x) + L(x) + \alpha\int_0^x f_n(x + z^*_{n+1}(x) - \xi)\varphi(\xi)\,d\xi + \alpha f_n(z^*_{n+1}(x))\int_x^\infty \varphi(\xi)\,d\xi \,.$$

We get

$$(45) \quad f'_{n+1}(x) = \begin{cases} \alpha\int_0^x f'_n(x + z^*_{n+1}(x) - \xi)\varphi(\xi)\,d\xi + L'(x) & x < \bar{x}_{n+1} \\ \alpha\int_0^x f'_n(x - \xi)\varphi(\xi)\,d\xi + L'(x) & x \geqq \bar{x}_{n+1} \,. \end{cases}$$

But

$$-L'(x) = -\int_0^x h'(x-\xi)\varphi(\xi)\,d\xi + p\int_x^\infty \varphi(\xi)\,d\xi \leqq p\int_x^\infty \varphi(\xi)\,d\xi$$

and $-f'_n(x) \leqq p$ by virtue of the induction hypothesis. Consequently,

$$(46) \qquad -f'_{n+1}(x) \leqq +\alpha p\int_0^x \varphi(\xi)\,d\xi + p\int_x^\infty \varphi(\xi)\,d\xi < p \quad \text{for all } x \,.$$

We have verified properties (i) and (ii) for the $(n+1)$-period inventory problem ; (iv) follows in a straightforward manner. We next show the truth of (iii). Let $x < \bar{x}_{n+1}$. From (45) we find

$$f''_{n+1}(x) = \left[\alpha\int_0^x f''_n(x + z^*_{n+1}(x) - \xi)\varphi(\xi)\,d\xi\right]\left[1 + \frac{dz^*_{n+1}(x)}{dx}\right] + \alpha f'_n(z^*_{n+1}(x))\varphi(x) + p\varphi(x) + \int_0^x h''(x-\xi)\varphi(\xi)\,d\xi + h'(0)\varphi(x) \,.$$

Except for $\alpha f'_n(z^*_{n+1}(x))\varphi(x)$, all terms are positive in view of what has been proved, the original hypotheses, and the induction hypotheses. Because of (ii), this term is dominated in magnitude by $p\varphi(x)$. Consequently, $f''_{n+1}(x) > 0$ for the range $x < \bar{x}_{n+1}$ as desired. Similar arguments show that $f''_{n+1}(x) > 0$ when $x \geqq \bar{x}_{n+1}$. The induction step has now been completed.

Our next objective is to proceed to a limit with $n \to \infty$. Since each $z^*_n(x)$ is a continuous differentiable decreasing function of x with

$$\frac{dz^*_n(x)}{dx} \leqq 0 \,,$$

we may select a limit $z^*(x)$ which is a continuous decreasing function of x. $z^*(x)$ has the form

$$(47) \qquad z^*(x) > 0 \quad x < \bar{x}, \qquad z^*(x) = 0 \quad x > \bar{x},$$

and satisfies

$$(48) \quad f(x) = c \cdot z^*(x) + L(x) + \alpha\int_0^x f(x + z^*(x) - \xi)\varphi(\xi)\,d\xi + f(z^*(x))\int_0^\infty \varphi(\xi)\,d\xi \,.$$

From (45) follows

$$f'(x) = \begin{cases} \alpha\int_0^x f'(x + z^*(x) - \xi)\varphi(\xi)\, d\xi + L'(x) & x < \bar{x} \\ \alpha\int_0^x f'(x - \xi)\varphi(\xi)\, d\xi + L'(x) & x > \bar{x}\,. \end{cases} \tag{49}$$

By consideration of difference quotients replacing differentiation of (49), we deduce, analogous to the argument of (47), that $f(x)$ is strictly convex. In fact, since $f_n(x)$ and $f_n'(x)$ converge uniformly to $f(x)$ and $f'(x)$, respectively, we infer that f is convex and $f'(x)$ exists everywhere and obeys the strict inequality $-f'(x) < p$. The last fact may be seen from examination of equation (45) after letting n tend to infinity.

The relation $K_n(x\,;\,z^*) = 0$ implies that $z^*(x)$ for $x < \bar{x}$ solves the equation

$$0 = c + \alpha\int_0^x f'(x + z - \xi)\varphi(\xi)\, d\xi + f'(z)\int_x^\infty \varphi(\xi)\, d\xi\,. \tag{50}$$

Since f is strictly convex, it follows that $z^*(x)$ for $x < \bar{x}$ is strictly decreasing and the slope of $z^*(x)$ is an absolute magnitude strictly smaller than 1.

All these arguments put together establish the validity of Theorem 4.

It is very difficult to find an efficient method of computing $z^*(x)$ explicitly. The qualitative structure of $z^*(x)$ as suggested by the assertion of Theorem 4 is the only guide to the form of $z^*(x)$. Rules for $z^*(x)$ which possess these characteristics depending on one or two parameters may be proposed, and the parameters can be selected to optimize a further criterion. One example of $z^*(x)$ which fits these prescriptions is

$$z^*(x) = \begin{cases} \beta(\bar{x} - x) & x < \bar{x} \\ 0 & x \geqq \bar{x}\,. \end{cases} \tag{51}$$

This procedure depends on the two parameters, β and $\bar{x}$ $(0 < \beta < 1$, according to Theorem 4), which are to be determined to satisfy two other conditions. The form of $z^*(x)$ as given in (51) cannot be the optimal rule. The optimal rule $z^*(x)$ is necessarily a continuously differentiable function, whereas this property is not satisfied for the example (51) when $x = \bar{x}$. Nevertheless, the simplicity of the form of $z^*(x)$ merits its consideration. The stationary structure of these simple policies will be studied in a separate publication.

4. The Analysis of "Simple" Policies in an Inventory Model with Time Lag

In the previous sections we were concerned with the characterization of optimal inventory policies in the presence of a time lag. It was shown that given a certain cost structure, the model II inventory problem (in which excess demand is deferred) has a simple optimal policy—i.e., the stock on hand plus incoming stock should be kept equal to a certain critical number $\bar{x}$. On the other hand, in the type I problem (in which excess demand is lost), the optimal policy is never of this simple form.

There are examples in both military and industrial areas in which the simple policy described above is used, even though the situation is most naturally thought of as a type I inventory problem. In this section we shall analyze the operating characteristics of such a policy in a type I situation, with a view to selecting an optimal value of $\bar{x}$.

Case 1: *One Stage Lag Stationary Inventory Policy.* We consider an inventory policy such that the quantity available now, plus the amount delivered next period, will equal $\bar{x}$. The transitions that transpire over one period can be described as follows: Let the pair $(x, \bar{x} - x)$ have the meaning that the first component represents the amount available at the start of the present period, and the second component is the quantity to be delivered at the start of the following period. If ξ (an observation according to the density $\varphi(\xi)$) is the demand of the period, then

$$(x, \bar{x} - x) \xrightarrow{T} (\bar{x} - x + \max(0, x - \xi), \min(x, \xi)) ; \tag{52}$$

that is, $\bar{x} - x + \max(0, x - \xi)$ will be the stock available next period and $\min(x, \xi)$ is the amount ordered to arrive one period in the future. If $\xi \leqq x$, all the demand was satisfied with available stock, while if $\xi > x$, $\xi - x$ represents lost sales. If current stock size x is a random variable with density $f(x)$, the distribution associated with the random variable measuring stock available at the next period, $g = Tf$, may be deduced easily from the expression of the transition law (52). In fact,

$$g(y) = f(\bar{x} - y)\int_{\bar{x}-y}^{\infty} \varphi(\xi)\, d\xi + \varphi(\bar{x} - y)\int_{\bar{x}-y}^{\bar{x}} f(\xi)\, d\xi \quad \text{for } 0 \leqq y \leqq \bar{x} . \tag{53}$$

The quantity $\bar{x} - x + \max(0, x - \xi)$ can equal y according to two different contingencies. $\xi > x$ requires that $\bar{x} - x = y$. This case corresponds to the first term of (53). When $\xi < x$, then $y = \bar{x} - \xi$ and we obtain the second term.

The process is said to be in an equilibrium state if $g(y) = f(y)$. Assuming that an equilibrium distribution of the form of a density exists, we get from (53)

$$f(y) = \frac{d}{dy}\left(\int_{\bar{x}-y}^{\infty} \varphi(\xi)\, d\xi\right)\left(\int_{\bar{x}-y}^{\bar{x}} f(u)\, du\right) . \tag{54}$$

If $F(y) = \int_0^y f(u)\, du$, integration of (54) gives

$$F(y) = [1 - F(\bar{x} - y)][1 - \Phi(\bar{x} - y)] . \tag{55}$$

This equation may be thought of as the stationary transition relation involving general distribution functions and may be derived independently of any restrictions as to the form of the equilibrium distribution F. Replacing y by $\bar{x} - y$ yields

$$F(\bar{x} - y) = [1 - F(y)][1 - \Phi(y)] \, . \tag{56}$$

Insertion of (56) into (55) gives

$$F(y) = \frac{\Phi(y)[1 - \Phi(\bar{x} - y)]}{1 - [1 - \Phi(y)][1 - \Phi(\bar{x} - y)]} \, . \tag{57}$$

The representation of (57) always expresses $F(y)$ in terms of Φ. It follows that the integral equation (53) possesses a unique solution exhibited by formula (57). The uniqueness of the stationary solution, together with the compact range of possible values allowed for x, implies (according to a standard method of stochastic processes) that the distribution of x converges after n periods of time as n tends to infinity, in the sense of average convergence, to the unique stationary distribution $F(y)$ given by (57).

The stationary distribution (57) having been determined explicitly, we next turn to the task of computing several quantities of direct interest in inventory analysis. To best illustrate the approach, we restrict attention to the special density $\varphi(\xi) = \lambda e^{-\lambda\xi}$. Our methods may be adapted readily to apply to the general case. Equation (57) reduces to

$$F(y) = \frac{e^{-\lambda\bar{x}}}{1 - e^{-\lambda\bar{x}}}\left[e^{\lambda y} - 1\right] = \frac{e^{\lambda y} - 1}{e^{\lambda\bar{x}} - 1} \, , \tag{58}$$

and the density is

$$f(y) = \frac{e^{-\lambda\bar{x}}}{1 - e^{-\lambda\bar{x}}} e^{\lambda y} \, . \tag{59}$$

The probability that demand exceeds supply is

$$\begin{aligned} \Pr\{\xi > y\} &= \iint_{\xi > y} f(y)\varphi(\xi)\, d\xi\, dy \\ &= \int_0^{\bar{x}} \varphi(\xi)[F(\xi)]\, d\xi + \int_{\bar{x}}^{\infty} \varphi(\xi)\, d\xi = \frac{e^{-\lambda\bar{x}}}{1 - e^{-\lambda\bar{x}}}[\bar{x} - 1 + 2e^{-\lambda\bar{x}}]. \end{aligned} \tag{60}$$

The expected quantity of lost sales is

$$\begin{aligned} E\,(\text{penalty}) &= \iint_{\xi > y} (\xi - y) f(y)\varphi(\xi)\, dy\, d\xi \\ &= \int_0^{\bar{x}} \varphi(\xi)\, d\xi \int_0^{\bar{x}} (\xi - y) f(y)\, dy + \int_{\bar{x}}^{\infty} \varphi(\xi)\, d\xi \int_0^{\bar{x}} (\xi - y) f(y)\, dy \\ &= \frac{\bar{x}}{e^{\lambda\bar{x}} - 1} \, . \end{aligned} \tag{61}$$

Another relevant quantity involves handling or storage. Two possible interpretations arise. In some circumstances, handling or carrying costs are functions of stock on hand at the start of each period. In other examples, the significant part of handling costs is associated with goods on hand at the end of each period. An intermediate case consists of treating handling cost as a function of average amount on hand for the full period. If handling is a function of original stock, then expected cost suffered from such sources is

$$\int yf(y)\,dy = \frac{\bar{x}e^{\lambda\bar{x}} - \frac{1}{\lambda}(e^{\lambda\bar{x}} - 1)}{e^{\lambda\bar{x}} - 1} . \tag{62}$$

If handling cost is evaluated as a function of excess supply over demand —or in other words, the amount on hand at the end of the period— then

$$\int_0^{\bar{x}}\int_0^{y}(y - \xi)\varphi(\xi)f(y)\,d\xi\,dy = \bar{x}\frac{e^{\lambda\bar{x}} + 1}{e^{\lambda\bar{x}} - 1} - \frac{2}{\lambda} = E\,(\text{handling}) . \tag{63}$$

Let p and h represent marginal penalty and handling costs. One important and useful criterion function is

$$hE\,(\text{handling}) + pE\,(\text{penalty}) = h\left[\bar{x}\frac{e^{\lambda\bar{x}} + 1}{e^{\lambda\bar{x}} - 1} - \frac{2}{\lambda}\right] + p\left[\frac{\bar{x}}{e^{\lambda\bar{x}} - 1}\right] . \tag{64}$$

The minimum with respect to $\bar{x}$, where $\bar{x} \geqq 0$, is achieved at the positive root of the equation

$$h\{(e^{2\lambda\bar{x}} - 1) - 2\lambda\bar{x}e^{\lambda\bar{x}}\} + p(e^{\lambda\bar{x}} - 1 - \lambda\bar{x}e^{\lambda\bar{x}}) = 0 . \tag{65}$$

This equation has a double zero at the origin, a single negative zero, and a single positive zero. The fact that the formula of (65) can have at most four zeros is a consequence of a known result concerning exponential sums. If $h = p$, then (65) yields $\bar{x} = 1.036/\lambda$.

Other optimization problems with different weighting attached to the penalty and holding costs can be performed by these same methods. The knowledge of the stationary distribution is of fundamental significance. Various cost structures may be added routinely to the format of the model.

Case 2: The k-Period Lag Stationary Inventory Policy. The model is the same as before except for the change that the lag in delivery lasts $k > 1$ periods. Ordering is always done at the close of a period and purchased orders are delivered at the start of a period. The state of goods can be represented by a $(k + 1)$-tuple

$$x,\ y_1,\ \cdots,\ y_{k-1},\ \bar{x} - x - y_1 - \cdots - y_{k-1}\ , \tag{66}$$

where x is the stock available now, y_1 is to be delivered at the start of the next period, y_2 will arrive two periods hence, etc., and finally $\bar{x} - x - y_1 - \cdots - y_{k-1}$ is scheduled to be delivered at the kth period. The ordering policy has the characteristic that total stock level (present + incoming) is maintained at the constant quantity $\bar{x}$. If the demand through the first period was ξ, the description of the state of goods becomes

$$(67) \qquad (y_1 + \max(0,\ x - \xi),\ y_2,\ y_3,\ \cdots,\ y_{k-1},\ \bar{x} - x - \sum_{i=1}^{k-1} y_i,\ \min(x, \xi)) .$$

This means that the stock level at period two is the random variable $y_1 + \max(0,\ x - \xi)$. The quantity to be delivered the following period would then be y_2, etc. The amount ordered at the end of period one, which will arrive after the lapse of k periods, is obviously $\min(x,\ \xi)$. The value $\max(\xi - x,\ 0)$ is a measure of lost sales and the random variable $\max(0,\ x - \xi)$ represents quantity of goods subject to holding costs.

If the process has been in operation infinitely long in the past, we may assume that the state of the system has reached equilibrium. Let $F(\alpha_1,\ \alpha_2,\ \alpha_3,\ \cdots,\ \alpha_k)$ denote the stationary distribution of the state $(x,\ y_1,\ y_2,\ \cdots,\ y_{k-1},\ \bar{x} - x - y_1 - \cdots - y_{k-1})$, where α_1 stands for x, α_2 for y_1, $\cdots$, α_k for y_{k-1}. Clearly, the variables are required to satisfy the inequalities of the system, $\alpha_i \geqq 0$ and

$$\sum_{i=1}^{k} \alpha_i \leqq \bar{x} .$$

The final variable α_{k+1} is evidently determined by the condition that

$$\sum_{i=1}^{k+1} \alpha_i = \bar{x}$$

and is not an independent variable. Hence, α_{k+1} may be suppressed in writing the distribution F as a function of its arguments. Suppose for simplicity that the stationary distribution has a density $f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_k)$. In most examples the equilibrium distribution is in fact a density. The general case may be dealt with analogously at the expense of more tedious complexities.

We now derive the integral equation that f must satisfy to qualify as an equilibrium density for the process. We must analyze how it is possible for the state of goods to be $(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_k)$. This can happen in two ways. First, if previously the state of the system was $(\beta_1,\ \beta_2,\ \cdots,\ \beta_k)$ and demand ξ was less than β_1, then in order for the state of the system to be $(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_k)$ after one period, it is necessary that

$$(68) \qquad \beta_2 + \beta_1 - \xi = \alpha_1, \quad \beta_3 = \alpha_2, \quad \beta_4 = \alpha_3, \quad \cdots, \quad \beta_k = \alpha_{k-1} ,$$

and

$$\bar{x} - \sum_{i=1}^{k} \beta_i = \alpha_k .$$

All the equations together imply that

$$\xi = \bar{x} - \sum_{i=1}^{k} \alpha_i \quad \text{and} \quad \beta_2 = \bar{x} - \sum_{i=2}^{k} \alpha_i - \beta_1 .$$

This is valid provided

$$\beta_1 > \xi = \bar{x} - \sum_{i=1}^{k}\alpha_i$$

(demand fell short of supply during the initial period).

The quantity β_1 can be thought of as a variable, and we have that the event of (68) occurs with probability

$$(69) \qquad \varphi(\bar{x} - \sum_{i=1}^{k}\alpha_i)\int_{\bar{x}-\sum_{i=1}^{k}\alpha_i}^{\bar{x}-\sum_{i=2}^{k}\alpha_i} f(u,\ \bar{x} - \sum_{i=2}^{k}\alpha_i - u,\ \alpha_2,\ \alpha_3,\ \cdots,\ \alpha_{k-1})\,du\ .$$

Finally, if demand ξ exceeds β_1, then

$$\beta_2 = \alpha_1, \quad \beta_3 = \alpha_2, \quad \cdots, \quad \beta_k = \alpha_{k-1}, \quad \bar{x} - \sum_{i=1}^{k}\beta_i = \alpha_k\ ,$$

and

$$\xi > \beta_1 = \bar{x} - \sum_{i=1}^{k}\alpha_i\ .$$

The probability of this event is

$$(70) \qquad f(\bar{x} - \sum_{i=1}^{k}\alpha_i,\ \alpha_2,\ \cdots,\ \alpha_{k-1})\int_{\bar{x}-\sum_{i=1}^{k}\alpha_i}^{\infty}\varphi(\xi)\,d\xi\ .$$

The terms (69) and (70) exhaust the various possibilities and taken in conjunction yield the integral equation for f:

$$(71) \qquad f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_k) = f(\bar{x} - \sum_{i=1}^{k}\alpha_i,\ \alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{k-1})\int_{\bar{x}-\sum_{i=1}^{k}\alpha_i}^{\infty}\varphi(\xi)\,d\xi$$

$$+\ \varphi(\bar{x} - \sum_{i=1}^{k}\alpha_i)\int_{\bar{x}-\sum_{i=1}^{k}\alpha_i}^{\bar{x}-\sum_{i=2}^{k}\alpha_i} f(u,\ \bar{x} - \sum_{i=2}^{k}\alpha_i - u,\ \alpha_2,\ \alpha_3,\ \cdots,\ \alpha_{k-1})\,du\ .$$

The fact that density solutions exist can be demonstrated by invoking standard ergodic theorems. The explicit computation of these solutions in general is very complicated. A more thorough investigation of the solutions of this equation will be undertaken in a later publication. In the remainder of this chapter we specialize to the specific density $\varphi(\xi) = \lambda e^{-\lambda\xi}$. A direct verification shows that

$$(72) \qquad f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_k) = Ce^{\lambda\alpha_1}$$

satisfies (71) where C is a normalizing constant. Uniqueness can also be established. The detailed arguments of this fact will be presented elsewhere. The marginal density of α_1 is

$$h(\alpha_1) = \int_{\alpha_2,\ldots,\alpha_k}\!\!\cdots\int f(\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_k)\,d\alpha_2,\ d\alpha_3,\ \cdots,\ d\alpha_k$$

$$= Ce^{\lambda\alpha_1}\int_{\alpha_2,\ldots,\alpha_k}\!\!\cdots\int d\alpha_2,\ d\alpha_3,\ \cdots,\ d\alpha_h\ ,$$

the integration being extended over the region

$$(\alpha_i \geqq 0, \quad i = 1,\ 2,\ 3,\ \cdots,\ k, \quad \text{and} \sum_{i=2}^{k} \alpha_i \leqq \bar{x} - \alpha_1)\ ,$$

or

$$h(\alpha_1) = C(\bar{x}) \frac{(\bar{x} - \alpha_1)^{k-1}}{(k-1)!} e^{\lambda \alpha_1} \tag{73}$$

and

$$C(\bar{x}) = \frac{\lambda^k}{e^{\lambda \bar{x}} - \left[1 + \lambda \bar{x} + \frac{(\lambda \bar{x})^2}{2!} + \cdots + \frac{(\lambda \bar{x})^{k-1}}{(k-1)!}\right]}\ . \tag{74}$$

The various steady state inventory quantities such as probability of shortage and average inventory level may be computed readily from the knowledge of (73) and (74). For example,

$$E\,(\text{shortage}) = \iint_{\xi > y} (\xi - y) h(y) \varphi(\xi)\, d\xi\, dy = \int_0^{\bar{x}} h(y) \int_y^{\infty} (\xi - y) \varphi(\xi)\, d\xi\, dy = \frac{C(\bar{x})}{\lambda} \frac{\bar{x}^k}{k!}\ . \tag{75}$$

The expected carrying costs are evaluated by the formula

$$E\,(\text{holding costs}) = \int_0^{\bar{x}} f(y) \int_0^{y} (y - \xi) \varphi(\xi)\, d\xi\, dy = \int_0^{\bar{x}} f(y) \left[y - \frac{(1 - e^{-y})}{\lambda}\right] dy = \bar{x} - \frac{(k+1)}{\lambda} + \frac{k+1}{\lambda} \frac{\bar{x}^k}{k!} C(\bar{x})\ . \tag{76}$$

We specialize to the case $k = 2$, and expected costs equal to

$$hE\,(\text{holding costs}) + pE\,(\text{shortage costs}) = h\left[\bar{x} - \frac{3}{\lambda} + \frac{3}{2} \frac{\bar{x}^2}{e^{\lambda \bar{x}} - 1 - \lambda \bar{x}}\right] + p\left[\frac{\bar{x}^2}{2!} \frac{\lambda}{e^{\lambda \bar{x}} - 1 - \lambda \bar{x}}\right]. \tag{77}$$

The minimum of this is achieved at the unique positive root of the equation

$$h(e^{\lambda \bar{x}} - 1 - \lambda \bar{x})^2 + \left(\frac{3}{2}\lambda h + \frac{\lambda p}{2}\right)(e^{\lambda \bar{x}}(2\bar{x} - \bar{x}^2 \lambda) - 2\bar{x} - \lambda \bar{x}^2) = 0\ . \tag{78}$$

For $h = p$, we have $\bar{x} = 1.35/\lambda$. This may be compared with the optimal policy in the one stage lag, where $\bar{x} = 1.04/\lambda$.

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