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INVENTORY MODELS AND RELATED STOCHASTIC PROCESSES

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1. Introduction

As several of the chapters in this book indicate, the discussion of inventory and production models gives rise to a number of stochastic processes of a complex character. In this chapter we shall discuss some additional questions connected to the two processes introduced by Scarf [5]. The relevance of these processes to inventory theory is elaborated in the introduction to Chapter 16, and will not be described in any detail in this chapter. The first of these processes may be viewed as a queue with an infinite number of servers, or a type II counter problem. The second process may be interpreted as a queue with N servers in which customers depart if all of the servers are busy [5]. In this chapter we shall describe some of the known results about processes of this sort, and demonstrate several new results. It should be mentioned that the interpretation in terms of inventory theory focuses attention on some novel aspects of these stochastic processes.

One physical situation which differs somewhat from those described in Chapter 16, to which the model is applicable, is as follows: We imagine a central mail order house which receives orders from a number of sources. The time interval between successive orders is assumed to be a random variable with a known distribution Ψ. Each order requires a random length of time to fill (referred to as the lag) which we assume to be sample observations based on a known distribution Ω. All orders originate independently and are handled independently.

One quantity of interest is the number N_t of unfilled orders at a given moment of time t, where initially the state of the system (number of unfilled orders) is zero. A second problem is the stationary analog of the first problem, namely, to determine the distribution of the number of unfilled orders N^* at the present time, given that the process has been going on infinitely long in the past. The distribution theory related to these variables can be obtained as an application of general theory concerned with the shot effect, developed by Takács [6], [7]. Our contribution in this case is to highlight the relevance of these ideas to the study of inventory models. This is the principal subject matter of Section 2. In particular, if $d\Psi = \mu e^{-\mu t}\, dt$, which means that orders in time constitute a Poisson process, the distribution of N_t is Poisson with parameter

$$\mu\int_0^t [1 - \Omega(s)]\, ds \,.$$

In Section 3 we shall introduce a new stochastic quantity which in the language of queueing theory represents the time necessary for all of the servers to finish their current work. A functional equation is obtained for the distribution of this random variable, which is solved completely when customers (orders) arrive according to a Poisson process, and which may be analyzed quite readily for other special cases (Ψ = gamma distribution of integral order). For the Poisson process, the stationary distribution function of this maximum is given by

$$P\{\omega \leqq x\} = e^{-\mu\int_x^\infty [1 - \Omega(\xi)]\, d\xi} \,.$$

In Section 4 this stochastic quantity is generalized so that instead of time necessary for service on all outstanding orders to be completed, we determine the distribution of the time necessary for all outstanding orders but one or two, etc., to be filled.

Section 5 is devoted to a study of a modified version of the model discussed in preceding sections of this chapter. We assume that only limited facilities for service are available, so that whenever N orders are being served, all additional orders are refused. This modification of the model examined in Section 2 cannot be treated by the methods of Takács. The problem was solved for an arbitrary interarrival distribution and a negative exponential service rate by Scarf (see also [8]). In this chapter we investigate the case where the input process is Poisson and the service distribution is general. The infinitesimal character of a Poisson process is exploited in order to derive partial differential equations satisfied by the distribution of random variables analogous to those studied in Sections 3 and 4. It is then shown (assuming a continuous service distribution) that the limiting distribution of N^* is a truncated Poisson. A special case of this result was treated in [3] (see also [8]).

Some results are appended at the close of the chapter discussing the circumstances in which the parameters of the Poisson process depend on t.

2. The Distribution of the Number of Outstanding Orders for General Input and Service Distributions

In this section we shall apply the method developed by Takács in [6] to a generalization of one of the inventory models discussed by Scarf

[5]. The reader is referred to [5] for a full discussion of this model, which may be summarized as follows. Demands for an item arrive singly with an interarrival distribution given by $\Theta(t)$. Every time that D items are sold out of current inventory, a shipment of D more is requested, and this shipment is delivered with a time lag whose distribution is $\Omega(\lambda)$. If we assume that the process begins with S items in current inventory and 0 items which have been requested but not delivered, then at any future date the sum of the items in current inventory plus the items requested will lie between $S - D + 1$ and S.

Two different models arise, depending on how we treat the situation, of a demand arriving when no current inventory is available. In this section we shall assume that such demands may be satisfied by subsequent deliveries. This permits the number of items in current inventory to be negative.

Let $w(t)$ represent the number of shipments requested but not yet delivered at time t. This may be represented as follows. Define

$$(1)\qquad f(u, v) = \begin{cases} 1 & 0 \leqq v - u \\ 0 & \text{otherwise} . \end{cases}$$

Then if $\tau_1, \tau_2, \cdots$ represent successive sums of independent observations from $\Psi(\tau)$ (the D-fold convolution of $\Theta(\tau)$), and $\lambda_1, \lambda_2, \cdots$ represent independent observations from $\Omega(\lambda)$:

$$(2)\qquad w(t) = \sum_{\tau_n \leq t} f(t - \tau_n, \lambda_n) .$$

For any y, define $\pi(y\,; t) = E(y^{w(t)})$. This is, of course, the generating function for the number of undelivered orders at time t. We shall obtain an integral equation for this function. Let us examine $w(t)$ under the condition that $\tau_1 = \tau$. Then under this condition

$$(3)\qquad w(t) = \begin{cases} f(t - \tau, \lambda_1) + \overline{w}(t - \tau) & \tau \leqq t \\ 0 & \tau > t , \end{cases}$$

where $\overline{w}$ is independent of $f(t - \tau; \lambda_1)$ and has the same distribution as w. Therefore

$$(4)\qquad E(y^{w(t)} | \tau_1 = \tau) = \begin{cases} \pi(y\,; t - \tau) E_\Omega y^{f(t-\tau, \lambda_1)} & \tau \leqq t \\ 1 & \tau > t . \end{cases}$$

This may be simplified to

$$(5)\qquad \begin{cases} \pi(y\,; t - \tau)\{\Omega(t - \tau) + y - y\Omega(t - \tau)\} & \tau \leqq t \\ 1 & \tau > t . \end{cases}$$

We obtain $\pi(y, t)$ by integrating τ with respect to its distribution $\Psi(\tau)$. Therefore

$$(6)\quad \pi(y, t) = 1 - \Psi(t) + \int_0^t \pi(y\,; t - \tau)\{(y - 1)[1 - \Omega(t - \tau)] + 1\}\, d\Psi(\tau) .$$

If we expand $\pi(y, t)$ as a power series in y about the point 1,

$$\pi(y, t) = \sum_0^\infty a_n(t)(y-1)^n , \tag{7}$$

then the factorial moments $a_n(t)$ satisfy the equations

$$\begin{aligned} &a_0(t) = 1, \\ &a_n(t) = \int_0^t a_n(t-\tau)\, d\Psi(\tau) + \int_0^t a_{n-1}(t-\tau)[1 - \Omega(t-\tau)]\, d\Psi(\tau) . \end{aligned} \tag{8}$$

This set of equations may be expressed very simply in terms of the renewal quantity $M(\tau)$ based on the distribution $\Psi(\tau)$, i.e.,

$$M(\tau) = \sum_1^\infty \Psi^{(n)}(\tau) , \tag{9}$$

where $\Psi^{(n)}(\tau)$ represents the n-fold convolution of the distribution $\Psi(\tau)$. We have, solving the Volterra equation (8) by the standard method,

$$\begin{aligned} &a_0(t) = 1, \\ &a_n(t) = \int_0^t a_{n-1}(t-\tau)[1 - \Omega(t-\tau)]\, dM(\tau) . \end{aligned} \tag{10}$$

These equations furnish a recursive procedure for computing the factorial moments, and hence the distribution of the number of shipments requested but not yet delivered at time t.

Let us, as a first example, apply these equations to the case where $\Psi(\tau) = 1 - e^{-u\tau}$, i.e., requests for shipments are a Poisson process. Then $M(\tau) = u\tau$, and

$$\begin{aligned} a_n(t) &= u\int_0^t a_{n-1}(t-\tau)[1 - \Omega(t-\tau)]\, d\tau \\ &= u\int_0^t a_{n-1}(\xi)[1 - \Omega(\xi)]\, d\xi, \end{aligned} \tag{11}$$

and hence

$$a_n'(t) = u a_{n-1}(t)[1 - \Omega(t)] .$$

Therefore

$$\frac{\partial \pi(y, t)}{\partial t} = u(y-1)[1 - \Omega(t)]\pi(y, t) , \tag{12}$$

and

$$\pi(y, t) = \exp\left\{ u(y-1)\int_0^t [1 - \Omega(\xi)]\, d\xi \right\} , \tag{13}$$

which is the generating function of a Poisson distribution with mean

$$u\int_0^t [1 - \Omega(\xi)]\, d\xi .$$

$\pi(y, \infty)$ is the generating function of a Poisson distribution with mean $uE_\Omega(\lambda)$.

As a second example, let us apply the above equations to the case in which $\Omega(\lambda) = 1 - e^{-\lambda/m}$, i.e., a negative exponential distribution for delivery time. This is the untruncated version of the problem treated by Scarf [5]. The equations become

$$a_n(t) = 1,$$
$$a_n(t) = \int_0^t a_{n-1}(t-\tau)e^{-t/m}e^{\tau/m}\,dM(\tau)\ .$$

If we denote the Laplace transform of the function $a_n(t)$ by $\tilde{a}_n(s)$, and the Laplace transform of the measure $M(t)$ by $\tilde{M}(s)$, then these equations become

$$\tilde{a}_0(s) = \frac{1}{s}\ ,$$

$$\tilde{a}_n(s) = \tilde{a}_{n-1}\left(s + \frac{1}{m}\right)\tilde{M}(s)\ ,$$

and therefore

$$\tilde{a}_n(s) = \frac{1}{s + \frac{n}{m}}\tilde{M}(s)\cdots\tilde{M}\left(s + \frac{n-1}{m}\right)\ . \tag{14}$$

Now we know that

$$a_n(\infty) = \lim_{s\to 0} s\tilde{a}_n(s)$$

(in this case the conditions for the application of the appropriate Abelian theorem are easily justified), so that

$$a_n(\infty) = \frac{m}{n}\prod_{j=1}^{n-1}\tilde{M}\left(\frac{j}{m}\right)\lim_{s\to 0} s\int_0^\infty e^{-st}\,dM(t)\ .$$

But

$$\lim_{s\to 0} s\int_0^\infty e^{-st}\,dM(t) = \lim_{s\to 0}\frac{s\int_0^\infty e^{-st}\,d\Psi(t)}{1 - \int_0^\infty e^{-st}\,d\Psi(t)} = \frac{1}{\int_0^\infty t\,d\Psi(t)} = \frac{u}{D}\ ,$$

where u is the average number of demands per unit time.

Therefore

$$a_n(\infty) = \frac{mu}{nD}\prod_{j=1}^{n-1}\tilde{M}\left(\frac{j}{n}\right)\ .$$

As a third example let us apply these techniques to the case in which the time lag is a constant λ. We shall show that if Ψ is not a lattice distribution, then

$$\lim_{t\to\infty}\pi_n(t) = \begin{cases} 1 - \frac{1}{\mu}\int_0^\lambda [1-\Psi]\,d\xi & n = 0 \\ \frac{1}{\mu}\int_0^\lambda [\Psi^{(n-1)} - 2\Psi^{(n)} + \Psi^{(n+1)}]\,d\xi & n > 0\ , \end{cases}$$

where $\pi_n(t)$ represents the probability of n undelivered orders at time t.

Let us begin by computing the factorial moments $a_n(t)$, by means of equations (10). If $t < \lambda$, we see that

$$a_n(t) = \int_0^t a_{n-1}(t-\tau)\, dM(\tau) ,$$

so that

$$a_n(t) = M^{(n)}(t) ,$$

the n-fold convolution of $M(t)$. On the other hand, for $t > \lambda$, we have

$$\begin{aligned} a_n(t) &= \int_{t-\lambda}^t a_{n-1}(t-\tau)\, dM(\tau) \\ &= \int_{t-\lambda}^t M^{(n-1)}(t-\tau)\, dM(\tau) = \int_\lambda^0 M^{(n-1)}(\xi)\, dM_\xi(t-\xi) . \end{aligned}$$

If we integrate this by parts, we obtain, for $t > \lambda$,

$$a_n(t) = M^{(n-1)}(\lambda)\{M(t) - M(t-\lambda)\} + \int_0^\lambda \{M(t-\xi) - M(t)\}\, dM^{(n-1)}(\xi) .$$

We now apply a theorem of Blackwell and Doob ([1], [2]) which implies that if Ψ is not a lattice distribution, then

$$\lim_{t\to\infty} \{M(t) - M(t-\xi)\} = \frac{\xi}{\mu} .$$

Therefore, if Ψ is not a lattice distribution,

$$\begin{aligned} \lim_{t\to\infty} a_n(t) &= M^{(n-1)}(\lambda)\frac{\lambda}{\mu} - \frac{1}{\mu}\int_0^\lambda \xi\, dM^{(n-1)}(\xi) \\ &= \frac{1}{\mu}\int_0^\lambda M^{(n-1)}\, d\xi = \frac{1}{\mu}\int_0^\lambda a_{n-1}(\xi)\, d\xi . \end{aligned}$$

In order to obtain $\lim_{t\to\infty} \pi_n(t)$, we examine the generating function $\pi(y, t)$. We have

$$\begin{aligned} \lim_{t\to\infty} \pi(y, t) &= 1 + \frac{(y-1)}{\mu} \sum_0^\infty (y-1)^n \int_0^\lambda a_n(\xi)\, d\xi \\ &= 1 + \frac{(y-1)}{\mu} \int_0^\lambda \pi(y, \xi)\, d\xi . \end{aligned}$$

Therefore

$$\pi_0(\infty) = 1 - \frac{1}{\mu}\int_0^\lambda \pi_0(\xi)\, d\xi ,$$

and

$$\pi_n(\infty) = \frac{1}{\mu}\int_0^\lambda [\pi_{n-1}(\xi) - \pi_n(\xi)]\, d\xi \quad \text{for } n > 0 .$$

Since it is clear that $\pi_n(\xi) = \Psi^{(n+1)}(\xi) - \Psi^{(n)}(\xi)$ for $\xi < \lambda$, the above result follows.

3. The Maximum Time for Service to be Completed

In this section we shall discuss another quantity associated with the process described in Section 2. We define the random variable $m(t)$ to be the time necessary for all of the shipments outstanding at time t to be delivered where initially there were no unfilled orders. In the notation of Section 2, we may write

$$m(t) = \max_{\tau_n \leq t} \{\lambda_n - (t - \tau_n), 0\} . \tag{15}$$

Let us define, for any t, $F(y, t)$ to be the distribution function corresponding to $m(t)$. This is in contrast to Section 2, in which a generating function rather than a distribution function was discussed. We shall obtain an integral equation for this function. Let us examine $m(t)$ under the condition that $\tau_1 = \tau$. Then under this condition

$$m(t) = \begin{cases} \max \{\bar{m}(t - \tau), \lambda_1 - (t - \tau)\} & \tau \leqq t \\ 0 & \tau > t , \end{cases} \tag{16}$$

where $\bar{m}$ is independent of λ_1 and has the same distribution as m. Therefore

$$\text{prob}\,\{m(t) \leqq y \mid \tau_1 = \tau\} = \begin{cases} F(y, t - \tau)\Omega(y + t - \tau) & \tau \leqq t \\ 1 & \tau > t , \end{cases}$$

and if we integrate out the condition, we obtain

$$F(y, t) = 1 - \Psi(t) + \int_0^t F(y, t - \tau)\Omega(y + t - \tau)\, d\Psi(\tau) . \tag{17}$$

It is to be noticed that generally there will be a non-zero probability that no shipments are outstanding at time t. This reflects itself in the fact that $F(0, t) > 0$, in most cases.

With relatively general assumptions on the functions Ω and Ψ, this equation may be solved by substituting a power series in t for $F(y, t)$ and equating coefficients. We shall not discuss this method in detail. Successive approximations may also be employed in order to obtain numerical evaluations of (17). Let us instead examine the above equation in the special case in which the time between successive requests for shipments is given by a negative exponential distribution. In other words, $\Psi(\tau) = 1 - e^{-u\tau}$. Then

$$F(y, t) = e^{-ut} + u\int_0^t F(y, t - \tau)\Omega(y + t - \tau)e^{-u\tau}\, d\tau . \tag{18}$$

Making the substitution $t - \tau = \xi$, this becomes

$$F(y, t) = e^{-ut} + u\int_0^t F(y, \xi)\Omega(y + \xi)e^{-ut}e^{u\xi}\, d\xi .$$

This shows us that $F(y, t)$ is differentiable in t, and we may write

$$\frac{\partial F(y,t)}{\partial t} = -u[1-\Omega(y+t)]F(y,t)\,. \tag{19}$$

Using the initial condition $F(y, 0) = 1$, we obtain

$$F(y,t) = \exp\left(-u\int_y^{t+y}[1-\Omega(\xi)]\,d\xi\right). \tag{20}$$

If the distribution of the time between successive requests for shipments is a member of the Γ family of distributions, then a procedure similar to the one above will give us an ordinary linear differential equation for $F(y, t)$, but of higher order than the first. Indeed, suppose

$$d\Psi(t) = \frac{u^k t^{k-1}e^{-ut}}{\Gamma(k)}\,dt \quad (k \geqq 1)\,.$$

The integral equation (17) becomes

$$e^{ut}F(y,t) = P_k(t) + \frac{u^k}{\Gamma(k)}\int_0^t F(y,\xi)\Omega(y+\xi)(t-\xi)^{k-1}e^{u\xi}\,d\xi\,, \tag{21}$$

where

$$P_k(t) = e^{ut}\int_t^\infty \frac{u^k\xi^{k-1}e^{-u\xi}}{\Gamma(k)}\,d\xi$$

is a polynomial of degree $k-1$. Differentiation of (21) k times and then cancellation of the common factor e^{ut} yields

$$\frac{\partial^k F}{\partial t^k} + \binom{k}{1}u\frac{\partial^{k-1}F}{\partial t^{k-1}} + \binom{k}{2}u^2\frac{\partial^{k-2}F}{\partial t^{k-2}} + \cdots + \binom{k}{i}u^i\frac{\partial^{k-i}F}{\partial t^{k-i}} + \cdots + \binom{k}{k}u^kF = \Omega(y+t)F\,. \tag{22}$$

The initial data are obtained after each successive differentiation of (21) by setting $t = 0$. We obtain $F(y, 0) \equiv 1$ and

$$\frac{\partial F}{\partial t}(y,0) = \frac{\partial^2 F}{\partial t^2}(y,0) = \cdots = \frac{\partial^{k-1}F}{\partial t^{k-1}}(y,0) = 0\,. \tag{23}$$

For general Ω, it is very difficult to explicitly solve (22) for F. If $k = 2$ and $\Omega(y+t) = 1 - e^{-\lambda(y+t)}$ (lag is an exponential distribution), then the solution of (22) may be represented as a combination of hypergeometric functions. For an arbitrary k, the representation of the solution can be expressed in terms of generalized hypergeometric functions. One particular case of importance is where

$$\Omega(u) = \begin{cases}1 & u > a\\ 0 & u < a\,.\end{cases}$$

This is the physical circumstance where the lag in delivery is of fixed length a. Of course, for this case $F(y, t) = 1$ for $y \geqq a$ and consequently $F(y, t)$ need be evaluated only for the range $y < a$. In this situation, (22) reduces to the pair of equations

$$(24)\qquad \frac{\partial^k F}{\partial t^k} + \binom{k}{1} u \frac{\partial^{k-1} F}{\partial t^{k-1}} + \cdots + u^{k-1} \binom{k}{k-1} \frac{\partial F}{\partial t} + u^k F = 0 \qquad y + t < a$$

$$(25)\qquad \frac{\partial^k F}{\partial t} + \binom{k}{1} u \frac{\partial^{k-1} F}{\partial t^{k-1}} + \cdots + u^{k-1} \binom{k}{k-1} \frac{\partial F}{\partial t} = 0 \qquad y + t \geqq a\,.$$

Inspection of (21) will uncover the fact that

$$F(y, t), \cdots, \frac{\partial^{k-1} F}{\partial t^{k-1}}$$

are all continuous for $y + t = a$. From (21), we deduce directly that

$$(26)\qquad F(y, t) = \int_t^\infty \frac{u^k \xi^{k-1} e^{-u\xi}}{\Gamma(k)} d\xi \quad \text{for } y + t < a\,.$$

Turning to the range $y + t \geqq a$, we observe that the algebraic characteristic equation associated with (25) is

$$(27)\qquad (\alpha + u)^k - u^k = 0\,.$$

If ω denotes a primitive kth root of unity, then the solutions of (27) are all distinct and are indeed

$$(28)\qquad \alpha_r = -u + u\omega^r \qquad r = 0, 1, 2, \cdots, k-1\,.$$

The general solution of (25) may now be explicitly constructed from knowledge of the roots α_r. In fact,

$$(29)\qquad F(y, t) = \sum_{r=0}^{k-1} A_r e^{\alpha_r t} \quad \text{for } y + t \geqq a\,,$$

where A_r represents arbitrary constants. These constants are to be determined so that

$$F, \frac{\partial F}{\partial t}, \cdots, \frac{\partial^{k-1} F}{\partial t^{k-1}}$$

are all continuous for $t = a - y$ $(y \leqq a)$. These conditions translate into the system of linear equations

$$(30)\qquad \sum_{r=0}^{k-1} A_r \alpha_r^l e^{\alpha_r (a-y)} = \frac{d^l}{dt^l} \left(\int_t^\infty \frac{u^k \xi^{k-1} e^{-u\xi}}{\Gamma(k)} d\xi \right) \Bigg|_{a-y}$$

where $l = 0, 1, 2, \cdots, k-1$, from which the constants A_r are uniquely determined since the matrix of the coefficients is the familiar Vandermonde matrix with α_r distinct. For the special case $k = 2$, we have in particular

$$(31)\quad F(y, t) = \begin{cases} (1 + ut)e^{-ut} & y + t < a \\ \left[1 + \left(u - \frac{u^2}{2}\right)(a - y)\right] e^{-u(a-y)} + \frac{u^2(a-y)}{2} e^{(2-u)(a-y)} e^{-2t} & y + t \geqq a \text{ and } y < a \\ 1 & y \geqq a\,. \end{cases}$$

4. Extensions

We shall now introduce a stochastic quantity which is somewhat more complex than the time necessary for all outstanding shipments to be delivered. Let us designate the random variable $m(t)$ by $m_0(t)$. Let $m_1(t)$ represent the time necessary for all outstanding orders but one to be delivered, and in the same way $m_k(t)$ is the random variable

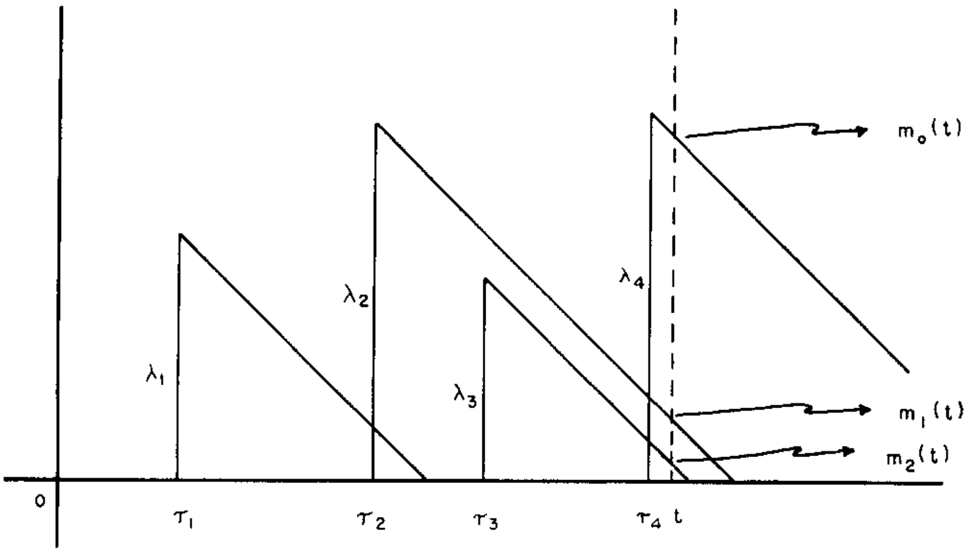

FIGURE 1

which represents the time for all of the outstanding shipments but k to be delivered. Of course, if there are n outstanding shipments at time t, the variables $m_k(t)$ will be zero for $k \geqq n$. All the random variables $m_k(t)$ are defined with $m_k(0) = 0$. Fig. 1 will clarify these definitions.

Let $F_k(x_0, \cdots, x_k, t)$ be the joint distribution of $m_0(t), \cdots, m_k(t)$. We shall use an argument similar to that given above, to obtain an integral equation for this quantity.

In order to simplify the exposition, let us begin with the case $k = 1$. Let us examine $(m_0(t), m_1(t))$ under the condition $\tau_1 = \tau$. If $\tau > t$, then $(m_0(t), m_1(t)) = (0, 0)$. On the other hand, if $\tau \leqq t$, then $(m_0(t), m_1(t))$ is equal to the highest two of the three quantities $\lambda_1 - (t - \tau)$, $\overline{m}_0(t - \tau)$, $\overline{m}_1(t - \tau)$, arranged in descending order. As before, λ_1 is independent of $(\overline{m}_0(t - \tau), \overline{m}_1(t - \tau))$ and this latter couple has the same distribution as its unbarred counterpart, at time $t - \tau$. We wish to obtain an expression for

$$\text{prob}\,\{m_0(t) \leqq x_0,\, m_1(t) \leqq x_1 \mid \tau_1 = \tau\}\ .$$

We shall only consider $x_0 \geqq x_1$. If $\tau > t$, then this probability is equal to 1. For the case in which $\tau < t$, let us examine this expression under three different assumptions.

1. $\lambda_1 - (t - \tau) > x_0$.

This case is incompatible with $m_0(t) \leqq x_0$.

2. $x_0 \geqq \lambda_1 - (t - \tau) > x_1$.

Under this condition, in order to have $m_0(t) \leqq x_0$ and $m_1(t) \leqq x_1$, we cannot have $\overline{m}_0(t - \tau) \geqq x_1$. For if this were true, we should have

$$m_1(t) \geqq \min \{\lambda_1 - (t - \tau), \overline{m}_0(t - \tau)\} > x_1 .$$

Hence we must have $\overline{m}_0(t - \tau) < x_1$, which implies that $m_1(t) = m_0(t - \tau)$. It follows that under this condition, the above probability is equal to prob $\{\overline{m}_0(t - \tau) \leqq x_1\}$, which is equal to $F_0(x_1 ; t - \tau)$.

3. $x_1 \geqq \lambda_1 - (t - \tau)$.

It is easy to see that under this condition, a necessary and sufficient condition that $m_0(t) \leqq x_0$ and $m_1(t) \leqq x_1$ is that $\overline{m}_0(t - \tau) \leqq x_0$ and $\overline{m}_1(t - \tau) \leqq x_1$. Hence the above probability is equal to $F_1(x_0, x_1; t - \tau)$.

Combining these conditions we see that

$$\text{prob}\, \{m_0(t) \leqq x_0, m_1(t) \leqq x_1 \mid \tau_1 = \tau\}$$
$$= \begin{cases} F_0(x_1 ; (t - \tau))[\Omega(x_0 + t - \tau) - \Omega(x_1 + t - \tau)] \\ \qquad + F_1(x_0, x_1 ; t - \tau)\Omega(x_1 + t - \tau) & \tau \leqq t \\ 1 & \tau > t . \end{cases}$$

If we integrate with respect to the distribution of τ, we obtain

$$F_1(x_0, x_1 ; t) = 1 - \Psi(t) + \int_0^t F_0(x_1 ; t - \tau)[\Omega(x_0 + t - \tau) - \Omega(x_1 + t - \tau)]\, d\Psi(t) + \int_0^t F_1(x_0, x_1 ; t - \tau)\Omega(x_1 + t - \tau)\, d\Psi(t) . \tag{32}$$

This is an integral equation for F_1 which may be solved if we know F_0.

The same argument may be applied for any value of k, and the following integral equation is obtained.

$$\begin{aligned} F_k(x_0, \cdots, x_k ; t) &= 1 - \Psi(t) \\ &+ \sum_{j=0}^{k-1} \int_0^t F_{k-1}(x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_k ; t - \tau) \\ &\qquad \times [\Omega(x_j + t - \tau) - \Omega(x_{j+1} + t - \tau)]\, d\Psi(\tau) \\ &+ \int_0^t F_k(x_0, \cdots, x_k ; t - \tau)\Omega(x_k + t - \tau)\, d\Psi(\tau), \quad \text{for } k \geqq 1 . \end{aligned} \tag{33}$$

Let us examine these equations in the special case in which $\Psi(\tau) = 1 - e^{-u\tau}$. We obtain

$$\begin{aligned} &\frac{\partial F_k(x_0, \cdots, x_k; t)}{\partial t} + u[1 - \Omega(x_k + t)]F_k(x_0, \cdots, x_k; t) \\ &= u\sum_{j=0}^{k-1} F_{k-1}(x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_k; t)[\Omega(x_j + t) - \Omega(x_{j+1} + t)] , \end{aligned} \tag{34}$$

which may be solved recursively.

These equations furnish an alternative procedure for computing the probability distribution of the number of shipments not yet delivered at time t. If we denote by $p_k(t)$ the probability that the number of outstanding shipments at time t is less than or equal to k, then clearly $p_k(t) = F_k(\infty, \cdots, \infty, 0; t)$, and therefore

$$p_k'(t) + u[1 - \Omega(t)]p_k(t) = +u[1 - \Omega(t)]p_{k-1}(t) . \tag{35}$$

The solution of this system of equations yields the partial sums of the exponential series

$$p_k(t) = e^{-a}\sum_{r=0}^{k}\frac{a^r}{r!}$$

with parameter

$$a = u\int_0^t [1 - \Omega(\xi)]\, d\xi .$$

5. Truncated Model with Poisson Input

This is the same model as discussed in Section 2 with the modification that whenever the state of the process (number of unfilled orders) reaches N, then all subsequent demands are turned away until the process falls back to the state $N-1$. We assume that the demands occur singly and arise as a Poisson process with parameter λ. Again, we shall analyze for this truncated variant of the model the random variables $m_0(t), m_1(t), \cdots, m_N(t)$ introduced in Section 4. The reader should observe that $m_i(t)$ $(i \geqq N+1)$ is not defined for the truncated model, since at most N orders can be unfilled at any given moment in time. The methods employed in deriving equations (17) and (33) are not immediately applicable in the present context. Instead of pursuing an approach generalizing equation (34) when the input is Poisson, it appears simpler to propose a method of analysis which exploits in a direct way the properties of the Poisson input. Let $q_N(t)$ be the probability that the state of the process (number of unfilled orders) at time t is smaller than N. Let

$$F_0^N(x; t) = \Pr_N\{m_0(t) \leqq x\} ,$$

where the superscript N is inserted to emphasize the level of truncation.

We shall assume in what follows that $\Omega(x)$ is a continuous function of x. Some remarks on the situation where $\Omega(x)$ may have jumps will be given later.

By the standard infinitesimal arguments valid for the case of Poisson input we show that $F_0^N(x; t)$ satisfies a first order partial differential equation. This is accomplished by considering what can happen during the interval of time $(t, t+h)$ with h sufficiently small. If $F_0^N(x; t)$ has continuous partial derivatives, then it follows that

$$(36) \qquad F_0^N(x; t+h) = (1-\lambda h)F_0^N(x+h; t) + \lambda h q_N(t)\Omega(x)F_0^N(x; t) + \lambda h(1-q_N(t))F_0^N(x; t) + o(h) .$$

Indeed, the lefthand side of $F_0^N(x; t+h)$ is equal to $\Pr\{m_0(t+h) \leqq x\}$ by definition. We investigate now the meaning of the terms of the righthand side of equation (36). The event $m_0(t+h) \leqq x$ can result in two ways. At time t, $m_0(t) \leqq x+h$ and no demand occurred during the time interval $(t, t+h)$. The first term of the righthand side is the probability of such a contingency. On the other hand, if a demand arises in the time interval $(t, t+h)$ (prob $\sim \lambda h$), then this demand can affect the possible values of $m_0(t)$ only if the state of the process is less than N. If this is the case, then $m_0(t+h) \leqq x$ if and only if $\max(y, m_0(t)) \leqq x+h$, where y is an observation from the distribution $\Omega(y)$. The second term of (36) is the probability of this event to within an order of magnitude $o(h)$. The third term corresponds to the circumstance when a demand occurs but the state of the system is N and therefore this demand is to be disregarded.

Dividing by h and letting h approach zero, we obtain

$$(37) \qquad \frac{\partial F_0^N(x; t)}{\partial t} = \frac{\partial F_0^N(x; t)}{\partial x} - q_N(t)\lambda F_0^N(x; t)[1-\Omega(x)] , \ F_0^N(x; 0) = 1 \qquad (x > 0) .$$

The fact that $F_0^N(x; t)$ possess continuous partial derivatives may be seen as follows. First consider the event that m pulses have occurred in time t according to the Poisson input. The conditional distributions of the locations t_i in the interval $[0, t]$ of the m events of a Poisson process are known to be beta distributions. We associate m observations y_i based on $\Omega(x)$ with these events. In terms of y_i one can compute the conditional $F_0^N(x; t)$, and it is readily seen that this distribution function has a continuous partial derivative with respect to x. This can be done for any m and since the convergence is uniform the existence of $\partial F_0^N(x; t)/\partial x$ is secured. A direct argument on (36) then establishes the existence of

$$\frac{\partial F_0(x; t)}{\partial t} .$$

The solution of (37) following standard procedures is obtained as

$$(38) \qquad F_0^N(x; t) = e^{-\lambda \int_x^{x+t} q_N(x+t-\xi)[1-\Omega(\xi)]d\xi} .$$

We prove now that $\lim_{t\to\infty} q_N(t)$ exists. This is the substance of the following two lemmas.

LEMMA A. $\lim_{t\to\infty} \int_0^t q_N(t-\xi)[1-\Omega(\xi)]d\xi$ *exists* .

PROOF. Let the random variable A_0 represent the duration of time starting with zero for which all servers are free. Let B_0 represent the subsequent time interval for which at least one server is busy. A_1 is defined as the following idle time (no servers are busy) and B_1 the next busy period. The random variables A_i and B_i are constructed analogously. Because of the character of the Poisson input the chance variables A_i are independent observations with a common exponential distribution. The B_i are likewise independent identically distributed random variables which are also independent of A_i.

A time value t is said to be covered by A if t falls in one of the A intervals. Invoking a result of renewal processes, we obtain

$$\lim_{t\to\infty} \Pr\{t \text{ is covered by } A\} = \frac{E(A)}{E(A)+E(B)} \tag{39}$$

(see [4], p. 292), where $E(A)$ and $E(B)$ are the average values of A_i and B_i respectively. From its interpretation the probability that t is covered by A is precisely $F_0^N(0, t)$. Hence the conclusion of the lemma follows on comparing (38) and (39).

Our next lemma is an application of the classical Wiener Tauberian theorem.

LEMMA B. $\lim_{t\to\infty} q_N(t)$ *exists.*

PROOF. It is easy to verify, since $\Omega(\xi)$ is continuous, that $1-\Omega(\xi)$ is integrable (because $\int \xi d\Omega(\xi) < \infty$) and possesses a Fourier transform which vanishes nowhere. Next we show that $q_N(t)$ is a slowly decreasing function, i.e.,

$$\varliminf_{\substack{t\to\infty\\ h\to 0}} [q_N(t+h) - q_N(t)] \geq 0 .$$

Recall that $q_N(t+h)$ is the probability that at time $t+h$ fewer than N servers are busy. The usual infinitesimal arguments show that

$$q_N(t+h) \geq q_N(t)(1-\lambda h) .$$

Clearly,

$$q_N(t+h) - q_N(t) \geq -\lambda h q_N(t) \geq -\lambda h,$$

and we deduce that $q_N(t)$ is slowly decreasing. In view of Lemma A and the Wiener theorem, the assertion of Lemma B is established [9].

Applying Lemma B and a simple Abelian theorem, we obtain the limit result

$$\lim_{t\to\infty} F_0^N(x; t) = F_0^N(x) = e^{-q^N\lambda\int_x^\infty [1-\Omega(\xi)]d\xi} \tag{40}$$

where $q^N = \lim_{t\to\infty} q_N(t)$.

Let $x_0 \geq x_1 \geq x_2 \geq \cdots \geq x_k \geq 0$. Then

$$F_k^N(x_0, x_1, \cdots, x_k; t) = \Pr_N\{m_0(t) \leq x_0,\ m_1(t) \leq x_1, \cdots, m_k(t) \leq x_k\} \tag{41}$$

for $k \leq N$, where the symbol N again means that we are considering the model truncated as described above. Similar to (37) we may derive a partial differential equation which F_k^N must satisfy. A calculation yields

$$\frac{\partial F_k^N}{\partial t} = \sum_{i=0}^{k} \frac{\partial F_k^N}{\partial x_i}(x_0, x_1, \cdots, x_k; t) + q_N(t)\lambda \sum_{j=0}^{k-1} F_{k-1}^N(x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_k; t)[\Omega(x_j) - \Omega(x_{j+1})] - q_N(t)\lambda F_k^N(x_0, x_1, \cdots, x_k; t)[1 - \Omega(x_k)], \quad \text{for } k \geqq 1, \tag{42}$$

with

$$F_k^N(x_0, x_1, \cdots, x_k; 0) \equiv 1 .$$

Equations (37) and (42) may be solved recursively by means of the recognized methods of the theory of first order linear partial differential equations. To describe the form of the distribution, we record the result for the case $k = 1$ (see also (38) for the case $k = 0$), viz.:

$$F_1^N(x_0, x_1; t) = e^{-\lambda \int_{x_1}^{t+x_1} q_N(t+x_1-\xi)[1-\Omega(\xi)]d\xi} \times \left[1 + \lambda \int_{x_1}^{t+x_1} q_N(t + x_1 - \xi)[\Omega(x_0 - x_1 + \xi) - \Omega(\xi)]\, d\xi\right] .$$

The remaining F_k^N may be obtained similarly. One can prove by solving (42) explicitly and citing the consequence of Lemma B that

$$\lim_{t\to\infty} F_k^N(x_1, x_2, \cdots, x_k; t)$$

converges to $F_k^N(x_1, x_2, \cdots, x_k)$. This last stationary distribution may be computed directly by recursively solving the equation

$$\sum_{i=0}^{k} \frac{\partial F_k^N}{\partial x_i}(x_0, x_1, \cdots, x_k) = -\lambda q^N \sum_{j=0}^{k-1} F_{k-1}^N(x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_k)[\Omega(x_j) - \Omega(x_{j+1})] + q^N \lambda F_k^N(x_0, x_1, \cdots, x_k)[1 - \Omega(x_k)] .$$

We shall now utilize the result of (42) to deduce the form of the limiting distribution of the probabilities

$$p_r = \lim_{t\to\infty} p_r(t)$$

where $p_r(t)$ is the probability that the number of busy servers at time t does not exceed r.

Let $x_0 = x_1 = \cdots = x_{k-1} = \infty$ (i.e., let $x_0 \to \infty$, then let $x_1 \to \infty$, etc., in (42)) and set $G_k(x, t) = F_k^N(\infty, \infty, \infty, \cdots, \infty, x; t)$. It follows from (42) that

$$\frac{\partial G_k}{\partial t} = \frac{\partial G_k}{\partial x}(x; t) + q_N(t)\lambda G_{k-1}(x; t)[1 - \Omega(x)] - q_N(t)\lambda G_k(x; t)[1 - \Omega(x)] . \tag{43}$$

If $t \to \infty$, (43) results in the expression

$$0 = \frac{d}{dx} G_k^*(x) + \lambda q^N G_{k-1}^*(x)[1 - \Omega(x)] - \lambda q^N G_k^*(x)[1 - \Omega(x)] , \tag{44}$$

where

$$G_k^*(x) = \lim_{t \to \infty} G_k(x; t) .$$

Solving equations (44) and successively setting $x = 0$ shows that $p_k = G_k^*(0)$ is a finite exponential polynomial

$$e^{-a}\sum_{r=0}^{k} \frac{a^r}{r!}$$

with

$$a = \lambda q^N \int_0^\infty [1 - \Omega(\xi)] \, d\xi .$$

This can also be seen on comparing (44) and (35). The constant q^N may be determined by the normalization requirement that $G_N^*(0) = 1$. This proves that the limiting distributions for the state (number of busy servers) of the process is a truncated Poisson distribution [8].

The result substantiates an old conjecture concerning the number of busy telephone lines. The special case where Ω is an exponential distribution was originally analyzed by Erlang [3]. It was stated by Takács in 1957 [8] that it is widely believed that the limiting distribution of the probabilities of the number of busy servers for a Poisson input is a truncated Poisson distribution depending only on the mean of $\Omega(x)$ and in all other respects independent of the form of $\Omega(x)$. Takács further claims that there does not seem to exist in the literature a proof of this fact. As a corollary of our study of the distributions F_k^N we have verified this conjecture in the circumstance where $\Omega(x)$ is continuous and otherwise arbitrary. Actually, if the above proof is examined, it may be seen to apply to the more general case in which Ω is a non-lattice distribution. It seems extremely intuitive that the same result holds with $\Omega(x)$ an arbitrary distribution, and this could probably be proved by approximating $\Omega(x)$ by continuous distributions. A rigorous proof of this limiting analysis has not been carried out.

6. Poisson Input Varying with Time

We consider again the model (with no truncation) discussed in Sections 2, 3, and 4, such that pulses arrive according to a Poisson process with parameter $\lambda(t)$, a continuous positive function of time. The service time distribution is $\Omega(x)$, which we assume initially to be a continuous function of x. The practical applications to inventory analysis in allowing $\lambda(t)$ to vary with time are clear.

We introduce, for $x_0 \geqq x_1 \geqq x_2 \geqq \cdots \geqq x_k$,

$$F_k(x_0, x_1, x_2, \cdots, x_k; t) = \Pr\{m_0(t) \leq x_0, m_1(t) \leqq x_1, \cdots, m_k(t) \leqq x_k\}$$

for $k = 0, 1, 2, \cdots$, where the $m_i(t)$ have the same meaning as previously. Since the input process is Poisson, we find that

$$(45)\qquad \frac{\partial F_k}{\partial t} = \sum_{i=0}^{k}\frac{\partial F_k}{\partial x_i} + \lambda(t)\sum_{j=0}^{k-1} F_{k-1}(x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_k; t)[\Omega(x_j) - \Omega(x_{j+1})] - \lambda(t)F_k(x_0, x_1, \cdots, x_k; t)[1 - \Omega(x_k)] \quad \text{for } k \geqq 0 ,$$

with

$$F_k(x_0, x_1, \cdots, x_k; 0) \equiv 1 \quad \text{and} \quad F_{-1} \equiv 0 .$$

The reader may observe that (45) is identical to (42) with $\lambda(t)$ replacing $\lambda q_N(t)$. We may solve for F_k recursively, which yields

$$(46)\qquad F_0(x; t) = e^{-\int_x^{x+t}\lambda(x+t-\xi)[1-\Omega(\xi)]d\xi} ,$$

$$F_1(x_0, x_1; t) = e^{-\int_{x_1}^{t+x_1}\lambda(t+x_1-\xi)[1-\Omega(\xi)]d\xi} \times \left[1 + \int_{x_1}^{x_1+t}\lambda(t + x_1 - \xi)\{\Omega(x_0 - x_1 + \xi) - \Omega(\xi)\}\, d\xi\right] , \text{ etc.}$$

If $\lambda(t)$, as t tends to ∞, converges to $\lambda > 0$, then a simple direct Abelian argument shows that the limit distribution for the number of busy servers is a Poisson distribution with parameter $\lambda\mu$ where

$$\mu = \int_0^\infty \eta d\Omega(\eta) .$$

In this case we can show that the limit distribution is Poisson even if $\Omega(x)$ is not continuous. Indeed, consider three service distributions obeying the inequalities $\Omega^1(x) \leqq \Omega(x) \leqq \Omega^2(x)$ where $\Omega^1(x)$ and $\Omega^2(x)$ are continuous, and let the input process be the same Poisson process with parameter $\lambda(t)$. It is very easy to see that

$$F_k^1(x_0, x_1, \cdots, x_k; t) \leqq F_k(x_0, x_1, \cdots, x_k; t) \leqq F_k^2(x_0, x_1, \cdots, x_k; t) .$$

In particular

$$e^{-\lambda\int_0^\infty[1-\Omega^1(\xi)]d\xi} \leqq \varliminf_{t\to\infty} F_0(0,t) \leqq \varlimsup_{t\to\infty} F_0(0,t) \leqq e^{-\lambda\int_0^\infty[1-\Omega^2(\xi)]d\xi} .$$

Letting $\Omega^1(\xi)$ and $\Omega^2(\xi)$ approach $\Omega(\xi)$, we deduce that

$$\lim_{t\to\infty} F_0(0, t) = e^{-\mu\lambda} ,$$

where

$$\mu = \int_0^\infty [1 - \Omega(\xi)]\, d\xi .$$

An extension of this argument gives

$$\lim_{t\to\infty} F_k(\infty, \infty, \cdots, \infty, 0; t) = e^{-\mu\lambda}[1 + (\mu\lambda) + \cdots + \frac{(\mu\lambda)^k}{k!}] ,$$

and we conclude that the limit distribution of the number of busy servers is Poisson with parameter $\lambda\mu$.

If $\lambda(t)$ does not converge, then it is easy to construct examples such that $F_0(0, t)$ does not converge even if $\lambda(t)$ is almost periodic.

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